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UNIQUENESS OF MINIMIZATION PROBLEMS

SHI YINGGUANG (史应光)
(*Computing Center, Academia Sinica*)

Abstract

In this paper we give a uniqueness theorem for a minimization problem. Meanwhile we provide a uniform uniqueness theorem for L_p $(1\leqslant p\leqslant\infty)$ simultaneous approximations including mean simultaneous approximation and Chebyshev simultaneous approximation.

Let $X\subset[a, b]$ be a compact set containing at least $n+1$ points and K an n-dimensional Haar subspace in $C[a, b]$, where n is a natural number. Let $F(x, y)$ be a nonnegative function of two variables from $X\times(-\infty, \infty)$ to $[-\infty, \infty]$ and suppose that there exists at least one element $P\in K$ such that $\|F(x, P)\|<\infty$, where $F(x, P)\equiv F(x, P(x))$ and $\|\cdot\|=\sup|\cdot|$.

Our minimization problem is to find an element $P\in K$ such that
$$\|F(x, P)\|=\inf_{Q\in K}\|F(x, Q)\|,$$
such an element P (if it exists) is called a minimum to F on K.

The author in [1] considered this problem and has established the main theorems in the Chebyshev theory including the theorems of existence, alternation and uniqueness under the following conditions:

(a) $\lim\limits_{|y|\to\infty} F(x, y)=\infty,\ \forall x\in X;$

(b) For each $x\in X$ and each y
$$\lim_{\substack{\eta\to y\\ F(x,\eta)\leqslant F(x,y)}} F(x, \eta)=F(x, y);$$

(c) For each $x\in X$ and arbitrary points $y_1\leqslant y_2\leqslant y_3$
$$F(x, y_2)\leqslant\max\{F(x, y_1), F(x, y_3)\};$$

(d) For each $x\in X$ and each y
$$\overline{\lim_{\substack{\xi\to x\\ \eta\to y}}} F(\xi, \eta)\leqslant F(x, y);$$

(e) For each $x\in X$ there exist two extended real numbers $f^-(x)$ and $f^+(x)$, $-\infty\leqslant f^-(x)\leqslant f^+(x)\leqslant\infty$, such that $F(x, y)$ is strictly decreasing with respect to y in $(-\infty, f^-(x)\rangle$ and strictly increasing in $\langle f^+(x), \infty)$, and $F(x, y)=F^*(x)\equiv\inf\limits_y F(x, y)$ in $\langle f^-(x), f^+(x)\rangle$, where the angular brackets $\langle$ or $\rangle$ is used to denote that $(-\infty, d\rangle$, for instance, means $(-\infty, d]$ whenever d is finite, and $(-\infty, d)$

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otherwise.

In this paper we mainly prove the following uniqueness theorem.

Theorem 1. *Under conditions (a)~(e), F has a unique minimum on K if and only if the cardinality of the set*

$$G=\{Q\in K: f_1\leqslant Q\leqslant f_2\}$$

is not greater than one, card $G\leqslant 1$, where

$$f_1(x)=\inf_{F(x,y)\leqslant\|F^*\|} y,\ f_2(x)=\sup_{F(x,y)\leqslant\|F^*\|} y.$$

Proof Necessity. First we show that each element of G (if any) must be a minimum to F from K. In fact, $P\in G$ implies $F(x, P)\leqslant\|F^*\|$ by (e). Hence $\|F(x, P)\|\leqslant\|F^*\|$. On the other hand we always have $\|F(x, P)\|\geqslant\|F^*\|$ from $F(x, P)\geqslant F^*(x)$. So $\|F(x, P)\|=\|F^*\|$ and P is a minimum to F.

Now we can easily prove the necessity of the theorem. Suppose on the contrary that card $G>1$. We can then assume $P_1, P_2\in G$ and $P_1\neq P_2$. This means that both P_1 and P_2 are minima to F on K. This is a contradiction.

Sufficiency. If possible, we assume that both P_1 and P_2 are minima to F on K and $P_1\neq P_2$. Without loss of generality we may further assume $P_1\bar{\in} G$ because card $G\leqslant 1$. Hence $\|F(x, P_1)\|>\|F^*\|$. According to Theorem 7 in [1], P_1 is the unique minimum to F on K. This contradiction completes the sufficiency of the theorem.

This uniqueness theorem is analogous to the one for another minimization problem with a mean norm by the author [2, Theorem 4], but in that paper

$$f_1=f^-,\quad f_2=f^+.$$

The content of this minimization problem includes as special cases a number of important approximation problems, such as

Generalized weight function approximation $F(x, y)=W(x, f(x)-y)$[3];

Weighted simultaneous approximation $F(x, y)=\left\{\sum_{j=1}^{\infty}\lambda_j|f_j(x)-y|^p\right\}^{\frac{1}{p}}$, $\lambda_j\geqslant 0$, $1\leqslant p<\infty$[4];

Dunham-type simultaneous approximation $F(x, y)=\sup_{f\in\mathscr{F}}|f(x)-y|$, where $\mathscr{F}$ is a set of bounded functions on X[5].

As an example we take $F(x, y)=\{|f^-(x)-y|^p+|f^+(x)-y|^p\}^{\frac{1}{p}}$, $1\leqslant p\leqslant\infty$ in which we suppose that $f^-\leqslant f^+$ and $f^-, f^+\in C[a, b]$ for simplicity, although our result is still valid for more general cases. Define

$$\|f\|\equiv\|f\|_p=\left\{\int_a^b|f(x)|^p dx\right\}^{\frac{1}{p}}.$$

The approximation problem is to find an element $P\in K$ such that

$$\|F(x, P)\|=\inf_{Q\ni K}\|F(x, Q)\|,$$

such an element P (if any) is called a best approximation to (f^-, f^+) on K. When

$p=\infty$, $F(x, y)=\max\{|f^-(x)-y|, |f^+(x)-y|\}$ and $\|\cdot\|$ is a Chebyshev norm, which was considered by Dunham[5]. When $p=1$, $F(x, y)=|f^-(x)-y|+|f^+(x)-y|$ and $\|\cdot\|$ is a mean norm, which was considered by Carroll and McLaughlin in [6]. We have the following uniqueness theorem, in which $\bar{f}=(f^+ + f^-)/2$ and $f^*=(f^+ - f^-)/2$.

Theorem 2. *(f^-, f^+) has a unique best approximation on K if and only if the cardinality of the set*

$$G=\{Q\in K: f_1\leqslant Q\leqslant f_2\}$$

is not greater than one, card $G\leqslant 1$, where

$$f_1=f^-,\ f_2=f^+ \qquad \textit{when } p=1,$$
$$f_1=f_2=\bar{f} \qquad \textit{when } 1<p<\infty,$$
$$f_1=f^+-\|f^*\|,\ f_2=f^-+\|f^*\| \quad \textit{when } p=\infty.$$

Proof As $1\leqslant p<\infty$ the theorem immediately follows from Theorem 4 in [2].

As $p=\infty$ we only need to determine f_1 and f_2 in Theorem 1. Now in our case $F^*=f^*$ and $F(x, y)\leqslant\|f^*\|$ means

$$\max\{|f^-(x)-y|, |f^+(x)-y|\}\leqslant\|f^*\|.$$

or equivalently, using the identity $\max\{|u|, |v|\}=(|u+v|+|u-v|)/2$,

$$|\bar{f}(x)-y|+f^*(x)\leqslant\|f^*\|.$$

Hence

$$\bar{f}(x)+f^*(x)-\|f^*\|\leqslant y\leqslant\bar{f}(x)+\|f^*\|-f^*(x),$$

or equivalently

$$f^+(x)-\|f^*\|\leqslant y\leqslant f^-(x)+\|f^*\|.$$

Thus

$$f_1(x)=\inf_{F(x,y)\leqslant\|f^*\|} y=f^+(x)-\|f^*\|,$$
$$f_2(x)=\sup_{F(x,y)\leqslant\|f^*\|} y=f^-(x)+\|f^*\|.$$

This theorem provides a somewhat uniform description for uniqueness of L_p $(1\leqslant p\leqslant\infty)$ simultaneous approximations.

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