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INITIAL BOUNDARY VALUE PROBLEMS FOR QUASILINEAR HYPERBOLIC-PARABOLIC COUPLED SYSTEMS IN HIGHER DIMENSIONAL SPACES

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Abstract

The initial bounary value problem for quasilinear hyperbolic-parabolic coupled systems in higher dimensional spaces, which arises in many mechanical problems is considered. Under the assumptions that the hyperbolic part of the coupled system is a quasilinear symmetric hyperbolic system and the parabolic part is a quasilinear parabolic system of second order and suitable assumptions of smoothness and compatibiliy conditions, the existence and uniqueness of local smooth solution is proved in the cases that the boundary of domain is noncharacteristic or uniformly characteristic with respect to the hyperbolic part.

As an application, the existence and uniqueness of local smooth solution for the initial boundary problem of the radiation hydrodynamic system, as well as of the viscous compressible hydrodynamic system, with solid wal' boundary, is obtained.

1. Introduction

In the recent years a great attention has been paid to the research of problems for the hyperbolic-parabolic coupled systems because of stimulation and motivation of radiation hydrodynamics, viscous compressible hydrodynamics and many other physical problems. The Cauchy paoblem for this kind of coupled systems has been considered in [1, 2]. In [1] it was done in higher dimensional case with the assumption of certain symmetry, i. e., the part of hyperoblic system was assumed to be symmetric. In [2] it was done in two-dimensional case with the general hyperbolic part, and in [3] for the initial boundary problem in two-dimensional case. In [5] the first initial boundary value problem of the viscous compressible hydrodynamic systems was considered. We should notice that the discussion benefited in [5] by the fact that the hyperbolic part is only one equation.

In the present paper we consider the following hyperbolic-parabolic coupled systems

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$$\begin{cases} a_0(t,\ x,\ u,\ v)\dfrac{\partial u}{\partial t}+\sum\limits_{i=1}^{n}a_i(t,\ x,\ u,\ v)\dfrac{\partial u}{\partial x_i}=f(t,\ x,\ u,\ v,\ v_x), \\ \dfrac{\partial v}{\partial t}-\sum\limits_{i,j=1}^{n}\dfrac{\partial}{\partial x_i}\left(a_{ij}(t,\ x,\ u,\ v)\dfrac{\partial v}{\partial x_j}\right)=g(t,\ x,\ u,\ v,\ u_x,\ v_x), \end{cases} \tag{1.1}$$

where u is the unknown k-vector function, v, for simplisity, is the unknown scalar function. The second part of (1.1) can be easily extended to the case of certain kind of parabolic systems for v (see the remark in the second section).

Let $\Omega\subset R^n$ be a bounded domain with C^∞ smooth boundary $\partial\Omega$. In $\Omega\times(0,\ h)$ we want to solve the following initial boundary value problem for (1.1)

$$\begin{cases} t=0:\quad u=0,\ v=0, \\ \text{on } \partial\Omega:\quad M(t,\ x)u=0,\ v=0, \end{cases} \tag{1.2}$$

where M is a given $l*k$ matrix function ($l\leqslant k$).

The second and third sections will be devoted to the problem (1.1), (1.2) in which $\partial\Omega$ is noncharacteristic with respect to the first part of (1.1). The section 4 will be devoted to the case that $\partial\Omega$ is characteristic with respect to the first part of (1.1). In both cases we prove the existence and uniqueness of local smooth solution for (1.1), (1.2). Finally, as an application, we discuss the initial boundary value problem for the radiation hydrodynamics system.

2. Estimates for auxiliary problems

In this section we make the following assumptions for (1.1), (1.2).

(i) $a_i(i=0,\ \cdots,n)$ are the $k\times k$ symmetric matrices and a_0 is a uniformly positive definite matrix. This means that the first part of (1.1) is a quasilinear symmetric hyperbolic system with respect to u.

(ii) There exists a positive number μ such that $\forall\xi_i\in R\ (i=1,\ \cdots,\ n)$

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j\geqslant\mu\sum_{i=1}^{n}\xi_i^2 \tag{2.1}$$

holds uniformly. This means that the second part of (1.1) is a quasilinear parabolic equation of second order with respect to v.

(iii) Assume that

$$\beta(t,\ x,\ u,\ v)=\sum_{i=1}^{n} a_i(t,\ x,\ u,\ v)n_i \tag{2.2}$$

is nonsingular in $[0,\ T]\times\bar{\Omega}\times\{|u|\leqslant A,\ |v|\leqslant B\}$ where A, B are certain positive numbers, and $n_i(i=1,\ \cdots,\ n)$ are the components of the unit exterior normal to $\partial\Omega$. This means that $\partial\Omega$ is noncharacteristic with respect to the first part of (1.1). We further assume that $MU=0$ is the admissible boundary condition in the Friedrichs' sense.

(iv) Let $m=\left[\dfrac{n+1}{2}\right]+1$ and p be the integer with $p\geqslant 2m+1$.

For simplicity, we assume that a_i, a_{ij}, f, g, M are in C^∞ (it is not difficult to prove that the conclusions in this paper remain valid when $M, f, a_i, g \in C^{p+1}$, $a_{ij} \in C^p$), and the compatibility conditions up to $p-1$ degree for (1.1), (1.2) are satisfied). That is to say, we can obtain successively from (1.1), (1.2)

$$\begin{aligned} u_1 &= \frac{\partial u}{\partial t}\bigg|_{t=0} = a_0^{-1}(0, x, 0, 0) f(0, x, 0, 0, 0), \quad v_1 = \frac{\partial v}{\partial t}\bigg|_{t=0} = g(0, x, 0, 0, 0, 0), \\ u_2 &= \frac{\partial^2 u}{\partial t^2}\bigg|_{t=0} = a_0^{-1}(0, x, 0, 0)\left[\frac{df}{dt}\bigg|_{t=0} - \frac{da_0}{dt}\,\frac{\partial u}{\partial t}\bigg|_{t=0} - \sum_{i=1}^n a_i(0, x, 0, 0)\frac{\partial u_1}{\partial x_i}\right], \\ v_2 &= \frac{\partial^2 v}{\partial t^2}\bigg|_{t=0} = \frac{dg}{dt}\bigg|_{t=0} + \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(0, x, 0, 0)\frac{\partial v_1}{\partial x_j}\right), \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{2.3}$$

We require that they should be compatible up to $p-1$ degree with the boundary condition (1.2) at $\{t=0, x\in\partial\Omega\}$.

We now introduce some notations. Let $D^l = \dfrac{\partial^l}{\partial t^{l_0}\partial x_1^{l_1}\cdots\partial x_n^{l_n}}$ $(l = l_0 + \cdots + l_n)$, $V_h = \Omega\times(0, h)$, $(h \leqslant T)$ and H^s be the s-order Sobolev space as usual. We denote H^s, L^2, C^s, C^0 norms by $\|\ \|_s$, $\|\ \|$, $|\ |_s$, $|\ |$, respectively. Let

$$H_p^h = \{u(x, t)\,|\,u(x, t)\in H^p(V_h),\ \|u\|_{H_p^h} = \|u\|_{H^p(V_h)} < +\infty\}, \tag{2.4}$$

and

$$H_{p+1,p}^h = \left\{v(x, t)\,|\,v(x, t)\in H_p^h,\ \frac{\partial v}{\partial x_i}\in H_p^h,\ (i=1, \cdots, n)\right\} \tag{2.5}$$

equipped with the norm

$$\|v\|_{H_{p+1,p}^h} = \left(\|v\|_p^2 + \sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|_p^2\right)^{\frac{1}{2}}. \tag{2.6}$$

It is easy to verify that $H_{p+1,p}^h$ is a Hilbert space.

Let

$$\begin{aligned} \Sigma(h) = \{&(u, v)\,|\,(u, v)\in C^\infty(\bar V_h)\times C^\infty(\bar V_h),\ \|u\|_{H_p^h}^2 \leqslant A_0,\ \|v\|_{H_p^h}^2 \leqslant B_0, \\ &\left\|\frac{\partial v}{\partial x}\right\|_{H_p^h}^2 = \sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|_{H_p^h}^2 \leqslant B_1,\ v|_{t=0} = 0,\ \frac{\partial u}{\partial t}\bigg|_{t=0} = u_1, \cdots, \frac{\partial^{p-1}}{\partial t^{p-1}}\bigg|_{t=0} = u_{p-1}, \\ &v\bigg|_{t=0} = 0,\ \frac{\partial v}{\partial t}\bigg|_{t=0} = v_1, \cdots, \frac{\partial^{p-1} v}{\partial t^{p-1}}\bigg|_{t=0} = v_{p-1}\Big\}, \end{aligned} \tag{2.7}$$

where A_0, B_0, B_1 are the positive numbers to be specified later on. It is easy to see that when A_0, B_0, B_1 are sufficiently large and h is sufficiently small, the set $\Sigma(h)$ is not empty.

Let $\overline{\Sigma(h)}$ be the closure of $\Sigma(h)$ in $H_p^h * H_{p+1,p}^h$. It is a bouded convex subset of $H_p^h * H_{p+1,p}^h$. For $(u, v) \in \Sigma(h)$ we consider the following auxiliary linear problems

$$\begin{cases} \sum_{i=0}^n a_i(t, x, u, v)\dfrac{\partial U}{\partial x_i} = f(t, x, u, v, v_x) \\ t=0:\ U=0, \\ \text{on } \partial\Omega:\ MU=0, \end{cases} \tag{2.8}$$

and

$$\begin{cases}\dfrac{\partial V}{\partial t}-\sum\limits_{i,j=1}^{n}\dfrac{\partial}{\partial x_i}\left(a_{ij}(t,\ x,\ u,\ v)\dfrac{\partial V}{\partial x_j}\right)=g(t,\ x,\ u,\ v,\ u_x,\ v_x),\\ V|_{t=0}=0,\ V|_{\partial\Omega}=0.\end{cases}\tag{2.9}$$

Before making the a priori estimates for the problems (2.8), (2.9), we introduce some expressions and estimates of the derivatives of composite functions.

Let $a(t,\ x,\ u,\ v)$, $u\in R^k$, $v\in R$, be a smooth function and $a^0=a(t,\ x,\ 0,\ 0)$. For $(u,\ v)\in\sum(h)$ we have

$$D^l a(t,\ x,\ u,\ v)=D^l a_0+D^l_{x,t}(a-a_0)+\sum_i a_{u_i}D^l u_i+a_v D^l v+G,\tag{2.10}$$

where we denote the partial derivatives with respect to x, t by $D^l_{x,t}$ and

$$a_{u_i}=\frac{\partial a}{\partial u_i},\quad a_v=\frac{\partial a}{\partial v}\ (i=1,\ \cdots,\ k).$$

From now on we denote the universal constants by C which are independent of h.

Lemma 1. *Suppose $(u,\ v)\in\sum(h)$ then $\forall l\leqslant p-m-2$ we have*

$$|D^l u\ (x,\ t)|\leqslant C_1+C_2\|u\|_{H^h_p}\tag{2.11}$$

$$|D^l v\ (x,\ t)|\leqslant C_1+C_2\|v\|_{H^h_p},\tag{2.12}$$

$$\left|\frac{\partial}{\partial x_i}\ D^l v(x,\ t)\right|\leqslant C_1+C_2\left\|\frac{\partial v}{\partial x_i}\right\|_{H^h_p}.\tag{2.13}$$

Proof Since

$$D^l u(x,\ t)=D^l u(x,\ 0)+\int_0^t\frac{\partial}{\partial t}\ D^l u(x,\ t)dt,$$

by the imbedding theorem we have

$$|D^l u(x,\ t)|\leqslant\max|D^l u(x,\ 0)|=Ch^{\frac{1}{2}}\|u\|_{H^h_p}.\tag{2.14}$$

Because $h\leqslant T$ and $D^l u(x,\ 0)$ can be explicitely expressed by $u_1,\ \cdots,\ u_l$, we obtain (2.11). Similarly, we have (2.13), (2.14). Thus the proof is completed.

Lemma 2. *For the expression (2.10) we have*

$$\|D^l_{x,t}(a-a^0)\|\leqslant C_0(\|u\|_l+\|v\|_l)=C_0(A_0,\ B_0),\tag{2.15}$$

$$\|G\|\leqslant C_1+C_2(A_0,\ B_0),\tag{2.16}$$

$$|D^l a|\leqslant C(A_0,\ B_0),\ \forall l\leqslant p-m-2.\tag{2.17}$$

Proof (2.15) is obvious. We can get (2.16), (2.17) from ((2.11), (2.12).

Now we turn to the problems (2.8), (2.9). For the initial boundary value problem (2.8) of symmetric hyperbolic systems we have

Lemma 3.[6,7] *There exists a unique solution $U(x,\ t)\in\bigcap_{r=0}^{p}C^r([0,\ h],\ H^{p-r}(\Omega))\subset H^p$. Furthermore, we have the estimate*

$$\|U\|^2_{H^h_p}\leqslant h(C+C_1(A_0,\ B_0)+\|f\|^2_{H^h_p})\leqslant h(C+C_1(A_0,\ B_0)+C_2(A_0,\ B_0)B_1),\tag{2.18}$$

where C is a constant dependent on $u_1,\ \cdots,\ u_p,\ v_1,\ \cdots,\ v_p$, but independent of A_0, B_0, and C_1, C_2 are the constants dependent on A_0, B_0.

For the initial boundary value problem (2.9) of parabolic equations, by the

Schauder theory, there exists a unique smooth solution $V\in C^{2(p-1),p-1}$. Moreover, $V\in \bigcap_{r=0}^{p} C^r([0, h], H^{2(p-r)}(\Omega))$, $\frac{\partial^{p+1}V}{\partial t^{p+1}}\in L^2(0, h, H^{-1}(\Omega))$ (see [13], Theorem 42.1).

Lemma 4. *Let V be the solution of (2.9), then there exists a constant M_1 dependent on μ such that*

$$\|V(t)\|_{L^2(\Omega)}\leqslant M_1\|g\|, \tag{2.19}$$

$$\|V\|\leqslant M_1 h^{\frac{1}{2}}\|g\|, \tag{2.20}$$

$$\sum_{i=1}^{n}\left\|\frac{\partial V}{\partial x_i}\right\|^2\leqslant M_1\|g\|^2. \tag{2.21}$$

Proof One can find the proof in [9]. It is also easy to give a direct proof. Multiplying the both sides of (2.9) by V and integrating on Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|V(t)\|^2_{L^2(\Omega)}+\mu\sum_{i=1}^{n}\left\|\frac{\partial V}{\partial x_i}\right\|^2_{L^2(\Omega)}\leqslant\frac{1}{2}(\|g\|^2_{L^2(\Omega)}+\|V\|^2_{L^2(\Omega)}). \tag{2.22}$$

Applying Gronwall inequality, we get (2.19). Integrating with respect to t, we get (2.20). From (2.22) we obtain (2.21).

Lemma 5. *For the smooth solution V of (2.9) we have*

$$\left\|\frac{\partial V}{\partial t}\right\|,\ \left\|\frac{\partial^2 V}{\partial x_i\partial x_j}\right\|\leqslant M_2\|g\| \quad (i, j=1, \cdots, n), \tag{2.23}$$

where M_2 depends on $|a_{ij}|$, and μ.

Proof See [10], Theorem 8.

We are now in a position to get the a priori estimates for the solution V of (2.9).

Theorem 1. *Let V be the smooth solution of (2.9), then there exists a positive number $\delta(A_0, B_0)$ dependent on A_0, B_0 such that when $h\leqslant\delta(A_0, B_0)$, we have*

$$\|V\|_p^2\leqslant hC_3(A_0, B_0), \tag{2.24}$$

$$\left\|\frac{\partial V}{\partial x}\right\|_p^2=\sum_{i=1}^{n}\left\|\frac{\partial V}{\partial x_i}\right\|_p^2\leqslant\tilde{C}+C_4(A_0, B_0), \tag{2.25}$$

where $\tilde{C}$ is a positive number independent of A_0, B_0, V, and $C_3(A_0, B_0)$, $C_4(A_0, B_0)$ are the positive numbers dependent on A_0, B_0, but independent of V.

Proof We first get the estimates of higher order tagential derivatives of solution by localization method, and then by differentiating the both sides of the equation get the estimates of higher order normal derivatives of solution. This is just the same as in the elliptic equation case.

Using localization, the problem is reduced to the one on $\bar{Q}_+\times[0, h]$:

$$\begin{cases}\dfrac{\partial V}{\partial t}-\displaystyle\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(t, x, u, v)\frac{\partial V}{\partial x_j}\right)=g(t, x, u, v, u_x, v_x),\\ V|_{\partial Q_+}=0, \quad V|_{t=0},\end{cases} \tag{2.26}$$

where Q_+ is a half cube in R^n and V vanishes near Γ (see Fig.1).

Since in the internal patch the situation is much simpler, it suffices to prove

(2.24), (2.25) for (2.26).

Fig. 1

Let

$$D^p=\frac{\partial^p}{\partial t^{p_0}\partial x_1^{p_1}\cdots\partial x_{n-1}^{p_{n-1}}}\ (p=p_0+\cdots+p_{n-1}). \tag{2.27}$$

Actting D^p on the both sides of (2.26), we have

$$\frac{\partial D^pV}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial D^pV}{\partial x_j}\Big)-\sum_{p\geq l\geq 1}\sum_{i,j}D^l(a_{ij})D^{p-l}\Big(\frac{\partial^2V}{\partial x_j\partial x_j}\Big)$$
$$-\sum_{p\geq l\geq 1}\sum_{i,j}D^l\Big(\frac{\partial a_{ij}}{\partial x_i}\Big)D^{p-l}\Big(\frac{\partial V}{\partial x_j}\Big)=D^pg. \tag{2.28}$$

According to the preceding differential expressions of composite function, we have

$$D^p\Big(\frac{\partial a_{ij}}{\partial x_i}\Big)=p_{ij\mu}\frac{\partial}{\partial x_i}D^pu_\mu+q_{ij}\frac{\partial}{\partial x_i}D^pv+G_0, \tag{2.28}$$

where

$$p_{ij\mu}=\frac{\partial a_{ij}}{\partial u_\mu},\quad q_{ij}=\frac{\partial a_{ij}}{\partial v}\ (1\leqslant\mu\leqslant k), \tag{2.30}$$

and we have the estimate for G_0

$$\|G_0\|\leqslant C+C_1(A_0,\ B_0), \tag{2.31}$$

where C is independent of A_0, B_0.

Similarly, we have

$$D^pg=\sum g_{\mu i}\frac{\partial D^pu_\mu}{\partial x_i}+\sum g_i\frac{\partial D^pv}{\partial x_i}+G_1, \tag{2.32}$$

where $g_{\mu i}$ and g_i are the partial derivatives of g with respect to $\frac{\partial u_\mu}{\partial x_i}$ and $\frac{\partial v}{\partial x_i}$ respectively, and

$$\|G_1\|\leqslant C+C_2(A_0,\ B_0). \tag{2.33}$$

Let $V_p=D^pV$, (2.28) can be rewritten as

$$\frac{\partial V_p}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial V_p}{\partial x_j}\Big)=\sum_{i,j,\mu}\Big(p_{ij\mu}\frac{\partial}{\partial x_i}D^pu_\mu+q_{ij}\frac{\partial}{\partial x_i}D^pv\Big)\frac{\partial V}{\partial x_j}$$
$$+\sum_{i,j}g_{\mu i}\frac{\partial D^pu_\mu}{\partial x_i}+\sum_i g_i\frac{\partial D^pv}{\partial x_i}+\sum_j G_0\frac{\partial V}{\partial x_j}+G_1+\sum_{i,j}D(a_{ij})\frac{\partial^2D^{p-1}V}{\partial x_i\partial x_j}$$
$$+\sum_{i,j}\sum_{p\geq l\geq 2}D^l(a_{ij})\frac{\partial^2D^{p-l}V}{\partial x_i\partial x_j}+\sum_{p-1\geq l\geq 1}D^l\Big(\frac{\partial a_{ij}}{\partial x_i}\Big)D^{p-l}\Big(\frac{\partial V}{\partial x_j}\Big). \tag{2.34}$$

On the other hand, from (2.26), we obtain

$$\begin{cases}V_p|_{\partial Q_+}=0,\\ V_p|_{t=0}=V_{0p}(x),\end{cases} \tag{2.35}$$

where $V_{0p}(x)$ is a given function and it can be derived completely from a_{ij}, g, u_1, $\cdots$, u_p and v_1, $\cdots$, v_p.

As in the proof of Lemma 4, multiplying the both sides of (2.34) by V_p and then integrating on Q_+, for the left hand side we have

$$\frac{1}{2}\frac{d}{dt}\|V_p(t)\|^2_{L^2(Q_+)}+\sum_{i,j}\int_{Q_+}a_{ij}\frac{\partial V_p}{\partial x_i}\frac{\partial V_p}{\partial x_j}\geqslant\frac{1}{2}\frac{d}{dt}\|V_p(t)\|^2_{L^2(Q_+)}+\mu\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2_{L^2(Q_+)}. \tag{2.36}$$

For the all terms in the right hand side

$$-\int_{Q_+}\Bigg[\sum_{i,j,\mu}\frac{\partial\left(p_{ij\mu}\frac{\partial V}{\partial x_j}V_p\right)}{\partial x_i}D^pu_\mu+\sum_{i,j}\frac{\partial\left(q_{ij}\frac{\partial V}{\partial x_i}V_p\right)}{\partial x_i}D^pv+\sum_{i,j}\frac{\partial(g_{\mu i}V_p)}{\partial x_i}D^pu_\mu$$
$$+\sum_i\frac{\partial(g_iV_p)}{\partial x_i}D^pv\Bigg]dx+\int_{Q_+}\left(\sum_j G_0\frac{\partial V}{\partial x_j}+G_1\right)V_pdx-\int_{Q_+}\frac{\partial(D(a_{ij})V_p)}{\partial x_i}\frac{\partial D^{p-1}V}{\partial x_j}dx$$
$$+\int_{Q_+}\Bigg[\sum_{p\geqslant l\geqslant 2}D^l(a_{ij})\frac{\partial^2D^{p-l}}{\partial x_i\partial x_j}+\sum_{\substack{p-1\geqslant l\geqslant 1\\ i,j}}D^l\left(\frac{\partial a_{ij}}{\partial x}\right)\frac{\partial D^{p-l}}{\partial x_j}\Bigg]V_pdx\triangleq I. \tag{2.37}$$

Integrating by parts, and then applying Lemmas 1,2 and the inequality $ab\leqslant\frac{a^2\varepsilon}{2}+\frac{b^2}{2\varepsilon}$ to make the estimates term by term, we arrive at

$$|I|\leqslant C_1(A_0,B_0)+\|V_p\|^2_{L^2(Q_+)}+C_2(A_0,B_0)\|V\|^2_{H^p(Q_+)}+\varepsilon\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2_{L^2(Q_+)}$$
$$+C_3|V|_2(\|u\|^2_p+\|v\|^2_p). \tag{2.38}$$

By the Schauder theory of parabolic equation and Lemma 1 we obtain for the solution V of (2.9)

$$|V|_2\leqslant C(A_0,B_0). \tag{2.39}$$

Choose $\varepsilon=\frac{\mu}{4}$ in (2.38), thus it follows from (2.36)—(2.38) that

$$\frac{d}{dt}\|V_p(t)\|^2_{L^2(Q_+)}+\frac{3\mu}{4}\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2_{L^2(Q_+)}\leqslant\tilde{C}_1(A_0,B_0)+\|V_p\|^2_{L^2(Q_+)}$$
$$+\tilde{C}_2(A_0,B_0)\|V\|^2_{H^p(Q_+)}. \tag{2.40}$$

Applying Gronwall inequality to (2.40), we obtain

$$\|V_p(t)\|^2_{L^2(Q_+)}+\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2\leqslant C\|V_{0p}\|^2+C_0(A_0,B_0)+C_1(A_0,B_0)\|V\|^2_p. \tag{2.41}$$

Therefore

$$\begin{cases}\|V_p\|^2\leqslant h(\tilde{C}+C_0(A_0,B_0)+C_1(A_0,B_0)\|V\|^2_p),\\ \sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2\leqslant\tilde{C}+C_0(A_0,B_0)+C_1(A_0,B_0)\|V\|^2_p.\end{cases} \tag{2.42}$$

Similarly, since V_{p-1} satisfies

$$\frac{\partial V_{p-1}}{\partial t}-\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial V_{p-1}}{\partial x_j}\right)=D^{p-1}g+\sum_{p-1\geqslant l\geqslant 1}D^l(a_{ij})D^{p-1-l}\left(\frac{\partial^2V}{\partial x_i\partial x_j}\right)$$
$$+\sum_{p-1\geqslant l\geqslant 1}D^l\left(\frac{\partial a_{ij}}{\partial x_i}\right)D^{p-1-l}\left(\frac{\partial V}{\partial x_j}\right)\triangleq\tilde{g}_{p-1}, \tag{2.43}$$

we obtain

$$\|V_{p-1}\|^2\leqslant h(C+C_0(A_0,B_0)+C_1(A_0,B_0)\|V\|^2_{p-1}). \tag{2.44}$$

On the other hand, since

$$\frac{\partial V_{p-1}}{\partial x_i}-\frac{\partial V_{p-1}}{\partial x_i}\bigg|_{t=0}=\int_0^t \frac{\partial^2}{\partial t\partial x_i}V_{p-1}dt, \tag{2.45}$$

we obtain

$$\left\|\frac{\partial V_{p-1}}{\partial x_i}\right\|^2_{L^2(Q_+)}\leqslant C+h\left\|\frac{\partial V_p}{\partial x_i}\right\|^2. \tag{2.46}$$

Hence from (2.42)

$$\sum_{i=1}^n\left\|\frac{\partial V_{p-1}}{\partial x_i}\right\|^2\leqslant h(C+C_0(A_0,\ B_0)+C_1(A_0,\ B_0)\|V\|_p^2). \tag{2.47}$$

Similarly, we have the same estimates (2.44), (2.47) for V_{p-2}. From

$$\frac{\partial V_{p-2}}{\partial t}-a_{nn}\frac{\partial^2 V_{p-2}}{\partial x_n^2}-\sum_{\substack{i\neq n\\ \text{or}\,j\neq n}}a_{ij}\frac{\partial^2 V_{p-2}}{\partial x_i\partial x_j}-\sum_{i,j}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial V_{p-2}}{\partial x_j}=\tilde{g}_{p-2} \tag{2.48}$$

and $a_{nn}\geqslant\mu>0$, we have

$$\left\|\frac{\partial^2 V_{p-2}}{\partial x_n^2}\right\|\leqslant\frac{1}{\mu}\left[\left\|\frac{\partial V_{p-2}}{\partial t}\right\|+\sum_{\substack{i\neq n\\ \text{or}\,j\neq n}}|a_{ij}|\left\|\frac{\partial^2 V_{p-2}}{\partial x_i\partial x_j}\right\|+\sum_{i,j}\left|\frac{\partial a_{ij}}{\partial x_i}\right|\left\|\frac{\partial V_{p-2}}{\partial x_j}\right\|+\|\tilde{g}_{p-2}\|\right]. \tag{2.49}$$

Using the same technigue as in (2.45), we have

$$\begin{cases}|a_{ij}(x,\ t)|\leqslant|a_{ij}(x,\ 0)|+h\cdot C(A_0,\ B_0),\\ \left|\dfrac{\partial a_{ij}}{\partial x_i}(x,\ t)\right|\leqslant\left|\dfrac{\partial a_{ij}(x,\ 0)}{\partial x_i}\right|+h\cdot C(A_0,\ B_0),\\ \|\tilde{g}_{p-2}\|^2\leqslant h\left(C+h\left\|\dfrac{\partial\tilde{g}_{p-2}}{\partial t}\right\|^2\right).\end{cases} \tag{2.50}$$

On the other hand, we can see from the expression of $\tilde{g}_{p-2}$ that the highest order of derivatives in $\dfrac{\partial\tilde{g}_{p-2}}{\partial t}$ with respect to u, v, V is not larger than p.

Hence

$$\left\|\frac{\partial\tilde{g}_{p-2}}{\partial t}\right\|\leqslant C_0(A_0,\ B_0)+C_1(A_0,\ B_0)\|V\|_p. \tag{2.51}$$

From (2.49), (2.44), (2.50), we obtain

$$\left\|\frac{\partial^2 V_{p-2}}{\partial x_n^2}\right\|^2\leqslant h(C+C_0(A_0,\ B_0)+C_1(A_0,\ B_0)\|V\|_p^2). \tag{2.52}$$

So far we have obtained the estimates of p-order derivatives of V which involve second order normal derivatives. Successively differentiating the both sides of (2.9), we can estimate the derivatives of V step by step which involve the higher order normal derivatives.In fact, actting $D_n^l D^{p-2-l}$ on the both sides of (2.9), we have

$$D_n^l D^{p-1-l}V-a_{nn}D_n^{l+2}D^{p-2-l}V-\sum_{\substack{i\neq n\\ \text{or}\,j\neq n}}a_{ij}D_n^l D^{p-2-l}\frac{\partial^2 V}{\partial x_i\partial x_j}$$
$$-\sum_{p-2\geqslant i\geqslant 1}D^i(a_{ij})D^{p-i}V-\sum_{p-1\geqslant i\geqslant 1}D^i(a_{ij})D^{p-i}V=D_n^l D^{p-2-l}g. \tag{2.53}$$

By induction, from (2.53) we can get the estimates for $D_n^{l+2}D^{p-2-l}V$ successively ($l=1,\ \cdots,\ p-2$):

$$\|D_n^{l+2}D^{p-2-l}V\|^2\leqslant h(C+C_0(A_0,\ B_0)+C_1(A_0,\ B_0)\|V\|_p^2). \tag{2.54}$$

From (2.42), (2.47), (2.52), (2.54), we have

$$\|V\|_p^2 \leqslant h(C_0(A_0, B_0) + C_1(A_0, B_0)\|V\|_p^2). \tag{2.55}$$

Choosing $\delta_0 = \frac{1}{2C_1(A_0, B_0)}$, when $h \leqslant \delta_0$ we have

$$\|V\|_p^2 \leqslant 2hC_0(A_0, B_0). \tag{2.56}$$

Substituting (2.56) into (2.42), we obtain

$$\sum_{i=1}^n \left\| \frac{\partial V_p}{\partial x_i} \right\|^2 \leqslant C + \widetilde{C}_0(A_0, B_0). \tag{2.57}$$

By the same procedure as above for $\|V_p\|$, we obtain

$$\sum_{i=1}^n \left\| \frac{\partial V}{\partial x_i} \right\|_p^2 \leqslant C + \widetilde{C}_1(A_0, B_0). \tag{2.58}$$

It is easy to extend (2.24), (2.25) from local patch to the whole domain $\Omega \times (0, h)$. Thus the proof is completed.

Remark 1. It is easy to see from the proof of Theorem 1 that the constant $\widetilde{C}$ in (2.25) depends on $\left.\frac{\partial V}{\partial t}\right|_{t=0}, \cdots, \left.\frac{\partial^p V}{\partial t^p}\right|_{t=0}$. If they vanish, then $\widetilde{C} = 0$.

Remark 2. It is also easy to see from the above proof that if (2.9) has the following form

$$\begin{cases} \frac{\partial V}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial V}{\partial x_j}\right) + \sum_{i=1}^n b_i \frac{\partial V}{\partial x_i} + C_0 V = \sum_{i=1}^n A_i \frac{\partial v}{\partial x_i} + \sum_{i,\mu} B_{i\mu} \frac{\partial u_\mu}{\partial x_i} + \sum_\mu E_\mu u_\mu + Fv \\ V|_{t=0} = 0, \ V|_{\partial\Omega} = 0, \end{cases} \tag{2.59}$$

then (2.24), (2.25) turn out to be

$$\|V\|_p^2 \leqslant M_1 h(C + \|u\|_p^2 + \|v\|_p^2), \tag{2.60}$$

$$\left\| \frac{\partial V}{\partial x} \right\|_p^2 = \sum_{i=1}^n \left\| \frac{\partial V}{\partial x_i} \right\|_p^2 \leqslant M_2(C + \|u\|_p^2 + \|v\|_p^2), \tag{2.61}$$

where the constants M_1, M_2 depend on the H^p norms of a_{ij}, b_i, C_0, A_i, B_i, E_μ, F_μ, and C is a constant indicated in Remark 1.

Remark 3. As in [1] our assertions are also valid for the following parabolic system

$$\frac{\partial V}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial V}{\partial x_j}\right) = g, \tag{2.62}$$

where V, g are the l-vector functions, (a_{ij}) is an $nl*nl$ symmetric matrix such that there exists $\mu > 0$, $\forall \xi_i \in R^l$ $(i = 1, \cdots, n)$.

$$\sum_{i,j=1}^n (a_{ij}, \xi_i, \xi_j) \geqslant \mu \sum_{i=1}^n |\xi_i|^2. \tag{2.63}$$

Here we denote the inner product and the norm in R^l by (,) and | | respectively.

3. Existence and Uniqueness

We are now in a position to prove the existence and uniqueness of the solution for the problem (1.1), (1.2) in which the boundary $\partial\Omega$ is noncharacteristic with respect to the first part of (1.1).

We first choose the positive numbers A_0, B_0 arbitrarily, then choose the positive number B_1 such that

$$B_1 \geqslant \widetilde{C} + C_4(A_0, B_0). \tag{3.1}$$

It follows from (2.18), (2.24) that there exists a positive number $\delta_1(A_0, B_0, B_1)$ such that when $h \leqslant \delta_1$

$$\|U\|_p^2 \leqslant A_0, \quad \|V\|_p^2 \leqslant B_0. \tag{3.2}$$

Thus when $h \leqslant \delta = \min(\delta_0(A_0, B_0), \delta_1(A_0, B_0, B_1))$, the linear auxiliary problem (2.8), (2.9) define a mapping $T: (u, v) \in \Sigma(h) \to (U, V) \in \overline{\Sigma(h)}$. Now what we want to do is to extend T from $\Sigma(h)$ to $\overline{\Sigma(h)}$.

For any $(u, v) \in \overline{\Sigma(h)}$ there exists a sequence $(u_n, v_n) \in \Sigma(h)$ such that $u_n \xrightarrow{H_p^h} u$, $v_n \xrightarrow{H_{p+1,p}^h} v$. Let $(U_n, V_n) = T(u_n, v_n)$, $\widetilde{U} = U_n - U_m$, $\widetilde{V} = V_n - V_m$, then $\widetilde{U}$, $\widetilde{V}$ satisfy

$$\begin{cases} \sum\limits_{i=0}^{n} a_i(t, x, u_n, v_n) \dfrac{\partial \widetilde{U}}{\partial x_i} = f(t, x, u_n, v_n, v_{n_x}) - f(t, x, u_m, v_m, v_{m_x}) \\ - \sum\limits_{i=0}^{n} [a_i(t, x, u_n, v_n) - a_i(t, x, u_m, v_m)] \dfrac{\partial U_m}{\partial x_i}, \\ M\widetilde{U}|_{t=0} = 0, \ \widetilde{U}|_{t=0} = 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \dfrac{\partial \widetilde{V}}{\partial t} - \sum\limits_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_{ij}(t, x, u_n, v_n) \dfrac{\partial \widetilde{V}}{\partial x_j}\right) = g(t, x, u_n, v_n, u_{n_x}, v_{n_x}) \\ -g(t, x, u_m, v_m, u_{m_x}, v_{m_x}) + \sum\limits_{i,j} [a_{ij}(t, x, u_n, v_n) - a_{ij}(t, x, u_m, v_m)] \dfrac{\partial^2 V_m}{\partial x_i \partial x_j} \\ + \sum\limits_{i,j} \left[\dfrac{\partial a_{ij}(t, x, u_n, v_n)}{\partial x_i} - \dfrac{\partial a_{ij}(t, x, u_m, v_m)}{\partial x_j}\right] \dfrac{\partial V_m}{\partial x_j}, \\ \widetilde{V}|_{\partial\Omega} = 0, \ \widetilde{V}|_{t=0} = 0, \end{cases} \tag{3.4}$$

respectively. Moreover

$$\left.\frac{\partial \widetilde{U}}{\partial t}\right|_{t=0} = \cdots = \left.\frac{\partial^{p-1} \widetilde{U}}{\partial t^{p-1}}\right|_{t=0} = 0, \quad \left.\frac{\partial \widetilde{V}}{\partial t}\right|_{t=0} = \cdots = \left.\frac{\partial^{p-1} \widetilde{V}}{\partial t^{p-1}}\right|_{t=0} = 0. \tag{3.5}$$

By the first two remarks in the preceding section, we have

$$\|\widetilde{U}\|_{p-1}^2 = \|U_n - U_m\|_{p-1}^2 \leqslant Ch\left(\|u_n - u_m\|_{p-1}^2 + \|v_n - v_m\|_{p-1}^2 + \left\|\frac{\partial(v_n - v_m)}{\partial x}\right\|_{p-1}^2\right) \to 0, \tag{3.6}$$

$$\left\|\frac{\partial \widetilde{V}}{\partial x}\right\|_{p-1}^2 = \sum_{i=1}^{n} \left\|\frac{\partial(V_n - V_m)}{\partial x_i}\right\|_{p-1}^2 \leqslant M_2(\|u_n - u_m\|_{p-1}^2 + \|v_n - v_m\|_{p-1}^2) \to 0, \tag{3.7}$$

$$\|\widetilde{V}\|_{p-1}^2 = \|V_n - V_m\|_{p-1}^2 \leqslant M_1 h(\|u_n - u_m\|_{p-1}^2 + \|v_n - v_m\|_{p-1}^2) \to 0. \tag{3.8}$$

This means that (U_n, V_n) converges in $H_{p-1} \times H_{p,p-1}$. We call (U, V) the limit function. Since $(U_n, V_n) \in \overline{\Sigma(h)}$, Banach-Saks theorem implies $(U, V) \in H_p \times H_{p+1,p}$ and $(U, V) \in \overline{\Sigma(h)}$. Thus for any $(u, v) \in \overline{\Sigma(h)}$ the solution (U, V) of (2.8), (2.9) is in $\overline{\Sigma(h)}$.

Theorem 2. *Under the assumptions* (i)—(iv) *in the second section there exists a sufficiently small number* $h>0$ *such that the problem* (1.1), (1.2) *has a unique solution* $(u, v)\in H_p^h\times H_{p+1,p}^h\subset C^{p-m}\times C^{p-m+1,p-m}$ *in* $V_h=\Omega\times(0, h)$. *Moreover, the uniqueness in* $C^1\times C^{2,1}$ *still holds.*

Proof We have seen from above that the operator T maps $\overline{\Sigma(h)}$ into itself. Moreover, for any (u_1, v_1), $(u_2, v_2)\in\overline{\Sigma(h)}$, let $(U_1, V_1)=T(u_1, v_1)$, $(U_2, V_2)=T(u_2, v_2)$, we have

$$\|U_2-U_1\|_{p-1}^2\leqslant Ch\left(\|u_2-u_1\|_{p-1}^2+\|v_2-v_1\|_{p-1}^2+\left\|\frac{\partial(v_2-v_1)}{\partial x}\right\|_{p-1}^2\right), \tag{3.9}$$

$$\|V_2-V_1\|_{p-1}^2\leqslant M_1h(\|u_2-u_1\|_{p-1}^2+\|v_2-v_1\|_{p-1}^2), \tag{3.10}$$

$$\left\|\frac{\partial(V_2-V_1)}{\partial x}\right\|_{p-1}^2\leqslant M_2(\|u_2-u_1\|_{p-1}^2+\|v_2-v_1\|_{p-1}^2). \tag{3.11}$$

We now introduce new equivalent norm in $H_{p-1}\times H_{p,p-1}(V_h)$ as follows:

$$\|(u, v)\|_{H_{p-1}\times H_{p,p-1}}^2=\|u\|_{p-1}^2+\|v\|_{p-1}^2+h^{\frac{1}{2}}\left\|\frac{\partial v}{\partial x}\right\|_{p-1}^2. \tag{3.12}$$

Thus

$$\begin{aligned}
\|T(u_2, v_2)-T(u_1, v_1)\|_{H_{p-1}\times H_{p,p-1}}^2 &= \|U_2-U_1\|_{p-1}^2+\|V_2-V_1\|_{p-1}^2\\
&+h^{\frac{1}{2}}\left\|\frac{\partial(V_2-V_1)}{\partial x}\right\|_{p-1}^2\leqslant Ch\left(\|u_2-u_1\|_{p-1}^2+\|v_2+v_1\|_{p-1}^2+\left\|\frac{\partial(v_2-v_1)}{\partial x}\right\|_{p-1}^2\right)\\
&+M_1h(\|u_1-u_1\|_{p-1}^2+\|v_1-v_1\|_{p-1}^2)+M_2h^{\frac{1}{2}}(\|u_2-u_1\|_{p-1}^2+\|v_2-v_1\|_{p-1}^2)\\
&\leqslant\tilde{C}h^{\frac{1}{2}}\left(\|u_2-u_1\|_{p-1}^2+\|v_2-v_1\|_{p-1}^2+h^{\frac{1}{2}}\left\|\frac{\partial(v_2-v_1)}{\partial x}\right\|_{p-1}^2\right)\\
&=\tilde{C}h^{\frac{1}{2}}\|(u_2, v_2)-(u_1, v_1)\|_{H_{p-1}\times H_{p,p-1}}^2.
\end{aligned} \tag{3.13}$$

This implies that when h is sufficiently small, T is a contractive operator in $H_{p-1}\times H_{p,p-1}$. As a consequence of the fixed point theorem in [4], T has a fixed point in $\overline{\Sigma(h)}$. It is also easy to give a direct proof. In fact, starting from any element (u_0, v_0) in $\overline{\Sigma(h)}$, the sequence defined by

$$(u_n, v_n)=T(u_{n-1}, v_{n-1})\quad(n=1, \cdots) \tag{3.14}$$

must converge in $H_{p-1}\times H_{p,p-1}$ as indicated before. This means that T has a fixed point $(u, v)\in H_{p-1}\times H_{p,p-1}$, by Banach-Saks theorem, which is also in $\overline{\Sigma(h)}$. By the imbedding theorem, $(u, v)\in C^{p-m}\times C^{p-m+1,p-m}$. This completes the proof of existence.

The uniqueness even holds in $C^1\times C^{2,1}$. In fact, if (u_2, v_2) and (u_1, v_1) are the solutions, then letting $U=u_2-u_1$, $V=v_2-v_1$, we have from (1.1)

$$\begin{aligned}
&\sum_{i=0}^n a_i(t, x, u_2, v_2)\frac{\partial U}{\partial x_i}=f(t, x, u_2, v_2, v_{2_x})-f(t, x, u_1, v_1, v_{1_x})\\
&-\sum_{i=0}^n[a_i(t, x, u_2, v_2)-a_i(t, x, u_1, v_1)]\frac{\partial u_1}{\partial x_i}=F_uU+F_vV\\
&+\sum_i F_{v_i}\frac{\partial V}{\partial x_i}-\sum_{i=0}^n(A_{u_i}U+A_{v_i}V)\frac{\partial u_1}{\partial x_i},
\end{aligned} \tag{3.15}$$

where F_u, F_v, etc. can be experessed as integrals in virtue of mean value theorem, e. g.,

$$F_v=\int_0^1 \frac{\partial f}{\partial \tau}(t,\ x,\ u_1,\ v_1+\tau(v_2-v_1),\ v_{2x})d\tau. \tag{3.16}$$

(3.15) can also be rewritten as follows

$$\sum_{i=0}^{n} \alpha_i \frac{\partial U}{\partial x_i}+KU=GV+\sum_{i=1}^{n} F_{v_i}\frac{\partial V}{\partial x_i} \tag{3.17}$$

with the initial boundary conditions

$$MU|_{\partial\Omega}=0,\ U|_{t=0}=0, \tag{3.18}$$

where

$$K=\sum_{i=0}^{n} A_{u_i}\frac{\partial u_1}{\partial x_i}-F_u,\quad G=F_v-\sum_{i=0}^{n} A_{v_i}\frac{\partial u_1}{\partial x_i}. \tag{3.19}$$

Thus from (3.17) (3.18) we have

$$\|U\|^2\leqslant C_1 h\Big(\|V\|^2+\sum_{i=1}^{n}\Big\|\frac{\partial V}{\partial x_i}\Big\|^2\Big). \tag{3.20}$$

Similarly, for V we have

$$\begin{aligned}\frac{\partial V}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(t,\ x,\ u_2,\ v_2)\frac{\partial V}{\partial x_j}\Big)&=g(t,\ x,\ v_2,\ v_2,\ u_{2x},\ v_{2x})\\ -g(t,\ x,\ u_1,\ v,\ u_{1x},\ v_{1x})+\sum_{i,j}\Big[&(a_{ij}(t,\ x,\ u_2,\ v_2)-a_{ij}(t,\ x,\ u_1,\ v_1))\frac{\partial^2 v_1}{\partial x_i\partial x_j}\\ +\Big(\frac{\partial a_{ij}(t,\ x,\ u_2,\ v_2)}{\partial x_i}-\frac{\partial a_{ij}(t,\ x,\ u_1,\ v_1)}{\partial x_i}\Big)\frac{\partial v_1}{\partial x_j}\Big],\end{aligned} \tag{3.21}$$

which can also be rewritten as follows:

$$\frac{\partial V}{\partial t}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial V}{\partial x_j}\Big)+\sum_{i=1}^{n} b_i\frac{\partial V}{\partial x_i}+C_0V=DU+\sum_{i=1}^{n} E_i\frac{\partial U}{\partial x_i} \tag{3.22}$$

with the initial boundary conditions

$$V|_{\partial\Omega}=0,\ V|_{t=0}=0. \tag{3.23}$$

It is easy to see that using the same technique as in Theorem 1, we have the following estimates

$$\|V\|^2\leqslant C_2 h\|U\|^2, \tag{3.24}$$

$$\sum_{i=1}^{n}\Big\|\frac{\partial V}{\partial x_i}\Big\|^2\leqslant C_3\|U\|^2. \tag{3.25}$$

It follows from (3.20), (3.24), (3.25) that

$$\|U\|^2\leqslant C_1 h\Big(\|V\|^2+\sum_{i=1}^{n}\Big\|\frac{\partial V}{\partial x_i}\Big\|^2\Big)\leqslant C_4 h\|U\|^2. \tag{3.26}$$

This implies that when h is sufficiently small, we have

$$\|U\|^2=0 \tag{3.27}$$

and from (3.24)

$$\|V\|^2=0. \tag{3.28}$$

This completes the proof.

4. Characteristic Boundary Case

In this section we deal with the problem (1.1), (1.2) in which the boundary $\partial\Omega$ is characteristic with respect to the first part of (1.1). In this case we preserve the assumptions (i) and (ii) made in the second section, but we need to revise the assumptions (iii) and (iv) as follows.

(iii)′ Let

$$\beta(t, x, u, v)=\sum_{i=1}^{n}\alpha_i(t, x, u, v)n_i, \tag{4.1}$$

where n_i are the components of unit exterior normal to $\partial\Omega$. Assume that for any smooth functions (u, v) which satisfy (1.2) $MU|_{\partial\Omega}=0$ is the admissible boundary condition in the Friedrichs' sense. Moreover, there exists a smooth matrix function $R(t, x)$ in the neighbourhood of $\partial\Omega\times[0, h]$ such that

$$\tilde{\beta}(t, x, u, v)=R^T(t, x)\beta(t, x, u, v)R(t, x)=\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}. \tag{4.2}$$

The rank of B_1 is a constant r near $\partial\Omega\times[0, h]$ and $B_2|_{\partial\Omega}=0$ (zero submatrix). Furthermore, by $\tilde{U}=RU$ the boundary condition (1.2) is transformed into

$$\tilde{U}_1=\cdots=\tilde{U}_L=0 \quad (L\leqslant r). \tag{4.3}$$

(iv)′ Let $m=\left[\frac{n+1}{2}\right]+1$, p be the integers with $p\geqslant 8m+8$. For simplicity, a_{ij}, α_i, f, g, $M\in C^\infty$ and the compatibility conditions up to $p-1$ degree are satisfied.

Under the assumptions (i), (ii), (iii)′, (iv)′ we are going to prove the existence and uniqueness of local solution for (1.1), (1.2).

We now introduce some notations. Let $V_h=\Omega\times(0, h)$, and $\{D_\sigma\}$ $(\sigma=0, \cdots, M_0)$, $D_0=I$, $D_\sigma=\sum_{i=0}^{n} d_\sigma^i\frac{\partial}{\partial x_i}$ $(\sigma\neq 0)$ (refer to [8]) be the smooth tagential derivative operator systems with respect to $\partial\Omega\times[0, h]$. As in [11], we denote the p-order generalized derivatives of u by $D_p^q u$ which involve q-order normal derivatives.

Let

$$B_p^h=\left\{u(x, t)\,\Big|\, \|u\|_{B_p^h}=\Big(\sum_{\delta\leqslant\frac{p}{2}}\sum_{\substack{s\leqslant p-\delta\\ t\leqslant\delta}}\|D_t^s u\|^2\Big)^{\frac{1}{2}}<+\infty\right\} \tag{4.4}$$

and

$$B_{p+1,p}^h=\left\{v(x, t)\,\Big|\, v\in B_p^h,\ \frac{\partial v}{\partial x_i}\in B_p^h(i=1, \cdots, n)\right\} \tag{4.5}$$

equipped with the norm

$$\|v\|_{B_{p+1,p}^h}=\Big(\|v\|_{B_p^h}^2+\Big\|\frac{\partial v}{\partial x}\Big\|_{B_p^h}^2\Big)^{\frac{1}{2}}=\Big(\|v\|_{B_p^h}^2+\sum_{i=1}^{n}\Big\|\frac{\partial v}{\partial x_i}\Big\|_{B_p^h}^2\Big)^{\frac{1}{2}}. \tag{4.6}$$

It is easy to verify that both B_p^h and $B_{p+1,p}^h$ are Hilbert sprces. Let

$$\Sigma_c^h=\Big\{(u,\ v)\mid (u,\ v)\in C^\infty(\bar{V}_h)\times C^\infty(\bar{V}_h),\ \|u\|^2_{B_p^h}\leqslant A_0,\ \|v\|^2_{B_p^h}\leqslant B_0,$$
$$\left\|\frac{\partial v}{\partial x}\right\|^2_{B_p^h}\leqslant B_1,\ u|_{t=0}=0,\ \cdots,\ \frac{\partial^{p-1}u}{\partial t^{p-1}}\Big|_{t=0}=u_{p-1},\ v|_{t=0}=0,\ \cdots,\ \frac{\partial^{p-1}v}{\partial t^{p-1}}\Big|_{t=0}=v_{p-1},$$
$$Mu|_{\partial\Omega}=0,\ v|_{\partial\Omega}=0\Big\} \tag{4.7}$$

and $\bar{\Sigma}_c^h$ be the closure of Σ_c^h which is not empty when h is appropriately small (refer to[7]).

From now on we will denote the universal constant by $\tilde{C}$ which depends on A_0, B_0.

Since B_p^h forms a Banach algebra when $p\geqslant 8m+8$ (refer to [11]), we have the following Lemmas, which are similar to Lemmas 1, 2 in the section 2.

Lemma 5. *Suppose* $(u,\ v)\in\Sigma_c^h$, $\forall s\leqslant p-m-1$, $q\leqslant\frac{p}{2}-m-1$, *we have*

$$\begin{cases}|D_q^s u(x,\ t)|\leqslant\tilde{C},\\ |D_q^s v(x,\ t)|\leqslant\tilde{C}.\end{cases} \tag{4.8}$$

From the following expression, similar to (2.10)

$$D_q^s a(t,\ x,\ u,\ v)=D_q^s a^0+D^s_{q,x,t}(a-a^0)+\sum_i a_{u_i}D_q^s u_i+a_v D_q^s v+G, \tag{4.9}$$

we have

Lemma 6. *For* $s\leqslant p-\delta$, $q\leqslant\delta$, $0\leqslant\delta\leqslant\frac{p}{2}$,

$$\begin{cases}\|D^s_{q,x,t}(a-a^0)\|\leqslant\tilde{C},\\ \|G\|\leqslant\tilde{C},\\ \|D_q^s a(t,\ x,\ u,\ v)\|\leqslant\tilde{C}.\end{cases} \tag{4.10}$$

In order to prove the existence of the solution for the problem (1.1), (1.2) we form again the linear auxiliary problem as follows:

$$\begin{cases}\sum\limits_{i=0}^{n}\alpha_i(t,\ x,\ u,\ v)\dfrac{\partial U}{\partial x}=f(t,\ x,\ u,\ v,\ v_x)\\ U|_{t=0}=0,\ MU|_{\partial\Omega}=0,\end{cases} \tag{4.11}$$

and

$$\begin{cases}\dfrac{\partial V}{\partial t}-\sum\limits_{i,j=1}^{n}\dfrac{\partial}{\partial x_i}\Big(a_{ij}(t,\ x,\ u,\ v)\dfrac{\partial V}{\partial x_j}\Big)=g(t,\ x,\ v,\ v,\ x_x,\ v_x),\\ V|_{\partial\Omega}=0,\ V|_{t=0}=0.\end{cases} \tag{4.12}$$

In what follows we will establish some a priori estimates of solutions U, V for $(u,\ v)\in\Sigma_c^h$, respectively.

For (4.10) we have

Lemma 7. *For* $(u,\ v)\in\Sigma_c^h$ *the problem* (2.11) *has a unique solution* $U\in B_p^h$. *Moreover*

$$\|U\|^2_{B_p^h}\leqslant\tilde{C}_1 h(C+\|f\|^2_{B_p^h})\leqslant\tilde{C}_1 h(C+\tilde{C}_2+\tilde{C}_3 B_1), \tag{4.13}$$

where C *is a constant independent of* A_0, B_0, B_1.

Proof Since $(u, v) \in \Sigma_c^h$, the problem (4.11) satisfies the compatibility conditions up to $p-1$ degree at $\{t=0,\ x\in\partial\Omega\}$.

Therefore, by the results in [4], the problem (4.11) has a unique solution $U\in B_p^h$. Moreover,

$$\|U\|_{B_p^h}^2 \leqslant \tilde{C}_1 h(C+\|f\|_{B_p^h}^2).$$

In virtue of Lemma 6 we have (4.13). This completes the proof. Now we establish the a priori estimates of solution V for (4.12).

As shown in the section 2, by localization it suffices to discuss the problem in a half cube of R^n. So in what follows we consider the problem in the half cube $Q_+=\{-1<x_i<1,\ 0<x_n<1,\ i=1,\ \cdots,\ n-1\}$.

The support of V is in the shade area (see Fig. 1) and the boundary condition is converted to

$$\text{On } x_n=0:\ V=0.$$

Lemma 8. *For the solution V of (4.12) we have*

$$\|V_p\|^2 \leqslant h(\tilde{C}_2+\tilde{C}_3\|V\|_{B_p^h}^2), \tag{4.14}$$

$$\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2 \leqslant C\|V_p(0)\|^2+\tilde{C}_0 h+\tilde{C}_1\|V\|_{B_p^h}^2. \tag{4.15}$$

Proof By the assumptions and the well known results for parabolic equations (refer to [10, 13]), the problem (4.12) has a unique solution $V\in H^{p,2p}$, $V_p\in C([0, h],\ L^2(\Omega))$, where

$$H^{p,2p}=\left\{V\ \middle|\ \frac{\partial^{k_1+k_2}V}{\partial t^{k_1}\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\in L^2,\ k_1\leqslant p,\ 2k_1+k_2\leqslant 2p\right\}.$$

Acting $D^p=\dfrac{\partial^p}{\partial t^{i_0}\partial x_1^{i_1}\cdots\partial x_{n-1}^{i_{n-1}}}$ on the both sides of (4.12), we get

$$\frac{\partial D^pV}{\partial t}-\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial D^pV}{\partial x_j}\right)=D^pg+\sum_{\substack{p\geqslant l\geqslant 1\\ i,j}}D^la_{ij}D^{p-l}\frac{\partial^2V}{\partial x_i\partial x_j}+\sum_{\substack{p\geqslant l\geqslant 1\\ i,j}}D^l\frac{\partial a_{ij}}{\partial x_i}D^{p-l}\frac{\partial V}{\partial x_j} \tag{4.16}$$

with the homogeneous boundary conditions for D^pV and

$$D^pV|_{t=0}=V_{0p}(x), \tag{4.17}$$

where $V_{0p}(x)$ are the given functions and can be expressed by $v_1,\ \cdots,\ v_p$.

As in the section 2, multipyling the both sides, by V_p and then integrating on $V_t=Q_+\times(0,\ t)$, we obtain

$$\frac{1}{2}\|V_p(t)\|^2+\mu\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2\leqslant\frac{1}{2}\|V_{0p}\|^2+\left|\int_0^t\int_{Q_+}D^pgV_p\,dx\,dt\right|$$
$$+\left|\int_0^t\int_{Q_+}\sum_{p\geqslant l\geqslant 1}D^la_{ij}D^{p-l}\frac{\partial^2V}{\partial x_i\partial x_j}V_p\,dx\,dt\right|+\left|\int_0^t\int_{Q_+}\sum_{p\geqslant l\geqslant 1}D^l\frac{\partial a_{ij}}{\partial x_i}D^{p-l}\frac{\partial V}{\partial x_j}V_p\,dx\,dt\right|. \tag{4.18}$$

Similarly, we have

$$\left|\int_0^t\int_{Q_+}D^pgV_p\,dx\,dt\right|\leqslant\tilde{C}_1+\|V_p\|^2+\frac{\varepsilon}{2}\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2. \tag{4.19}$$

Let

$$I_1=\int_0^t\int_{Q_+}\sum_{p\geq l\geq 1} D^l\frac{\partial a_{ij}}{\partial x_i}D^{p-l}\frac{\partial V}{\partial x_j}V_p\,dx\,dt=\int_0^t\int_{Q_+}\sum_{p\geq l\geq p-1}\frac{\partial D^l a_{ij}}{\partial x_i}\frac{\partial D^{p-l}V}{\partial x_j}V_p\,dx\,dt$$

$$=\int_0^t\int_{Q_+}\sum_{p-2\geq l\geq 2}\frac{\partial D^l a_{ij}}{\partial x_i}\frac{\partial D^{p-l}V}{\partial x_j}V_p\,dx\,dt+\int_0^t\int_{Q_+}\frac{\partial Da_{ij}}{\partial x_i}\frac{\partial D^{p-1}V}{\partial x_j}V_p\,dx\,dt$$

$$=I_{11}+I_{12}+I_{13}. \tag{4.20}$$

It follows from Lemma 6 that

$$\|D^l a_{ij}\|^2\leq\widetilde{C},\quad p-1\leq l\leq p. \tag{4.21}$$

On the other hand, by the Schauder type estimates for (4.12) (see [9]), we have

$$|V|_0,\ |V|_1,\ |V|_2,\ |V|_3\leq\widetilde{C}. \tag{4.22}$$

Thus

$$|I_{11}|\leq\widetilde{C}_1+\frac{\|V_p\|^2}{3}+\frac{\varepsilon}{4}\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2, \tag{4.23}$$

and

$$|I_{12}|\leq\widetilde{C}_2+\widetilde{C}_3\|V\|^2_{B^h_p}+\frac{1}{3}\|V_p\|^2, \tag{4.24}$$

$$|I_{13}|=\left|-\int_0^t\int_{Q_+}\frac{\partial^2 a_{ij}}{\partial x_i\partial x_j}D^{p-1}VV_p\,dx\,dt-\int_0^t\int_{Q_+}\frac{\partial Da_{ij}}{\partial x_i}D^{p-1}V\frac{\partial V_p}{\partial x_j}\,dx\,dt\right|$$

$$\leq\widetilde{C}_4+\frac{1}{3}\|V_p\|^2+\frac{\varepsilon}{4}\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2. \tag{4.25}$$

It follows from (4.23)—(4.25) that

$$|I_1|\leq\widetilde{C}_5+\widetilde{C}_3\|V\|^2_{B^h_p}+\|V_p\|^2+\frac{\varepsilon}{2}\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2. \tag{4.26}$$

Similarly, we get

$$\left|\int_0^t\int_{Q_+}\sum_{p\geq l\geq 1}D^l\frac{\partial a_{ij}}{\partial x_i}D^{p-l}\frac{\partial V}{\partial x_j}V_p\,dx\,dt\right|\leq\widetilde{C}_6+\widetilde{C}_7\|V\|^2_{B^h_p}+\|V_p\|^2+\frac{\varepsilon}{2}\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2. \tag{4.27}$$

Therefore, it follows from (4.19), (4.26), (4.27) that

$$\frac{1}{2}\|V_p(t)\|^2_{L^2(Q_+)}+\mu\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2\leq\widetilde{C}_8+\frac{3\varepsilon}{2}\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2+3\|V_p\|^2+\widetilde{C}_9\|V\|^2_{B^h_p}. \tag{4.28}$$

Choosing $\varepsilon=\frac{\mu}{2}$, we obtain from (4.28)

$$\|V_p(t)\|^2_{L^2(Q_+)}\leq 2(\widetilde{C}_8+3\|V_p\|^2+\widetilde{C}_9\|V\|^2_{B^h_p}), \tag{4.29}$$

$$\frac{\mu}{4}\sum_{i=1}^n\left\|\frac{\partial V_p}{\partial x_i}\right\|^2\leq\widetilde{C}_8+3\|V_p\|^2+\widetilde{C}_9\|V\|^2_{B^h_p}. \tag{4.30}$$

Integrating (4.29) with respect to t, when h is appropriately small, we have (4.14). Substituting it into (4.30), we obtain (4.15). Thus the proof is completed.

Lemma 8. *For the solution V of* (4.12) *we have*

$$\sum_{i=1}^n\left\|\frac{\partial V_{p-1}}{\partial x_i}\right\|^2\leq h(C+\widetilde{C}_1+\widetilde{C}_2\|V\|^2_{B^h_p}). \tag{4.31}$$

Proof From

By induction it is not difficult to prove that (4.42) and (4.44) hold for $1\leqslant l\leqslant\frac{p}{2}$. We omit the details.

Thus we have

$$\|V\|^2_{B^h_{\frac{p}{2}}}\leqslant h(\tilde{C}_9+\tilde{C}_{10}\|V\|^2_{B^h_{\frac{p}{2}}}), \tag{4.45}$$

$$\sum_{i=1}^n\left\|\frac{\partial V}{\partial x_i}\right\|^2_{B^h_{\frac{p}{2}}}\leqslant\tilde{C}_{11}+\tilde{C}_{12}\|V\|^2_{B^h_{\frac{p}{2}}}+\tilde{C}_{13}h\sum_{i=1}^n\left\|\frac{\partial V}{\partial x_i}\right\|^2_{B^h_{\frac{p}{2}}}. \tag{4.46}$$

These imply that there exists a positive number $\delta_1(A_0, B_0)$ such that when $h\leqslant\delta_1(A_0, B_0)$, (4.40), (4.41) hold. Thus the proof is completed.

Remark 4. We have the same conclusions as indicated in Remarks 1—3. The only thing we have to do is to change the notaton of norm from H^p into B^h_p.

In what follows we use the a priori estimates obtained before to prove the existence and uniqueness of local smooth solution for (1.1), (1.2). To this end, we go along the same line as in the section 3.

For any fixed positive numbers A_0, B_0, we choose the positive number B_1 sufficiently large such that in (4.41)

$$\tilde{C}_2\leqslant B_1. \tag{4.47}$$

(4.13) and (4.40) imply that there exists a positive number $\delta_2(A_0, B_0, B_1)\leqslant\delta_1(A_0, B_0)$ such that when $h\leqslant\delta_2$,

$$\|U\|^2_{B^h_{\frac{p}{2}}}\leqslant A_0,\ \|V\|^2_{B^h_{\frac{p}{2}}}\leqslant B_0,\ \sum_{i=1}^n\left\|\frac{\partial V}{\partial x_i}\right\|^2_{B^h_{\frac{p}{2}}}\leqslant B_1. \tag{4.48}$$

Thus the operator $T:(u, u)\to(U, V)$ is a nonlinear mapping from Σ^h_c into Σ^h_c.

As in the section 3 if we introduce the new equivalent norm in $B^h_{p-2}\times B^h_{p-1,p-2}$

$$\|(u, v)\|^2_{B^h_{p-2}\times B^h_{p-1,p-2}}=(\|u\|^2_{B^h_{p-2}}+\|v\|^2_{B^h_{p-2}})+h^{\frac{1}{2}}\sum_{i=1}^n\left\|\frac{\partial v}{\partial x_i}\right\|^2_{B^h_{p-2}}, \tag{4.49}$$

then T is a contractive operator with respect to the new norm.

Applying the theorem in [4] or repeating the detailed discussion indicated in the third section, we arrive at

Theorem 4. *Under the assumptions* (i), (ii), (iii)′, (iv)′ *there exists a positive number* δ *such that when* $h\leqslant\delta$, *the problem* (1.1) (1.2) *admitsa unique solution* $(u, v)\in B^h_p\times B^h_{p+1,p}\subset C^{\frac{p}{2}-m}\times C^{\frac{p}{2}-m+1,\frac{p}{2}-m}$. *Moreover, the uniqueness is still valid in* $C^1\times C^{2,1}$.

5. Applications

Since the hyperbolic-parabolic coupled system arise in many physiscal and mechanical problems, our preceding results have wide applications. In this section we briefly describe the applications to the radiation hydrodynamic problem and the compressible viscous hydrodynamic problem with solid wall boundary which one usually meets in practice.

(I) Radiation hydrodynamic problem with solid wall boundary.

As was pointed out in [2], the radiation hydrodynamic equations can be written as follows:

$$\begin{pmatrix} 1 & & & \\ & \frac{\rho^2}{RT} & & \\ & & \frac{\rho^2}{RT} & \\ & & & \frac{\rho^2}{RT} \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} \rho \\ u \\ v \\ w \end{pmatrix} + \begin{pmatrix} u & \rho & & \\ \rho & \frac{\rho^2 u}{RT} & & \\ & & \frac{\rho^2 u}{RT} & \\ & & & \frac{\rho^2 u}{RT} \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \rho \\ u \\ v \\ w \end{pmatrix}$$

$$+\begin{pmatrix} v & & \rho & \\ & \frac{\rho^2 v}{T^2 R} & & \\ \rho & & \frac{\rho^2 v}{RT} & \\ & & & \frac{\rho^2 v}{RT} \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} \rho \\ u \\ v \\ w \end{pmatrix} + \begin{pmatrix} w & & & \rho \\ & \frac{\rho^2 w}{RT} & & \\ & & \frac{\rho^2 w}{RT} & \\ \rho & & & \frac{\rho^2 w}{RT} \end{pmatrix} \frac{\partial}{\partial z}\begin{pmatrix} \rho \\ u \\ v \\ w \end{pmatrix}$$

$$=\begin{pmatrix} 0 \\ -\left(R+\frac{16\sigma}{3\rho C}T^3\right)\frac{\rho^2}{RT}\frac{\partial T}{\partial x} \\ -\left(R+\frac{16\sigma}{3\rho C}T^3\right)\frac{\rho^2}{RT}\frac{\partial T}{\partial y} \\ -\left(R+\frac{16\sigma}{3\rho C}T^3\right)\frac{\rho^2}{RT}\frac{\partial T}{\partial z} \end{pmatrix}, \tag{5.1}$$

$$\left(\frac{R\rho}{\gamma-1}+\frac{16\sigma}{C}T^3\right)\frac{\partial T}{\partial t}-\frac{\partial}{\partial x}\left(\frac{16\sigma A}{3}T^{3+\alpha}\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{16\sigma A}{3}T^{3+\alpha}\frac{\partial T}{\partial y}\right)$$
$$-\frac{\partial}{\partial z}\left(\frac{16\sigma A}{3}T^{3+\alpha}\frac{\partial T}{\partial z}\right)+\left(\frac{R\rho}{\gamma-1}+\frac{16\sigma}{C}T^3\right)\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)$$
$$+\left(\rho RT+\frac{16\sigma}{3C}T^4\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0, \tag{5.2}$$

with the initial boundary conditions

$$\begin{cases} \text{on } \partial\Omega: un_1+vn_2+wn_3=0, \ T=T_1(x,t)>0, \\ t=0: \rho=\rho_0(x)>0, \ T=T_0(x)>0, \ u=u_0(x), \ v=v_0(x), \ w=w_0(x), \end{cases} \tag{5.3}$$

where (u, v, w) is the velocity vector, n_1, n_2, n_3 are the components of unit exterior normal to $\partial\Omega$, ρ is the density, T is the absolute temperature, σ, A, γ, α, R are the positive constants, and C is the light speed.

It is easy to see that (5.1) is a quasilnear symmetric hyperbolic system for (ρ, u, v, w) and (5.2) is a quasilinear second order parabolic equation for T.

From (5.1) we get

$$\beta=\begin{pmatrix} u_n & \rho n_1 & \rho n_2 & \rho n_3 \\ \rho n_1 & \dfrac{\rho^2 u_n}{RT} & & \\ \rho n_2 & & \dfrac{\rho^2 u_n}{RT} & \\ \rho n_3 & & & \dfrac{\rho^2 u_n}{RT} \end{pmatrix}. \tag{5.4}$$

For (u, v, w, T) satisfying the boundary conditions (5.10),

$$\beta=\begin{pmatrix} 0 & \rho n_1 & \rho n_2 & \rho n_3 \\ \rho n_1 & 0 & 0 & 0 \\ \rho n_2 & 0 & 0 & 0 \\ \rho n_3 & 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

As shown in [11], the assumption (iii)′ is satisfied. Moreover, the boundary conditions (5.3) are admissible. Thus Theorem 4 implies the existence and uniqueness of local smooth solution provided that the initial and boundary conditions satisfy the assumptions of smoothness and the compatibility conditions indicated in the section 4.

(II) The compressible viscous hydrodynamic problem with sol id wall boudary.

This problem has been investigated in [5]. it is not difficult to verify that without considering the differences in smoothness, as a consequence of the Theorem 4 for the general system (1.1), we also get the existence and uniqueness of local smooth solution.

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