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SINGULAR INTERGRALS IN SEVERAL COMPLEX VARIABLES (III)—CAUCHY INTEGRALS OF CLASSICAL DOMAINS

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Abstract

In [1], we discussed the Henkin-Ramirez integrals and Stein-Kerzman integrals of strictly pseudoconvex domains. We found that there are many different ways to define the *principal values of* H-R *integrals or* S-K integrals, so that there are many different Plemelj formulas. In the present paper, the similar problems of Cauchy integrals of classical domains are considered. The conclusions similar to [1] and the corresponding Plemelj formulas are obtained. Finally, we discuss the same problems for the Bochner-Martinelli integrals.

§ 0. Introduction

In [1], we discussed the Henkin-Ramirez (H-R) integrals and Stein-Kerzman (S-K) integrals of strictly pseudoconvex domains and gave the general Plemelj formula. We found that there are many different ways to define the principal values of H-R integrals or S-K integrals, so that there are many different Plemelj formulas.

Can the above situations happen for the other integral representations? This is the problem that will be discussed in the present paper.

Let Ω be a unit ball. Then the S-K kernel of Ω is $(1-z\bar{u}')^{-n}$, and we have the Plemelj formula of the unit ball in [1]. When the deleted neighborhood around the boundary point is an "ellipse" or a "rectangle", the Plemelj formulas can be written as Theorem 1.1 and Theorem 1.2 in the present paper respectively. In § 1, we shall give other proofs for these two theorems. These direct proofs can be easily generalized to a more general situation. This is the content of § 2.

In [4], the Cauchy integral of Lie sphere hyperbolic space was considered. The elements of Lie sphere hyperbolic spaces $R_{IV}(N)$ are the all complex vectors $z=(z_1, \cdots,$

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z_N) satisfying the following conditions

$$\begin{cases} 1+|zA_0z'|^2-z\bar{z}'>0 \\ 1-|zA_0z'|>0, \end{cases} \quad A_0=\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\dotplus\frac{1}{2}I^{N-2}.$$

By B_{IV} we denote the boundary of R_{IV}, and by L_{IV} the characteristic manifold of R_{IV}.

Let R^*_{IV} be the set of all complex vectors $z=(z_1, \cdots, z_N)$ satisfying

$$\begin{cases} 1+|zA_0z'|^2-z\bar{z}'>0, \\ 1-|zA_0z'|<0, \end{cases}$$

L_{IV} is also the characteristic manifold of R^*_{IV}.

If $\varphi(\xi)$ is continuous in L_{IV}, then when $z\in R_{IV}$ or $z\in R^*_{IV}$, the Cauchy integral

$$V(L_{IV})^{-1}\int_{L_{IV}} H_{IV}(z, \xi)\varphi(\xi)\dot{\xi} \tag{0.1}$$

exists, where H_{IV}, $V(L_{IV})$ and $\dot{\xi}$ stand for the Cauchy kernel $(1+zA_0z'\overline{\xi A_0\xi'}-z\bar{\xi}')^{-\frac{N}{2}}$, the volume of L_{IV} and the volume element of L_{IV} respectively.

In [4], the limit values of (0.1), as z approaches $B_{IV}-L_{IV}$ or L_{IV} from the interior of R_{IV}, were obtained. By the same way we can obtain the limit value of (0.1) as z approaches the boundary from the interior of R^*_{IV}. These limit values can be represented by the Cauchy principal values.

In the section 3, using the results of § 1 and § 2, we prove that the limit value of (0.1) can be represented by a lot of distinct Cauchy principal values when z approaches $B_{IV}-L_{IV}$ from the interior of R_{IV}, but it cannot be represented as the above when z approaches L_{IV} from the interior of R_{IV}, because the computation of the limit value of (0.1) in this case must reduce to that of the Cauchy integrals of polydisc, however the polydisc is the topological product of unit discs, it cannot give the various results. It is the same for R^*_{IV}.

In [5, 6], the Cauchy integrals of matrix hyperbolic spaces were considered.

Let $R_I(m, n)$ and $R^*_I(m, n)$ stand for the matrix hyperbolic spaces $I-Z\bar{Z}'>0$ and $I-Z\bar{Z}'<0$ respectively, where Z is the $m\times n(m\leqslant n)$ complex matrix. If $\varphi(U)$ is continuous on the characteristic manifold $L_I(m, n)$ of $R_I(m, n)$, then the Cauchy integrals of $R_I(m, n)$ and $R^*_I(n)=R^*_I(n, n)$

$$V(L_I(m, n))^{-1}\int_{L_I(m,n)}\varphi(U)\det{}^{-n}(I-Z\bar{U}')\dot{U},\ Z\in R_I(m, n), \tag{0.2}$$

$$V(L_I(n))^{-1}\int_{L_I(n)}\varphi(U)\det{}^{-n}(I-Z\bar{U}')\dot{U},\ Z\in R^*_I(n) \tag{0.3}$$

exist. Here $V(L_I(m, n))$ denotes the volume of $L_I(m, n)$.

In [5], the limit values of (0.2) and (0.3), when Z approaches $L_I(n)$ from the interior of $R_I(n)$ and $R^*_I(n)$ respectively, were obtained. By the same reason as above, it cannot give the various definitions of Cauchy principal value. But when Z approaches $L_I^{(m-1)}$ ($L_I^{(m-1)}$ is a part of the boundary of $R_I(m, n)$, it is formed by the matrix satisfying $\operatorname{rank}(I-Z\bar{Z}')=m-1$) from the interior of $R_I(m,n)$, by the results in § 1, we can obtain a lot of distinct definitions of Cauchy principal value and Plemelj formulas.

Another important kind of Cauchy-Fantappie kernels is of Bochner-Martinelli kernels. Lu qikeng and Zhong tongde[7] gave the Plemelj formulas of Bochner-Martinelli integrals when z approaches the boundary from the interior or exterior of domain. It is natural to ask whether one can obtain the various Cauchy principal values and Plemelj formulas if one deletes the various neighborhoods as in [1]. The answer is negative, this will be illustrated in § 5.

§ 1. The Cauchy integrals on sphere

Since the ball is a strictly pseudoconvex domain, it is easy to see, the Stein-Kerzman kernel is just $(1-z\bar{u}')^{-n}$. Therefore, the Theorems 1.1 and 1.2 below are the special cases of the Plemelj formulas in § 2.1 and § 3.1 of [1]. We shall give a direct proof here in order to prove the Lemmas 2.1 and 2.2 in § 2.

We first consider the case that the deleted neighborhood is an ellipse.

Lemma 1.1. *Suppose* $u=(u_1, \cdots, u_n)$, $u\bar{u}'=1$ *then*

$$\lim_{\varepsilon\to 0}\omega_{2n-1}^{-1}\int_\sigma (1-\bar{u}_1)^{-n}\dot{u}=1-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1},$$

where $\sigma=\{u\,|\,u\bar{u}'=1,\ \alpha^2(\operatorname{Re}(1-\bar{u}_1))^2+\beta^2(\operatorname{Im}\bar{u}_1)^2>\varepsilon^2\}$, $\alpha\geqslant 0$, $\beta\geqslant 0$, $\alpha+\beta\neq 0$.

Proof We first assume that $\alpha>0$, $\beta>0$. Since the case of $\alpha=\beta$ was proved in [2], it is enough to prove it for the case of $\alpha\neq\beta$.

Pick $0<\rho<1$, we have

$$1=\omega_{2n-1}^{-1}\int_\sigma (1-\rho\bar{u}_1)^{-n}\dot{u}+\omega_{2n-1}^{-1}\int_{\sigma'} (1-\rho\bar{u}_1)^{-n}\dot{u}=I+I',$$

where $\sigma'=\{u\,|\,u\bar{u}'=1,\ \alpha^2(\operatorname{Re}(1-\bar{u}_1))^2+\beta^2(\operatorname{Im}\bar{u}_1)^2\leqslant\varepsilon^2\}$.

Let

$$I'=\omega_{2n-1}^{-1}\int_{\sigma'}(1-\rho\bar{u}_1)^{-n}\dot{u}.$$

Set $\bar{u}_1=re^{i\theta}$, $u_2=v_2, \cdots, u_n=v_n$, $v=(v_2, \cdots, v_n)$, $\alpha^2=t$, $\beta^2=1-t\,(0<t<1)$, then the points of σ' satisfy

$$t(1-r\cos\theta)^2+(1-t)r^2\sin^2\theta\leqslant\varepsilon^2,\quad v\bar{v}'=1-r^2,$$

so

$$\cos\theta\geqslant\frac{t-\sqrt{(1-t)(t+r^2(1-2t))+(2t-1)\varepsilon^2}}{r(2t-1)}. \tag{1.1}$$

Let

$$a=\arccos\frac{t-\sqrt{(1-t)(t+r^2(1-2t))+(2t-1)\varepsilon^2}}{r(2t-1)},$$

then $-a\leqslant\theta\leqslant a$. By (1.1), we have

$$1 \geqslant \frac{t-\sqrt{(1-t)(t+r^2(1-2t))+(2t-1)\varepsilon^2}}{r(2t-1)}.$$

It is not hard to see $1-r^2 \leqslant \frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}$, thus

$$I' = \omega_{2n-1}^{-1} \int_{E(\varepsilon)} \dot{v} \int_{-a}^{a} (1-\rho r e^{i\theta})^{-n}\, d\theta = 2\omega_{2n-1}^{-1} \operatorname{Im}\left\{\sum_{p=1}^{n-1} K_p + K_0\right\} + 2\omega_{2n-1}^{-1} \int_{E(\varepsilon)} a\dot{v}, \quad (1.2)$$

where $E(\varepsilon) = \left\{v=(v_2, \cdots, v_n) \mid v\bar{v}' \leqslant \frac{2\varepsilon}{\sqrt{t}} - \frac{\varepsilon^2}{t}\right\}$,

$$K_p = \int_{E(\varepsilon)} \frac{\dot{v}}{p(1-\rho r e^{ia})^p}, \quad K_0 = \int_{E(\varepsilon)} \log \frac{1}{1-\rho r e^{ia}}\, \dot{v}.$$

Obviously

$$\frac{t-\sqrt{(1-t)(t+r^2(1-2t))+(2t-1)\varepsilon^2}}{r(2t-1)} - 1 = O(\varepsilon^2),$$

so $a=O(\varepsilon^2)$, and it follows that

$$2\omega_{2n-1}^{-1} \int_{E(\varepsilon)} a\dot{v} = O(\varepsilon^{n+1}).$$

Let

$$Q = \sqrt{(1-t)(t+r^2(1-2t))+(2t-1)\varepsilon^2},$$

then $r\cos a = \frac{t-Q}{2t-1}$, so

$$r\sin a = \frac{\sqrt{r^2(2t-1)t - t - (2t-1)\varepsilon^2 + 2Qt}}{|2t-1|}.$$

When $p \geqslant 1$

$$K_p = \frac{1}{p}\int_{E(\varepsilon)} \left\{1-\rho\frac{t-Q}{2t-1} - i\rho\frac{\sqrt{r^2(2t-1)t-t-(2t-1)\varepsilon^2+2Qt}}{|2t-1|}\right\}^{-p} \dot{v}.$$

Set $v=(x_1, x_2, \cdots, x_{2n-2})$, using the sphere polar coordinates

$$x_1 = s\cos\varphi_1,\ x_2 = s\sin\varphi_1\cos\varphi_2,\ \cdots,\ x_{2n-2} = s\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{2n-3},$$
$$\dot{v} = s^{2n-3}\sin^{2n-4}\varphi_1\sin^{2n-5}\varphi_2\cdots\sin\varphi_{2n-4}\,ds\,d\varphi_1\cdots d\varphi_{2n-3},$$

we have

$$K_p = \frac{2\pi^{n-1}}{p\Gamma(n-1)} \int_0^{\sqrt{\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}}} \left\{1-\rho\frac{t-Q}{2t-1} - i\rho\frac{\sqrt{t(1-s^2)(2t-1)-t-(2t-1)\varepsilon^2+2Qt}}{|2t-1|}\right\}^{-p} s^{2n-3}\, ds,$$

the absolute value of integrand is

$$\left\{\left(1-\rho\frac{t-Q}{2t-1}\right)^2 + \rho^2\frac{t(1-s^2)(2t-1)-t-(2t-1)\varepsilon^2+2Qt}{(2t-1)^2}\right\}^{-\frac{p}{2}} s^{2n-3},$$

and its limit value is

$$|2t-1|^p s^{2n-3}\left[(t+Q-1)^2 + t(1-s^2)(2t-1) - t - (2t-1)\varepsilon^2 + 2Qt\right]^{-\frac{p}{2}}$$

as $\rho \to 1$. It is not hard to prove that there are $\eta_0>0$, $\delta>0$, such that

$$(t+Q-1)^2 + t(1-s^2)(2t-1) - t - (2t-1)\varepsilon^2 + 2Qt > \eta_0 s^2 \quad (1.3)$$

for $t\in\left(0,\ \sqrt{\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}}\right)$, $\varepsilon\in(0,\ \delta)$.

Thus

$$J_p=\lim_{\rho\to1}K_p=O\left(\varepsilon^{-p}\int_0^{\sqrt{\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}}}s^{2n-3}\,ds\right)=O(\varepsilon^{n-p-1}).$$

When $p<n-1$

$$\lim_{\varepsilon\to0}J_p=\lim_{\varepsilon\to0}\lim_{\rho\to1}K_p=0.$$

By the Lebesgue theorem

$$J_{n-1}=\frac{2\pi^{n-1}}{(n-1)!}\int_0^{\sqrt{\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}}}\Bigg[\frac{t+Q-1}{2t-1}-i\frac{\sqrt{t(2t-1)(1-s^2)-t-(2t-1)\varepsilon^2+2Qt}}{|2t-1|}\Bigg]^{-(n-1)}s^{2n-3}\,ds$$

$$=\frac{2\pi^{n-1}}{(n-1)!}\int_0^1\Bigg[\frac{t+Q-1}{2t-1}-i\frac{\sqrt{t(2t-1)(1-\eta^2R^2)-t-(2t-1)\varepsilon^2+2Qt}}{|2t-1|}\Bigg]^{-(n-1)}\eta^{2n-2}R^{2n-3}\,dR,$$

where $\eta=\sqrt{\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}}$, $s=\eta R$.

Since

$$\eta^{-2}\,\frac{t-1+Q}{2t-1}=\eta^{-2}\,\frac{t-1+\sqrt{(1-t)^2+(2t-1)(1-t)\eta^2R^2+(2t-1)\varepsilon^2}}{2t-1}=\frac{R^2}{2}+O(\varepsilon)$$

and

$$\frac{1}{\eta^2|2t-1|}\left\{t(2t-1)(1-\eta^2R^2)-t(2t-1)\varepsilon^2-2Qt\right\}^{\frac12}$$

$$=\left(\frac{t}{1-t}\right)^{\frac12}\left\{\frac1t\,\frac{1}{\left(\frac{2}{\sqrt{t}}-\frac{\varepsilon}{t}\right)^2}-\frac{R^4}{4}+O(\varepsilon)\right\}^{\frac12}=\frac12\left(\frac{t}{1-t}\right)^{\frac12}(1-R^4)^{\frac12}+O(\varepsilon),$$

we have

$$\lim_{\varepsilon\to0}J_{n-1}=\frac{2^n\pi^{n-1}}{\Gamma(n)}\int_0^1\frac{R^{2n-3}dR}{\left[R^2-i\left(\frac{t}{1-t}\right)^{\frac12}\sqrt{1-R^4}\right]^{n-1}}=\frac{(2\pi)^{n-1}}{\Gamma(n)}\int_0^1\frac{u^{n-2}du}{(u-ib\sqrt{1-u^2})^{n-1}},$$

where $b=\left(\frac{t}{1-t}\right)^{\frac12}$. Let $u=\cos\theta$, then

$$\lim_{\varepsilon\to0}J_{n-1}=\frac{(2\pi)^{n-1}}{i\Gamma(n)(1+b)^{n-1}}\int_0^{\frac{\pi}{2}}\frac{(e^{2i\theta}+1)^{n-2}(e^{2i\theta}-1)}{\left(1+\frac{1-b}{1+b}e^{2i\theta}\right)^{n-1}}\,d\theta.$$

Since $b>0$, $-1<\frac{1-b}{1+b}<1$, we have

$$\lim_{\varepsilon\to0}J_{n-1}=\frac{(2\pi)^{n-1}}{\Gamma(n)}\left\{\frac{-1}{2(1+b)^{n-1}}\sum_{p=0}^{n-2}\sum_{q=0}^{\infty}(-1)^qC_{n-2}^pC_{n+q-2}^q\left(\frac{1-b}{1+b}\right)^q\times\left[\frac{(-1)^{p+q+1}-1}{p+q+1}-\frac{(-1)^{p+q}-1}{p+q}\right]-\frac{1}{i(1+b)^{n-1}}\,\frac{\pi}{2}+\frac{1}{(1+b)^{n-1}}\right\}.$$

Then

$$\operatorname{Im}(\lim_{\varepsilon\to 0} J_{n-1}) = \lim_{\varepsilon\to 0}(\operatorname{Im} J_{n-1}) = \frac{\pi^n}{(n-1)!}\,\frac{1}{2}\left(\frac{2}{1+b}\right)^{n-1}.$$

Now we calculate

$$J_0 = \lim_{\rho\to 1} K_0 = \int_{E(\varepsilon)} \log\frac{1}{1-re^{ia}}\,\dot{v} = \frac{-2\pi^{n-1}}{\Gamma(n-1)}\int_0^{\sqrt{\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}}} s^{2n-3}\log(1-re^{ia})\,ds.$$

Since $\operatorname{Im}\log(1-re^{ia}) = O(1)$, so $J_0 = O(\varepsilon^{n-1})$.

In summary, we have

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1} I' = \frac{2}{\omega_{2n-1}}\,\frac{\pi^n}{(n-1)!}\,\frac{1}{2}\left(\frac{2}{1+b}\right)^{n-1} = \frac{1}{2}\left(\frac{2}{1+b}\right)^{n-1}.$$

Substituting $b = \left(\frac{t}{1-t}\right)^{\frac{1}{2}} = \frac{\alpha}{\beta}$ into the above equality, we have proved Lemma 1.1 for the case $\alpha>0$, $\beta>0$.

When $\alpha=0$, $\beta>0$, this was proved in [8].

When $\alpha>0$, $\beta=0$, Lemma 1.1 becomes

$$\lim_{\varepsilon\to 0}\omega_{2n-1}^{-1}\int_{\sigma_1}(1-\bar{u}_1)^{-n}\dot{u} = 1, \tag{1.4}$$

where $\sigma_1 = \{u \mid u\bar{u}'=1,\ \operatorname{Re}(1-\bar{u}_1)>\varepsilon\}$.

The proof is similar to the case $\beta>0$. Taking $0<\rho<1$, we have

$$1 = \omega_{2n-1}^{-1}\int_{\sigma_1}(1-\rho\bar{u}_1)^{-n}\dot{u} + \omega_{2n-1}^{-1}\int_{\sigma_1'}(1-\rho\bar{u}_1)^{-n}\dot{u} = I_1 + I_1',$$

where $\sigma_1' = \{u \mid u\bar{u}'=1,\ \operatorname{Re}(1-\bar{u}_1)\leqslant\varepsilon\}$.

Similar to (1.2)

$$I_1' = 2\omega_{2n-1}^{-1}\operatorname{Im}\left\{\sum_{p=1}^{n-1}K_p + K_0\right\} + 2\omega_{2n-1}^{-1}\int_{v\bar{v}'\leqslant 2\varepsilon-\varepsilon^2} a\dot{v},$$

where $a = \arccos\frac{1-\varepsilon}{r}$. As in the case $\beta>0$, we can prove

$$\int_{v\bar{v}'\leqslant 2\varepsilon-\varepsilon^2} a\dot{v} = o(1),\ J_p = O(\varepsilon^{n-p-1}),\ J_0 = o(1),\ p=1,\cdots,n-1.$$

At this case

$$J_{n-1} = \frac{2\pi^{n-1}}{(n-1)!}\int_0^{\sqrt{2\varepsilon-\varepsilon^2}} \frac{s^{2n-3}ds}{(\varepsilon - i\sqrt{2\varepsilon-\varepsilon^2-s^2})^{n-1}}$$

$$= \frac{2\pi^{n-1}}{(n-1)!}\left(\frac{2\varepsilon-\varepsilon^2}{\varepsilon}\right)^{n-1}\int_0^1 \frac{R^{2n-3}dR}{(1-i\eta\varepsilon^{-1}\sqrt{1-R^2})^{n-1}},$$

where $\eta = \sqrt{2\varepsilon-\varepsilon^2}$, $s=\eta R$.

The absolute value of the denominator of the integrand is

$$(1+\eta^2\varepsilon^{-2}(1-R^2))^{\frac{n-1}{2}} = [1+(2\varepsilon-\varepsilon^2)\varepsilon^{-2}(1-R^2)]^{\frac{n-1}{2}} \geqslant 1.$$

By Lebesgue theorem, $\lim_{\varepsilon\to 0} J_{n-1}=0$, so $\lim_{\varepsilon\to 0} I_1'=0$. The proof of Lemma 1.1 is complete.

Using this Lemma, as in [2], we can prove the following Plemelj formula:

Theorem 1.1. *Suppose f is in* Lipα *on $u\bar{u}'=1$, v is an arbitrary point on*

$u\bar{u}'=1$, then

$$(K\text{-}\lim_{z\to v})\ \omega_{2n-1}^{-1}\int_{u\bar{u}'=1}\frac{f(u)}{(1-z\bar{u}')^n}\dot{u}=\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}f(v)+p.\,v.\,\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1(\gamma)}\frac{f(u)\dot{u}}{(1-v\bar{u}')^n}.$$

Here the Cauchy principal value is defined as

$$p.\,v.\,\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1(\gamma)}\frac{f(u)\dot{u}}{(1-v\bar{u}')^n}=\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{\sigma}\frac{f(u)\dot{u}}{(1-v\bar{u}')^n},\quad \gamma=\frac{\alpha}{\beta},$$

where $\sigma=\{u\,|\,u\bar{u}'=1,\ \alpha^2(1-\mathrm{Re}v\bar{u}')^2+\beta^2(\mathrm{Im}v\bar{u}')^2\geqslant\varepsilon^2\}$ *(The definition of K-limit, see* [9]).

It is clear that this theorem is a generalization of Theorem 1.3 in [2] (see [2, 3]).

Now we consider the case that the deleted neighborhood is a rectangle.

Theorem. 1.2 *Suppose f satisfies the same condition as in Theorem 1.1, then*

$$(K\text{-}\lim_{z\to v})\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-z\bar{u}')^n}=bf(v)+p.\,v.\,\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1(b)}\frac{f(u)\dot{u}}{(1-v\bar{u}')^n},$$

where $b=\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-\int_0^{\operatorname{arctg}\frac{\beta}{\alpha}}\cos^{n-2}t\,\frac{\sin(n-1)t}{\sin t}dt\right\}$, *and the Cauchy principal value is defined as*

$$p.v.\,\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1(b)}\frac{f(u)\dot{u}}{(1-v\bar{u}')^n}=\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{D^*(v,\varepsilon)}\frac{f(u)}{(1-v\bar{u}'^n)}\dot{u},$$

$D^*(v,\varepsilon)$ is the complementary set of $D(v,\varepsilon)$ on $u\bar{u}'=1$ and

$$D(v,\varepsilon)=\{u\,|\,u\bar{u}'=1,\ 1-\mathrm{Re}v\bar{u}'<\alpha\varepsilon,\ |\mathrm{Im}v\bar{u}'|<\beta\varepsilon\},\ \alpha\geqslant0,\ \beta\geqslant0.$$

To prove Theorem 1.2, it is enough to prove the following

Lemma 1.2.

$$\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{D^*(v,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_1)^n}=1-\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-\int_0^{\operatorname{arctg}\frac{\beta}{\alpha}}\cos^{n-2}t\,\frac{\sin(n-1)t}{\sin t}dt\right\}.\tag{1.5}$$

Proof It is only need to prove

$$\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{D(v,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_1)^n}=\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-\int_0^{\operatorname{arctg}\frac{\beta}{\alpha}}\cos^{n-2}t\,\frac{\sin(n-1)t}{\sin t}dt\right\}.$$

Since

$$D(v,\varepsilon)=\{u\,|\,u\bar{u}'=1,\ 1-\mathrm{Re}v\bar{u}'<\alpha\varepsilon\}\cup\{u\,|\,u\bar{u}'=1,\ |\mathrm{Im}\,v\bar{u}'|<\beta\varepsilon\}$$
$$-\{u\,|\,u\bar{u}'=1\}\cup\{u\,|\,u\bar{u}'=1,\ 1-\mathrm{Re}v\bar{u}'>\alpha\varepsilon,\ |\mathrm{Im}v\bar{u}'|>\beta\varepsilon\}.$$

it is known that (see [8])

$$\lim_{\varepsilon\to0}\omega_{2n-1}^{-1}\int_{\substack{u\bar{u}'=1\\1-\mathrm{Re}u_1<\alpha\varepsilon}}(1-\bar{u}_1)^{-n}\dot{u}=0,\tag{1.6}$$

$$\lim_{\varepsilon\to0}\omega_{2n-1}^{-1}\int_{\substack{u\bar{u}'=1\\|\mathrm{Im}u_1|<\beta\varepsilon}}(1-\bar{u}_1)^{-n}\dot{u}=2^{n-2}.\tag{1.7}$$

Thus we only need to prove

$$\lim_{\varepsilon\to0}\omega_{2n-1}^{-1}\int_{\substack{u\bar{u}'=1\\1-\mathrm{Re}u_1>\alpha\varepsilon,\ |\mathrm{Im}u_1|>\beta\varepsilon}}(1-\bar{u}_1)^{-n}\dot{u}=1-\frac{2^{n-1}}{\pi}\int_0^{\operatorname{arctg}\frac{\beta}{\alpha}}\cos^{n-2}t\,\frac{\sin(n-1)t}{\sin t}dt.\tag{1.8}$$

When $\beta=0$, (1.8) is just (1.6). When $\alpha=0$, by [1]

$$\int_0^{\frac{\pi}{2}} \cos^{n-2} t \frac{\sin(n-1)t}{\sin t} dt=\frac{\pi}{2}.$$

(1.8) is just (1.7). So it is enough to prove the theorem for the case $\alpha>0$, $\beta>0$.

It is not hard to see that

$$\frac{1}{\omega_{2n-1}}\int_{S(\varepsilon)} \frac{\dot{u}}{(1-u_1)^n}=\frac{1}{\omega_{2n-1}}\int_{M(\varepsilon)} \dot{v}\left(\int_{-(\pi-b)}^{-a}+\int_a^{\pi-b}\right)\frac{d\theta}{(1-re^{i\theta})^n}$$

$$+\frac{1}{\omega_{2n-1}}\int_{N(\varepsilon)} \dot{v}\left(\int_{-(\pi-b)}^{-b}+\int_b^{\pi-b}\right)\frac{d\theta}{(1-re^{i\theta})^n}=I_1+I_2,$$

where

$$S(\varepsilon)=\{u\,|\,u\bar{u}'=1,\ 1-\operatorname{Re} u_1>\alpha\varepsilon,\ |\operatorname{Im} u_1|>\beta\varepsilon\},$$

$$M(\varepsilon)=\{v\,|\,v\bar{v}'<2\alpha\varepsilon-(\alpha^2+\beta^2)\varepsilon^2\},$$

$$N(\varepsilon)=\{v\,|\,2\alpha\varepsilon-(\alpha^2+\beta^2)\varepsilon^2\leqslant v\bar{v}'<1-\beta^2\varepsilon^2\}.$$

Since

$$\int_{-(\pi-b)}^{-a}(1-re^{i\theta})^{-n}\,d\theta+\int_a^{\pi-b}(1-re^{i\theta})^{-n}d\theta=2\operatorname{Im}(B_1-B_2)+2(\pi-a-b),$$

where

$$B_1=\sum_{k=1}^{n-1}\frac{1}{k(1+re^{-ib})^k}-\log(1+re^{-ib}),$$

$$B_2=\sum_{k=1}^{n-1}\frac{1}{k(1-re^{ia})^k}-\log(1-re^{ia}),$$

and

$$a=\arccos\frac{1-\alpha\varepsilon}{r},\quad b=\sin\frac{\beta\varepsilon}{r},$$

heuce

$$I_1=2\omega_{2n-1}^{-1}\operatorname{Im}\int_{M(\varepsilon)}(B_1-B_2)\dot{v}+2\omega_{2n-1}^{-1}\int_{M(\varepsilon)}(\pi-a-b)\dot{v}$$

$$=2\omega_{2n-1}^{-1}\operatorname{Im}\left[\left(\sum_{k=1}^{n-1}J_k-J_0\right)-\left(\sum_{k=1}^{n-1}K_k-K_0\right)\right]+2\omega_{2n-1}^{-1}\int_{M(\varepsilon)}(\pi-a-b)\dot{v},$$

where

$$J_0=\int_{M(\varepsilon)}\log(1+re^{-ib})\dot{v},\quad J_k=\frac{1}{k}\int_{M(\varepsilon)}(1+re^{-ib})^{-k}\dot{v},\quad k=1,\ \cdots,\ n-1,$$

$$K_0=\int_{M(\varepsilon)}\log(1-re^{ia})\dot{v},\quad K_k=\frac{1}{k}\int_{M(\varepsilon)}(1-re^{ia})^{-k}\dot{v},\quad k=1,\ \cdots,\ n-1.$$

Since $1+re^{-ib}=1+\sqrt{r^2-\beta^2\varepsilon^2}-i\beta\varepsilon$, we have

$$J_k=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_0^{\sqrt{2\alpha\varepsilon-(\alpha^2+\beta^2)\varepsilon^2}} s^{2n-3}\left(1+\sqrt{r^2-\beta^2\varepsilon^2}-i\beta\varepsilon\right)^{-k}ds.$$

The absolute value of integrand does not exceed s^{2n-3}, so $J_k=O(\varepsilon^{n-1})$ and

$$J_0=\int_{M(\varepsilon)}\log(1+re^{-ib})\dot{v}=\frac{2\pi^{n-1}}{\Gamma(n-1)}\int_0^{\sqrt{2\alpha\varepsilon-(\alpha^2+\beta^2)\varepsilon^2}} s^{2n-3}\log\left(1+\sqrt{r^2-\beta^2\varepsilon^2}-i\beta\varepsilon\right)ds.$$

Thus $\operatorname{Im} J_0=O(\varepsilon^{n-1})$. On the other hand, since $1-re^{ia}=\alpha\varepsilon-i\sqrt{r^2-(1-\alpha\varepsilon)^2}$, we have $\operatorname{Im}K_0=O(\varepsilon^{n-1})$. By the inequality

$$|s^{2n-3}(\alpha s-i\sqrt{1-s^2-(1-\alpha s)^2})^{-k}|\leqslant(\alpha^2+\beta^2)^{-\frac{k}{2}}s^{-k}(\sqrt{2\alpha s-(\alpha^2+\beta^2)s^2})^{2n-3},$$

$$K_k=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_0^{\sqrt{2\alpha s-(\alpha^2+\beta^2)s^2}}s^{2n-3}(\alpha s-i\sqrt{1-s^2-(1-\alpha s)^2})^{-k}ds=O(s^{n-k-1}).$$

Finally

$$K_{n-1}=\frac{2\pi^{n-1}}{\Gamma(n)}\int_0^{\sqrt{2\alpha s-(\alpha^2+\beta^2)s^2}}s^{2n-3}(\alpha s-i\sqrt{1-s^2-(1-\alpha s)^2})^{-(n-1)}ds$$

$$=\frac{2\pi^{n-1}}{\Gamma(n)}\left(\frac{\eta^2}{\alpha s}\right)^{n-1}\int_0^1 R^{2n-3}\left[1-\frac{i}{\alpha s}\sqrt{1-\eta^2R^2-(1-\alpha s)^2}\right]^{-(n-1)}dR,$$

where $\eta=\sqrt{2\alpha s-(\alpha^2+\beta^2)s^2}$, $s=\eta R$. Since

$$\frac{1}{\alpha s}\sqrt{1-\eta^2R^2-(1-\alpha s)^2}=\left[\frac{1}{\alpha s}(1-R^2)(2-\alpha s)+\frac{\beta^2R^2}{\alpha^2}\right]^{\frac{1}{2}}=O(s^{-\frac{1}{2}}),$$

it follows that $\lim\limits_{s\to0}K_{n-1}=0$. Using the same method, we have $\int_{M(s)}(\pi-a-b)\dot{v}=o(1)$. Hence $\lim\limits_{s\to0}I_1=0$.

Now we calculate I_2

$$I_2=\omega_{2n-1}^{-1}\int_{N(s)}\dot{v}\left(\int_{-(\pi-b)}^{-b}+\int_b^{\pi-b}\right)\frac{d\theta}{(1-re^{i\theta})^n}$$

$$=2\omega_{2n-1}^{-1}\operatorname{Im}\left\{\left(\sum_{k=1}^{n-1}P_k-P_0\right)-\left(\sum_{k=1}^{n-1}Q_k-Q_0\right)\right\}+2\omega_{2n-1}^{-1}\int_{N(s)}(\pi-2b)\dot{v},$$

where $N(s)=\{v\,|\,\eta^2\leqslant v\bar{v}'\leqslant1-\beta^2s^2\}$

$$P_0=\int_{N(s)}\log(1+re^{-ib})\dot{v},\quad P_k=\frac{1}{k}\int_{N(s)}(1+re^{-ib})^{-k}\dot{v},\quad k=1,\ \cdots,\ n-1,$$

$$Q_0=\int_{N(s)}\log(1-re^{ib})\dot{v},\quad Q_k=\frac{1}{k}\int_{N(s)}(1-re^{ib})^{-k}\dot{v},\quad k=1,\ \cdots,\ n-1.$$

It is clear that

$$\int_{N(s)}b\dot{v}=o(1)$$

and

$$\lim_{s\to0}\int_{N(s)}\pi\dot{v}=\lim_{s\to0}\frac{\pi^{n-1}}{\Gamma(n)}\pi[(\sqrt{1-\beta^2s^2})^{2n-2}-\eta^{2n-2}]=\frac{\pi^n}{\Gamma(n)},$$

so

$$\lim_{s\to0}2\omega_{2n-1}^{-1}\int_{N(s)}(\pi-2b)\dot{v}=1.$$

On the other hand

$$P_k=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_\eta^{\sqrt{1-\beta^2s^2}}s^{2n-3}(1+\sqrt{r^2-\beta^2s^2}-i\beta s)^{-k}\,ds$$

$$=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_0^{\sqrt{1-\beta^2s^2}}s^{2n-3}(1+\sqrt{r^2-\beta^2s^2}-i\beta s)^{-k}ds+o(1)$$

$$=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_0^1\gamma^{2n-2}R^{2n-3}(1-\gamma\sqrt{1-R^2}-i\beta s)^{-k}dR+o(1),$$

where $\gamma=\sqrt{1-\beta^2s^2}$, $s=\gamma R$. When $s\to0$, P_k tends to a real number and $\lim\limits_{s\to0}\operatorname{Im}P_k=0$, $(1\leqslant k\leqslant n-1)$. Similarly, $\operatorname{Im}P_0=\operatorname{Im}Q_0=o(1)$. Since $1-re^{ib}=1-\sqrt{r^2-\beta^2s^2}-i\beta s$,

$$Q_k=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_\eta^{\sqrt{1-\beta^2\varepsilon^2}} s^{2n-3}(1-\sqrt{1-s^2-\beta^2\varepsilon^2}-i\beta\varepsilon)^{-k}ds.$$

It is clear that

$$\int_0^\eta s^{2n-3}(1-\sqrt{1-s^2-\beta^2\varepsilon^2}-i\beta\varepsilon)^{-k}ds=O(\varepsilon^{n-k-1}),$$

$$\left|\int_0^{\sqrt{1-\beta^2\varepsilon^2}} s^{2n-3}(1-\sqrt{1-s^2-\beta^2\varepsilon^2}-i\beta\varepsilon)^{-k}ds\right|=\left|\int_0^1\frac{\gamma^{2n-2}R^{2n-3}dR}{(1-\gamma\sqrt{1-R^2}-i\beta\varepsilon)^k}\right|$$
$$\leqslant\int_0^1\frac{R^{2n-3}dR}{(1-\sqrt{1-R^2})^k},$$

when $k<n-1$, the integral of right side converges, so

$$\lim_{\varepsilon\to 0}Q_k=\text{real number}.$$

it follows that $\lim\limits_{\varepsilon\to 0}\operatorname{Im}Q_k=0$ $(k<n-1)$. Finally

$$Q_{n-1}=\frac{2\pi^{n-1}}{\Gamma(n)}\int_\eta^\gamma s^{2n-3}(1-\sqrt{\gamma^2-s^2}-i\beta\varepsilon)^{-(n-1)}ds=\frac{2\pi^{n-1}}{\Gamma(n)}\left(\int_0^\gamma-\int_0^\eta\right)=T_1-T_2.$$

Note

$$T_1=\frac{2\pi^{n-1}}{\Gamma(n)}\int_0^1\gamma^{2n-2}R^{2n-3}(1-\gamma\sqrt{1-R^2}-i\sqrt{1-\gamma^2})^{-(n-1)}dR$$

so

$$\lim_{\varepsilon\to 0}\operatorname{Im}T_1=\frac{2^{n-2}\pi^n}{\Gamma(n)}$$

and

$$T_2=\frac{2\pi^{n-1}}{\Gamma(n)}\int_0^1\eta^{2n-2}R^{2n-3}(1-\sqrt{\gamma^2-\eta^2R^2}-i\beta\varepsilon)^{-(n-1)}dR.$$

Since

$$(\gamma^2-\eta^2R^2)^{\frac{1}{2}}=(1-\beta^2\varepsilon^2-\eta^2R^2)^{\frac{1}{2}}=1-\alpha R^2\varepsilon+O(\varepsilon^2),$$
$$(1-\sqrt{\gamma^2-\eta^2R^2}-i\beta\varepsilon)^{n-1}=(\alpha R^2-i\beta)^{n-1}\varepsilon^{n-1}+O(\varepsilon^n),$$

and $\eta^{2n-2}R^{2n-3}=(2\alpha\varepsilon-(\alpha^2+\beta^2)\varepsilon^2)^{n-1}R^{2n-3}$, hence

$$\lim_{\varepsilon\to 0}T_2=\frac{2\pi^{n-1}}{\Gamma(n)}\int_0^1\frac{(2\alpha)^{n-1}}{(\alpha R^2-i\beta)^{n-1}}R^{2n-3}dR=\frac{(2\pi)^{n-1}}{\Gamma(n)}\int_0^1\frac{u^{n-2}du}{(u-ic)^{n-1}},$$

where $c=\dfrac{\beta}{\alpha}$. An easy computation shows that

$$\operatorname{Im}\left(\int_0^1\frac{u^{n-2}du}{(u-ic)^{n-1}}\right)=\operatorname{arctg}\frac{1}{c}-\sum_{k=1}^{n-2}\frac{1}{k}C_{n-2}^k\operatorname{Im}\left(\frac{ic}{1-ic}\right)$$
$$=\operatorname{arctg}\frac{\alpha}{\beta}-\sum_{k=1}^{n-2}\frac{(-1)^{k+1}C_{n-2}^k\beta^k}{k(\alpha^2+\beta^2)^k}\sin k\left(\operatorname{arctg}\frac{\alpha}{\beta}\right).$$

and

$$\lim_{\varepsilon\to 0}\operatorname{Im}Q_{n-1}=\frac{2\pi^{n-1}}{\Gamma(n)}\left(2^{n-3}\pi-2^{n-2}\operatorname{arc\,tg}\frac{\alpha}{\beta}+2^{n-2}h_n\right),$$

where $h_n=\sum\limits_{k=1}^{n-2}(-1)^{k+1}\dfrac{C_{n-2}^k\beta^k}{k(\alpha^2+\beta^2)^{k/2}}\sin k\left(\operatorname{arc\,tg}\dfrac{\alpha}{\beta}\right)$. Hence

$$\lim_{\varepsilon\to 0}I_2=1-2^{n-2}\left(1-\frac{2}{\pi}\operatorname{arc\,tg}\frac{\alpha}{\beta}+\frac{2}{\pi}h_n\right)$$

By [1], (1.8) holds.

§ 2. Two Lemmas

It is not hard to generalize Lemmas 1.1 and 1.2 to the following Lemmas 2.1 and 2.2.

Lemma 2.1. *Suppose* $y=(y_1, \cdots, y_N)$, $y\bar{y}'=1$, *then*

$$\lim_{\varepsilon\to 0}\omega_{N-1}^{-1}\int_{\sigma(\varepsilon)}[1-(y_1+iy_2)]^{-\frac{N}{2}}\dot{y}=1-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{\frac{N}{2}-1},$$

where $\sigma(\varepsilon)=\{y\,|\,y\bar{y}'=1,\ \alpha^2(1-y_1)^2+\beta^2y_2^2\geqslant\varepsilon^2\}$, ω_{N-1}, $\dot{y}$ *stand for the volume and the element of volume of* $y\bar{y}'=1$ *respectively.*

Proof When $N=2n$, this is just Lemma 1.1, so it is enough to prove it when $N=2n+1$. It is known[4] that

$$\omega_{N-1}^{-1}\int_{y\bar{y}'=1}(1-\rho(y_1+iy_2))^{-\frac{N}{2}}\dot{y}=1,$$

so we only need to prove that

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\omega_{N-1}^{-1}\int_{\sigma'(\varepsilon)}(1-\rho(y_1+iy_2))^{-\left(n+\frac{1}{2}\right)}\dot{y}=\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-\frac{1}{2}},$$

where $\sigma'(\varepsilon)=\{y\,|\,y\bar{y}'=1,\ \alpha^2(1-y_1)^2+\beta^2y_2^2\leqslant\varepsilon^2\}$. Let

$$I=\omega_{2n}^{-1}\int_{\sigma'(\varepsilon)}(1-\rho(y_1+iy_2))^{-\left(n+\frac{1}{2}\right)}\dot{y}$$

and suppose $\alpha>0$, $\beta>0$. Set $y_1+iy_2=re^{i\theta}$, $v=(y_3, \cdots, y_{2n+1})$, then

$$I=\omega_{2n}^{-1}\int_{P(\varepsilon)}\dot{v}\int_{-a}^{a}(1-\rho re^{i\theta})^{-\left(n+\frac{1}{2}\right)}d\theta,$$

where $P(\varepsilon)=\left\{v\,\middle|\,v\bar{v}'\leqslant\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t^2}\right\}$

$$a=\arccos\frac{t-\sqrt{(1-t)(t+r^2(1-2t))+(2t-1)\varepsilon^2}}{r(2t-1)},\quad t=\alpha^2,\ 1-t=\beta^2.$$

Using the same method as that of the proof of Lemma 2.1 in [4], we have

$$\int_{-a}^{a}(1-\rho re^{i\theta})^{-\frac{N}{2}}d\theta=2\,\mathrm{Im}\left\{\sum_{k=1}^{n}\frac{1}{\left(k-\frac{1}{2}\right)(1-\rho re^{ia})^{k-\frac{1}{2}}}+2\log\frac{2}{1+\sqrt{1-\rho re^{ia}}}\right\}+2a,$$

so

$$I=2\omega_{N-1}^{-1}\,\mathrm{Im}\left\{\sum_{k=1}^{n}L_k+L_0\right\}+L_*,$$

where

$$L_*=2a\omega_{N-1}^{-1}\int_{P(\varepsilon)}\dot{v}=O\left(\varepsilon^{\frac{N}{2}}\right),$$

$$L_0=2\int_{P(\varepsilon)}\log\frac{2}{1+\sqrt{1-\rho re^{ia}}}\dot{v},$$

$$L_k=\int_{P(\varepsilon)}\frac{\dot{v}}{\left(k-\frac{1}{2}\right)(1-\rho re^{ia})^{k-\frac{1}{2}}},\quad k=1, \cdots, n.$$

Similar to the proof of Lemma 1.1, it is easy to show

$$\operatorname{Im} L_0=o(1), \lim_{\rho\to1} L_k=O(\varepsilon^{n-k}),\ 1\leqslant k\leqslant n-1.$$

Since

$$re^{i\alpha}=\frac{t-Q}{2t-1}+i\frac{\sqrt{r^2(2t-1)t-t-(2t-1)\varepsilon^2+2Qt}}{|2t-1|},$$

where $Q=\sqrt{(1-t)(t+r^2(1-2t))+(2t-1)\varepsilon^2}$, hence

$$J_n=\lim_{\rho\to1} L_n=\frac{2\pi^{n-\frac12}}{\left(n-\frac12\right)\Gamma\left(n-\frac12\right)}\int_0^{\left(\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}\right)^{\frac12}}\Bigg[\frac{t+Q-1}{2t-1}$$

$$-i\frac{\sqrt{t(2t-1)(1-s^2)-t-(2t-1)\varepsilon^2+2Qt}}{|2t-1|}\Bigg]^{-n+\frac12}s^{2n-2}ds$$

$$=\frac{2\pi^{n-\frac12}}{\left(n-\frac12\right)\Gamma\left(n-\frac12\right)}\int_0^1\Bigg[\frac{t+Q-1}{2t-1}$$

$$-i\frac{\sqrt{t(2t-1)(1-\eta^2R^2)-t-(2t-1)\varepsilon^2+2Qt}}{|2t-1|}\Bigg]^{-n+\frac12}\eta^{2n-1}R^{2n-2}dR,$$

where $\eta=\left(\frac{2\varepsilon}{\sqrt{t}}-\frac{\varepsilon^2}{t}\right)^{\frac12}$, $s=\eta R$. As the proof in § 1

$$\eta^{-2}\frac{t-1+Q}{2t-1}=\frac{R^2}{2}+O(\varepsilon),$$

$$\eta^{-2}\frac{1}{|2t-1|}\{t(2t-1)(1-\eta^2R^2)-t-(2t-1)\varepsilon^2+2Qt\}^{\frac12}=\frac12\left(\frac{t}{1-t}\right)^{\frac12}(1-R^4)^{\frac12}+O(\varepsilon).$$

Thus

$$J=\lim_{\varepsilon\to0} J_n=\frac{2\pi^{n-\frac12}}{\left(n-\frac12\right)\Gamma\left(n-\frac12\right)}\int_0^1\frac{R^{2n-2}dR}{\left[R^2-i\left(\frac{t}{1-t}\right)^{\frac12}\sqrt{1-R^4}\right]^{n-\frac12}}$$

$$=\frac{(2\pi)^{n-\frac12}}{\Gamma\left(n+\frac12\right)}\int_0^{\frac{\pi}{2}}\frac{\cos^{n-\frac32}\theta\sin\theta\, d\theta}{(\cos\theta-ib\sin\theta)^{n-\frac12}}\quad\left(b=\sqrt{\frac{t}{1-t}}\right).$$

Note that

$$\frac{\cos^{n-\frac32}\theta\sin\theta}{(\cos\theta-ib\sin\theta)^{n-\frac12}}=\frac1i\,\frac{1}{(1+b)^{n-\frac12}}\,\frac{(e^{2i\theta}+1)^{n-\frac32}(e^{2i\theta}-1)}{(1+de^{2i\theta})^{n-\frac12}}\quad\left(d=\frac{1-b}{1+b}\right),$$

$$J=\frac{1}{i(1+b)^{n-\frac12}}\int_0^{\frac{\pi}{2}}\frac{(e^{2i\theta}-1)(e^{2i\theta}+1)^{n-\frac32}d\theta}{(1+de^{2i\theta})^{n-\frac12}}$$

$$=\frac{1}{i(1+b)^{n-\frac12}}\lim_{\xi\to1^-}\int_0^{\frac{\pi}{2}}\frac{(e^{2i\theta}-1)(1+\xi e^{2i\theta})^{n-\frac32}}{(1+de^{2i\theta})^{n-\frac12}}d\theta$$

$$=\frac{1}{i(1+b)^{n-\frac12}}\lim_{\xi\to1^-}\int_0^{\frac{\pi}{2}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\left(n-\frac12+q-1\right)\cdots\left(n-\frac12-p\right)}{p!\,q!}(-1)^q d^q\xi^p e^{2i(p+q)\theta}(e^{2i\theta}-1)d\theta$$

$$=\text{real number}+i\,\frac{\pi}{2}\,\frac{1}{(1+b)^{n-\frac12}}.$$

Then we have

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1} I=\frac{2}{\omega_{N-1}}\frac{(2\pi)^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)}\frac{\pi}{2}\frac{1}{(1+b)^{n-\frac{1}{2}}}=\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-\frac{1}{2}}.$$

Lemma 2.1 is proved when $\alpha>0$, $\beta>0$.

When $\alpha=\beta$, it is just Lemma 2.1 of [4].

When $\alpha=0$, $\beta>0$ and $\alpha>0$, $\beta=0$, the proofs are similar, we omit the details.

Lemma 2.2. *Suppose* $y=(y_1, \cdots, y_N)$, $y\bar{y}'=1$, *then*

$$\lim_{\varepsilon\to 0}\omega_{N-1}^{-1}\int_{D^*(p_1,\varepsilon)}(1-(y_1+iy_2))^{-\frac{N}{2}}\dot{y}=1-\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-\int_0^{\operatorname{arc\,tg}\frac{\beta}{\alpha}}\cos^{n-2}t\,\frac{\sin(n-1)t}{\sin t}\,dt\right\},$$

where $D^*(p_1, \varepsilon)$ *is the complementary set of* $\{y\,|\,y\bar{y}'=1, 1-y_1<\alpha\varepsilon, |y_2|<\beta\varepsilon\}$ *on* $y\bar{y}'=1$.

The proof is similar to that of Lemmas 1.2 and 2.1.

§ 3. The Cauchy integrals of Lie sphere hyperbolic spaces

Replacing Lemma 2.1 in [4] by Lemma 2.1 in the last section and using the same method in [4], we can obtain the following results, which are the generalizations of [4]. Because the details of proofs are similar, we only state these theorems and omit their proofs.

Theorem 3.1. *Suppose* $\varphi(\xi)$ *is in* Lip p $(0<p\leqslant 1)$ *on* L_{IV}, *then the Cauchy principal value*

$$p.v.V(L_{\mathrm{IV}})^{-1}\int_{L_{\mathrm{IV}}(\gamma)}H_{\mathrm{IV}}(\eta_0, \xi)\varphi(\xi)\dot{\xi}=\lim_{\varepsilon\to 0}V(L_{\mathrm{IV}})^{-1}\int_{G(\varepsilon)}H_{\mathrm{IV}}(\eta_0, \xi)\varphi(\xi)\dot{\xi}$$

exists. Here $\eta_0=e^{i\theta_0}(t, 1, 0, \cdots, 0)P_0^{-1}\Gamma P_0\in B_{\mathrm{IV}}-L_{\mathrm{IV}}$, $P_0=\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\dot{+}\sqrt{2}\,I^{(N-2)}$ *is a real orthogonal matrix,* $z_0=e^{i\theta_0}(t, 0, \cdots, 0)P_0^{-1}\Gamma P_0$, $\gamma=\dfrac{\alpha}{\beta}$

$$G(\varepsilon)=\{\xi\,|\,\xi\in L_{\mathrm{IV}},\ \alpha^2[\mathrm{Re}(1+\eta_0A_0\eta_0'\overline{\xi A_0\xi'}-\eta_0\bar{\xi}')(1-\bar{z}_0\xi')]^2 +\beta^2[\mathrm{Im}(1+\eta_0A_0\eta_0'\overline{\xi A_0\xi'}-\eta_0\bar{\xi}')(1-\bar{z}_0\xi')]^2\geqslant\varepsilon^2\}.$$

Theorem 3.2. *Suppose* $\varphi(\xi)$ *is in* Lip $p(0<p\leqslant 1)$ *on* L_{IV}. *Let*

$$F(z)=V(L_{\mathrm{IV}})^{-1}\int_{L_{\mathrm{IV}}}H_{\mathrm{IV}}(z, \xi)\varphi(\xi)\xi,\ z\in R_{\mathrm{IV}}.$$

If z *approaches* $\eta_0=e^{i\theta_0}(t, 1, 0, \cdots, 0)P_0^{-1}\Gamma P_0\in B_{\mathrm{IV}}-L_{\mathrm{IV}}$ *from the interior of* R_{IV} *satisfying the following condition*

$$|1+\eta_0A_0\eta_0'\overline{\xi A_0\xi'}-\eta_0\bar{\xi}'|\,|1+zA_0z'\overline{\xi A_0\xi'}-z\bar{\xi}'|^{-1}\leqslant M\ (M=\text{constant}),$$

then

$$\lim_{z\to\eta_0}F(z)=p.v.V(L_{\mathrm{IV}})^{-1}\int_{L_{\mathrm{IV}}(\gamma)}H_{\mathrm{IV}}(\eta_0, \xi)\varphi(\xi)\dot{\xi}+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{\frac{N}{2}-1}$$
$$\times\frac{1}{2\pi}\int_0^{2\pi}\varphi\left(e^{i\theta_0}\left(e^{i\theta_0}\frac{1+te^{-i\theta}}{1+te^{i\theta}}, 1, 0, \cdots, 0\right)P_0^{-1}\Gamma P_0\right)\frac{d\theta}{(1+te^{i\theta})^{N/2}},$$

where $p.v.\int_{L_{IV}(\gamma)}$ *is defined in Theorem* 3.1.

In particular, when $\beta=0$, *we have*

$$\lim_{z\to\eta_0} F(z)=p.v.\int_{L_{IV}(\infty)} H_{IV}(\eta_0,\ \xi)\varphi(\xi)\dot{\xi}=\lim_{\varepsilon\to 0} V(L_{IV})^{-1}\int_{G'(\varepsilon)} H_{IV}(\eta_0,\ \xi)\varphi(\xi)\dot{\xi},$$

where

$$G'(\varepsilon)=\{\xi\,|\,\xi\in L_{IV},\ \mathrm{Re}\{(1+\eta_0A_0\eta_0'\overline{\xi A_0\xi'}-\eta_0\bar{\xi}')(1-\bar{z}_0\xi')\}\geqslant\varepsilon\}.$$

Namely, the limit value of Cauchy integral can be represented by a certain Cauchy principal value.

Theorem 3.3. *Suppose* $f(z)=u(z)+iv(z)$ *is holomorphic in* R_{IV} *continuous in* $R_{IV}\cup L_{IV}$ *and* $u(\xi)$ *is in* Lip $p(0<p\leqslant 1)$ *on* L_{IV}, $\eta=e^{i\theta_0}(t,\ 1,\ 0,\ \cdots,\ 0)P_0^{-1}\Gamma P_0\in B_{IV}-L_{IV}$, *then*

$$v(\eta)=p.v.V(L_{IV})^{-1}\int_{L_{IV}(\gamma)} u(\xi)\frac{1}{i}(H_{IV}(\eta,\ \xi)-H_{IV}(\xi,\ \eta))\dot{\xi}$$

$$+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{\frac{N}{2}-1}\frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta_0}\left(e^{i\theta}\frac{1+te^{-i\theta}}{1+te^{i\theta}},\ 1,\ 0,\ \cdots,\ 0\right)P_0^{-1}\Gamma P_0\right)$$

$$\times\frac{1}{i}(T(t,\ \theta)-T(t,\ -\theta))d\theta+v(0),$$

where $T(t,\ \theta)=(1+te^{i\theta})^{-\frac{N}{2}}$, $p.v.V(L_{IV})^{-1}\int_{L_{IV}(\gamma)}$ *is defined in Theorem* 3.1.

In particular, when $\beta=0$ *we have*

$$v(\eta)=p.v.V(L_{IV})^{-1}\int_{L_{IV}(\infty)} u(\xi)\frac{1}{i}(H_{IV}(\eta,\ \xi)-H_{IV}(\xi,\ \eta))\dot{\xi}+v(0),$$

where $p.\ v.\ V(L_{IV})^{-1}\int_{S_{IV}(\infty)}$ *is defined as* $\lim_{\varepsilon\to 0} V(L_{IV})^{-1}\int_{G'(\varepsilon)}$.

That is to say, under the conditions of Theorem 3.3, v, the imaginary part of f, can be represented by a certain Cauchy principal value of Cauchy integral of u, the real part of f.

Theorem 3.4. *Under the conditions of Theorem* 3.1, *the Cauchy principal value*

$$p.\ v.\ V(L_{IV})^{-1}\int_{L_{IV}(b)} H_{IV}(\eta_0,\ \xi)\varphi(\xi)\dot{\xi}=\lim_{\varepsilon\to 0} V(L_{IV})^{-1}\int_{D_b(\eta_0,\varepsilon)} H_{IV}(\eta_0,\ \xi)\varphi(\xi)\dot{\xi}$$

exists, where $D_b(\eta_0,\ \varepsilon)$ *is the complementary set of*

$$\{\xi\,|\,\xi\in L_{IV},\ \mathrm{Re}\{(1+\eta_0A_0\eta_0'\overline{\xi A_0\xi'}-\eta_0\bar{\xi}')(1-\bar{z}_0\xi')\}<\alpha\varepsilon\},$$

$$\mathrm{Im}\{(1+\eta_0A_0\eta_0'\overline{\xi A_0\xi'}-\eta_0\bar{\xi}')(1-\bar{z}_0\xi')\}<\beta\varepsilon\}$$

on L_{IV}, $z_0=e^{i\theta_0}(t,\ 0,\ \cdots,\ 0)P_0^{-1}\Gamma P_0$, $b=\dfrac{2^{n-1}}{\pi}\left\{\dfrac{\pi}{2}-\displaystyle\int_0^{\operatorname{arctg}\frac{\beta}{\alpha}}\cos^{n-2}t\,\frac{\sin(n-1)t}{\sin t}\,dt\right\}$.

Theorem 3.5. *Under the conditions of Theorem* 3.2, *we have*

$$\lim_{z\to\eta_0} F(z)=p.\ v.\ V(L_{IV})^{-1}\int_{L_{IV}(b)} H_{IV}(\eta_0,\ \xi)\varphi(\xi)\dot{\xi}$$

$$+b\cdot\frac{1}{2\pi}\int_0^{2\pi}\varphi\left(e^{i\theta_0}\left(e^{i\theta_0}\frac{1+te^{-i\theta}}{1+te^{i\theta}},\ 1,\ 0,\ \cdots,\ 0\right)P_0^{-1}\Gamma P_0\right)\frac{d\theta}{(1+te^{i\theta})^{N/2}}$$

where $p.\ v.\ V(L_{\mathrm{IV}})^{-1}\int_{L_{\mathrm{IV}}(b)}$ *is defined in Theorem* 3.4.

Theorem 3.6. *Under the conditions of Theorem* 3.3, *we have*

$$v(\eta)=p.\ v.\ V(L_{\mathrm{IV}})^{-1}\int_{L_{\mathrm{IV}}(b)}u(\xi)\frac{1}{i}(H_{\mathrm{IV}}(\eta,\xi)-H_{\mathrm{IV}}(\xi,\eta))\dot{\xi}$$

$$+b\frac{1}{2\pi}\int_0^{2\pi}u\left(e^{i\theta_0}\left(e^{i\theta}\frac{1+te^{-i\theta}}{1+te^{i\theta}},1,0,\cdots,0\right)P_0^{-1}\Gamma P_0\right)\frac{1}{i}(T(t,\theta)-T(t,-\theta))\,d\theta+v(0),$$

where $p.\ v.\ V(L_{\mathrm{IV}})^{-1}\int_{L_{\mathrm{IV}}(b)}$ *and* T *are defined in Theorems* 3.4 *and* 3.3 *respectively*

§ 4. The Cauchy integrals of matrix hyperbolic spaces

Replacing Theorem 1.3 in [2] by Theorem 1.1 of § 1,and using the same method as that in the proofs of Theorems 3.4.1, 3.4.2, 3.5.1, and 3.5.2 in [5], we can obtain the following results, which are generalization of the corresponding theorems in [5]. In order to save space, we omit their proofs.

Theorem 4.1. *Suppose* $\varphi(U)$ *is in* Lipα $(0<\alpha\leqslant 1)$ *on* $L_I(m, n)$, *then when* $Z=U_0'\begin{pmatrix}\rho & 0\\ 0 & Z_1\end{pmatrix}V_0\in R_I(m,n)$ *approaches* $Q=U_0'\begin{pmatrix}1 & 0\\ 0 & Z_1\end{pmatrix}V_0\in L_I^{(m-1)}$, *the limit value of the Cauchy integral*

$$F(Z)=V(L_I(m,n))^{-1}\int_{L_I(m,n)}\varphi(U)\det(I-Z\bar{U}')^{-n}\dot{U}$$

exists and equals

$$p.\ v.\ V(L_I(m,n))^{-1}\int_{L_I(m,n)(\gamma)}\varphi(U)\det(I-Q\bar{U}')^{-n}\dot{U}$$

$$+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}\det(I-Z_1\bar{Z}_1')^{-1}V(L_I(m-1,n-1))^{-1}$$

$$\times\int_{L_I(m-1,n-1)}\varphi\left(U_0'\begin{pmatrix}1 & 0\\ 0 & U_1\end{pmatrix}V_0\right)\frac{\det(I-U_1\bar{Z}_1')}{\det(I-Z_1\bar{U}_1')^{n-1}}\dot{U}_1,$$

where the definition of the principal value is

$$\lim_{\varepsilon\to 0}V(L_I(m,n))^{-1}\int_{D(\varepsilon)}\varphi(U)\det(I-Q\bar{U}')^{-n}\dot{U}$$

and

$$D(\varepsilon)=\{U\,|\,U\in L_I(m,n),\ \alpha^2(\mathrm{Re}\det S_1)^2+\beta^2(\mathrm{Im}\det S_1)^2\geqslant\varepsilon^2\},$$

$$S_1=I-Q\bar{U}'-U\bar{T}_0'+Q\bar{T}_0',\ T_0=U_0'\begin{pmatrix}0 & 0\\ 0 & Z_1\end{pmatrix}V_0,\ \gamma=\frac{\alpha}{\beta}.$$

In particular, when $\beta=0$, *we have*

$$\lim_{z\to Q}F(Z)=p.\ v.\ V(L_I(m,n))^{-1}\int_{L_I(m,n)(\infty)}\varphi(U)\det(I-Q\bar{U}')^{-n}\dot{U},$$

where the definition of principal value is

$$\lim_{\varepsilon\to 0}V(L_I(m,n))^{-1}\int_{\substack{L_I(m,n)\\ \mathrm{Re}\det S_1\geqslant\varepsilon}}\varphi(U)\det(I-Q\bar{U}')^{-n}\dot{U}.$$

Namely, the limit value of Cauchy integral may be represented by a certain Cauchy principal value.

Theorem 4.2. *Suppose* $\varphi(U)$ *is in* Lipα $(0<\alpha\leqslant 1)$ *on* $L_I(n)$, *then when* $Z=U_0'\begin{pmatrix}\rho & 0\\ 0 & Z_1\end{pmatrix}V_0\in R_I^*(n)$, $(\rho>1)$ *approaches* $Q^*=U_0'\begin{pmatrix}1 & 0\\ 0 & Z_1\end{pmatrix}V_0$, *the limit value of Cauchy integral* $F(Z)=V(L_I(n))^{-1}\int_{L_I(n)}\varphi(U)\det(I-Z\overline{U}')^{-n}\dot{U}$ *exists and equals*

$$
\begin{aligned}
&p.\ v.\ V(L_I(n))^{-1}\int_{L_I(n)}\varphi(U)\det(I-Q^*\overline{U}')^{-n}\dot{U}\\
&-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}[V(L_I(n-1))\det(I-Z_1\overline{Z}_1')]^{-1}\\
&\times\int_{L_I(n-1)}\varphi\left(U_0'\begin{pmatrix}1 & 0\\ 0 & U_1\end{pmatrix}V_0\right)\det(I-U_1\overline{Z}_1')\det(I-Z_1\overline{U}_1')^{1-n}\dot{U}_1,
\end{aligned}
$$

where the definition of principal value is

$$\lim_{\varepsilon\to 0}V(L_I(n))^{-1}\int_{D'(\varepsilon)}\varphi(U)\det(I-Q^*\overline{U}')^{-n}\dot{U}$$

and $D'(\varepsilon)=\{U\,|\,U\in L_I(n),\ \alpha^2(\mathrm{Re}\det S_2)^2+\beta^2(\mathrm{Im}\det S_2)^2\leqslant\varepsilon^2\}$,

$$S_2=Q^*\overline{Q}^{*\prime}+\overline{Q}_0'^*-U\overline{Q}'^*-Q^*\overline{U}'\overline{Q}_0'^*,\quad Q_0^*=U_0'\begin{pmatrix}0 & 0\\ 0 & I^{(n-1)}\end{pmatrix}\overline{U}_0.$$

when $\beta=0$, *we have*

$$\lim_{z\to Q^*}F(Z)=p.\ v.\ V(L_I(n))^{-1}\int_{L_I(n)}\varphi(U)\det(I-Q^*\overline{U}')^{-n}\dot{U},$$

where the difinition of principal value is

$$\lim_{\varepsilon\to 0}V(L_I(n))^{-1}\int_{\substack{L_I(n)\\ \mathrm{Re}\det S_2\geqslant\varepsilon}}\varphi(U)\det(I-Q^*\overline{U}')^{-n}\dot{U}.$$

That is to say, in this case, the limit value of Cauchy integral may also be represented by a certain Cauchy principal value.

Theorem 4.3. *Suppose* $f(Z)=u(Z)+iv(Z)$ *is holomorphic in* $R_I(m,n)$ *continuous in* $R_I(m,n)\cup L_I(m,n)$ *and* $u(U)$ *is in* Lip $\alpha(0<\alpha\leqslant 1)$ *on* $L_I(m,n)$, *then for an arbitrary point* $Q=U_0'\begin{pmatrix}1 & 0\\ 0 & Z_1\end{pmatrix}V_0$ *on* $L_I(m,n)$, $Z_1\overline{Z}'<I^{(n-1)}$ *we have*

$$
\begin{aligned}
v(Q)=&p.\ v.\ V(L_I(m,n))^{-1}\int_{L_I(m,n)(\gamma)}\frac{1}{i}u(U)(H(Q,\overline{U})-H(U,\overline{Q}))\dot{U}\\
&+\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}\frac{\det(I-Z_1\overline{Z}_1')^{-1}}{V(L_I(m-1,n-1))}\\
&\times\int_{L_I(m-1,n-1)}u\left(U_0'\begin{pmatrix}1 & 0\\ 0 & U_1\end{pmatrix}V_0\right)\frac{1}{i}(T(Z_1,\overline{U}_1)-T(U_1,\overline{Z}_1))\dot{U}_1+v(0),
\end{aligned}
$$

where $H(Q,\overline{U})=\det(I-Q\overline{U}')^{-n}$, $T(Z_1,\overline{U}_1)=\det(I-U_1\overline{Z}_1')\det(I-Z_1\overline{U}_1')^{1-n}$.

When $\beta=0$, *we have*

$$v(Q)=p.\ v.\ V(L_I(m,n))^{-1}\int_{L_I(m,n)(\infty)}u(U)\frac{1}{i}(H(Q,\overline{U})-H(U,\overline{Q}))\dot{U}+v(0).$$

Namely v can be represented by a certain Cauchy principal value of u. Here the definition of principal value is the same as that of Theorem 4.1.

Theorem 4.4. *Suppose $f(Z)=u(Z)+iv(Z)$ is holomorphic in $R_I^*(n)$, continuous on $R_I^*(n)\cup L_I(n)$, and $u(U)$ is in* Lip $\alpha(0<\alpha\leqslant 1)$ *on $L_I(n)$, then for an arbitrary point $Q^*=U_0'\begin{pmatrix}1 & 0\\ 0 & Z_1\end{pmatrix}V_0$, $Z_1\bar{Z}_1'>I^{(n-1)}$ on $L_I^*(n-1)$, we have*

$$v(Q^*)=p.\ v.\ (V(L_I(n)))^{-1}\int_{L_I(n)(\gamma)}u(U)\ \frac{1}{i}H^*(Q^*,\ \bar{U})-H^*(U,\ \bar{Q}^*))\dot{U}$$

$$-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}\frac{\det(I-Z_1\bar{Z}_1')^{-1}}{V(L_I(n))}$$

$$\times\int_{L_I(n-1)}u\left(U_0'\begin{pmatrix}1 & 0\\ 0 & U_1\end{pmatrix}V_0\right)\frac{1}{i}(T^*(Z_1,\bar{U}_1)-T^*(U_1,\ \bar{Z}_1))\dot{U}_1+v(Z_\infty),$$

where $H^*(Q^*,\ \bar{U})=(-1)^n\det(Q^*\bar{U})^n\det(I-Q^*\bar{U}')^{-n}$,

$$T^*(Z_1,\ \bar{U}_1)=(-1)^n\det(Z,\ \bar{U}_1)\det(I-U_1\bar{Z}_1')\det(I-Z_1\bar{U}_1')^{1-n},$$

Z_∞ denotes the infinite point of homogenous coordinates $(Z_1,\ Z_2)=(O^{(n)},\ I^{(n)})$ in the Grassmann manifold.

When $\beta=0$, we have

$$v(Q^*)=p.\ v.\ V(L_I(n))^{-1}\int_{L_I(n)(\infty)}u(U)\frac{1}{i}(H^*(Q^*,\ \bar{U})-H^*(U,\ \bar{Q}^*))\dot{U}+v(Z_\infty),$$

i. e. v may be represented by a certain Cauchy principal value of u. The definition of principal value is as that of Theorem 4.2.

Replacing Theorem 1.3 in [2] by Theorem 1.2 of § 1, and using hte same method as that in the proofs of Theorems 3.4.1, 3.4.2, 3.5.1 and 3.5.2 in [5], we also obtain the analogues of Theorem 4.1, 4.2, 4.3 and 4.4. The details are omited here.

§ 5. On the Bochner-Martinelli integrals

On the Bochner-Martinelli integrals, Lu Qikeng and Zhong Tongde[7] proved the following Plemelj formula.

Suppose D is a domain in $\mathbf{R}^{2n}$, $\Omega=\partial D$ is an orientable, C^2 manifold, $f(z)$ is a continuous complex function in Ω and satisfies Lipschtz condition, and $K_{(2n-1)}(z,\ \zeta)$ is the Bochner-Martinelli kernel. Then, when w approaches $w_0\in\Omega$ from the interior or exterior of D, the limit values of Cauchy integral

$$F(w)=\int_\Omega f(z)K_{(2n-1)}(z,\ w)$$

are expressed by the following formulas respectively

$$F_i(w_0)=p.\ v.\int_\Omega f(z)K_{(2n-1)}(z,\ w_0)+\frac{1}{2}f(w_0),$$

$$F_e(w_0) = p.\ v.\int_\Omega f(z) K_{(2n-1)}(z,\ w_0) - \frac{1}{2} f(w_0).$$

where $p.\ v.\int_\Omega$ is defined as $\lim_{\delta\to 0}\int_{\Omega-\Omega\cap S_\delta(w_0)}$, and $S_\varepsilon(w_0)$ is the ball with the center W_0 and raduis ε.

For the Bochner-Martinelli kernel, we can raise the same questions: Are there various definitions for the Cauchy principal value as above? Will the value $\frac{1}{2}$ also change into different values by virture of the different definitions?

The answer is negative, the reason is as follows: In the proof of the above theorem, the deleted neighborhood around w_0 is $\Omega\cap S_\varepsilon(w_0)$. Now if the deleted neighborhood around w_0 is $\Omega\cap M_{\varepsilon_1}(w_0)$ rather than $\Omega\cap S_\varepsilon(w_0)$ ($M_{\varepsilon_1}(w_0)$ is a neighborhood of w_0, and it contracts to the point w_0 as $\varepsilon_1\to 0$), then we can take ε_1 sufficiently small, such that $\Omega\cap M_{\varepsilon_1}(w_0)\subset\Omega\cap S_\varepsilon(w_0)$. Since the Bochner-Martinelli kernel is well defined in $\mathbf{C}^n-\{w_0\}$, and it is a closed form, hence, by Stokes formula, the integral on $\Omega\cap S_\varepsilon(w_0)$ is equal to that on $\Omega\cap M_{\varepsilon_1}(w_0)$.

That is to say, no matter what neighborhood is deleted, the results are the same.

The reason why the kernels discussed above have the different Cauchy principal values is that either these kernels are only well defined in the interior or they are not closed forms.

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