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GENERALIZATIONS OF THE MEAN VALUES AND THEIR INEQUALITIES

SUN JIACHANG (孙家昶)
(*Computing Center, Academia Sinica*)

Abstract

Some generalizations of the classical Mean Values and their inequalities are presented in this paper, using B-spline functions. The new generalized Means can be applied to nonlinear difference schemes for solving O. D. E. and P. D. E. with singularities ([6, 7, 8]).

§ 1. Generalized Mean Values $S(r, t; a)$

For an arbitrary finite and positive sequence

$$(a)=(a_1, a_2, \cdots, a_n) \tag{1}$$

and a positive power sequence

$$(p)=(p_1, p_2, \cdots, p_n), \ \sum_{j=1}^{n} p_j=1$$

the Mean Value of order r of (a) with the power (p) $M(r; a, p)$ is defined in [1] as

$$M_r \equiv M(r; a, p)=\left(\sum_{j=1}^{n} p_j a_j^r\right)^{\frac{1}{r}} \tag{2}$$

and there are some main results as follows ([1, 2]):

1° $$m \leqslant M(r; a, p) \leqslant M$$

holds for all real number r, where

$$m=\min_{1\leqslant j\leqslant n}(a_j), \ M=\max_{1\leqslant j\leqslant n}(a_j).$$

2° $\lim\limits_{r\to-\infty} M(r; a, p)=m, \ \lim\limits_{r\to+\infty} M(r; a, p)=M, \ \lim\limits_{r\to\infty} M(r; a, p)=G\equiv\prod\limits_{j=1}^{n} a_j^{p_j}.$

3° $\dfrac{d}{dr} M(r; a, p)\geqslant 0$, where the equality holds if and only if $m=M$.

In this paper we generalize the above concept of Mean Values.

For simplicity, let $p_j=\dfrac{1}{n}(j=1, 2, \cdots, n)$.

Definition 1. *A positive number $S_{r,k}$ is called the k-th Means of order $r(r\neq 0)$ of (a) if for any natural number k*

$$S_{r,k}\equiv S(r, k; a)=\left\{\frac{k!(n-1)!}{(k+n-1)!}\sum_{j_v\in\sigma}\prod_{i=1}^{k} a_{j_i}^r\right\}^{\frac{1}{kr}}, \tag{3}$$

where the summation set $j_v \in \sigma$ means

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$$1\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_k\leqslant n.$$

In particular, for $k=1$ we have $S_{r,1}=M_r$.

In order to extend the concept of the Means further, we apply the divided difference in numerical analysis as a main tool.

The divided difference of order $n-1$ of a function f at the points $a_1, \cdots, a_n$ is denoted by $[a_1, \cdots, a_n]f$ and defined as

$$[a_1, \cdots, a_n]f=\begin{cases}\dfrac{1}{(n-1)!}f^{(n-1)}(a_i), \text{ if } a_1=\cdots=a_n,\\ \dfrac{1}{a_j-a_k}([a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n]f\\ \quad-[a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n]f), \text{ if } a_j\neq a_k.\end{cases} \tag{4}$$

Lemma 1[5]. *If k is a natural number, then*

$$[a_1, \cdots, a_n]y^{n-1+k}=\sum_{j_\nu\in\sigma}\prod_{i=1}^{k}a_{j_i}. \tag{5}$$

Thus, we may further extend the concept of the Means as follow.

Definition 2. *$S(r, t)$ is the Means of order (r, t) with two parameters of (a) if for all points in the plane (r, t) $S(r, t)$ is defined by*

$$S(r, t)\equiv S(r, t; a)=\left\{\frac{(n-1)!\,\Gamma(t+1)}{\Gamma(t+n)}[a_1^r, \cdots, a_n^r]y^{n-1+t}\right\}^{\frac{1}{rt}}, \tag{6}$$

except for $r=0$ or $t=-k\,(k=0, 1, \cdots, n)$, where $S(r, t)$ is additionally defined by their limit values as follows

$$S(0, t)\equiv\lim_{r\to0}S(r, t)=G\equiv\prod_{j=1}^{n}a_j^{\frac{1}{n}},$$

$$S(r, -n)\equiv\lim_{t\to-n}S(r, t)=G,$$

$$S(r, 0)\equiv\lim_{t\to0}S(r, t)=\exp\left\{\frac{1}{r}[a_1^r, \cdots, a_n^r](y^{n-1}\log y)-\frac{n}{2r(n-2)!}\right\},$$

$$s(r, -k)\equiv\lim_{t\to-k}S(r, t)=\left\{\frac{(n-1)!(-1)^{k-1}}{(k-1)!(n-k-1)!}\times[a_1^r, \cdots, a_n^r](y^{n-k-1}\log y)\right\}^{-\frac{1}{kr}},$$

$$(k=1, 2, \cdots, n-1). \tag{7}$$

It is interesting that in the whole plane (r, t) the generalized Means on the two lines $r=0$ and $t=-n$ both equal to the geometric Means and vice versa. Therefore, the two lines of $r=0$ and $t=-n$ play a role as "coordinate axes" in the parameter plane (r, t) of the generalized Means. There exists a close connection between this fact and some inequalities about the generalized Means.

First, we show that $S(r, t)$ defined in Definition 2 is really a Mean Value.

Theorem 1. *For any real pair (r, t), $S(r, t)$ given in Definition 2 satisfies the following inequality*

$$m\leqslant S(r, t)\leqslant M, \tag{8}$$

where the equality holds if and only if $m=M$.

Proof First we assume $r\neq 0,\ t\neq -k(k=0,\ 1,\ \cdots,\ n)$. Based on the well-known Peano Theorem on an error estimation of linear operators in approximation theory, it is not difficult to obtain the following

Lemma 2. *For* $r\neq 0,\ t\neq -k(k=0,\ 1,\ \cdots,\ n)$ *and*

$$a(r)=\min_{1\leq j\leq n}(a_j^r),\ b(r)=\max_{1\leq j\leq n}(a_j^r),$$

there exists an identity

$$\frac{(n-1)!\Gamma(t+1)}{\Gamma(t+n)}[a_1^r,\ \cdots,\ a_n^r]x^{n-1+t}=\int_{a(r)}^{b(r)}x^t M_{n-1}(x;\ a_1^r,\ \cdots,\ a_n^r)dx, \tag{9}$$

where $M_{n-1}(x;\ a_1^r,\ \cdots,\ a_n^r)$ *is the* $(n-1)$*-th B-spline at the knots* $a_1^r,\ \cdots,\ a_n^r$.

Applying the two main properties of B-spline

1° $M_{n-1}(x;\ a_1^r,\ \cdots,\ a_n^r)\begin{cases}=0, \text{ if } x\bar{\in}[a(r),\ b(r)],\\ >0, \text{ if } x\in(a(r),\ b(r)),\end{cases}$

2° $\int_{a(r)}^{b(r)}M_{n-1}(x;\ a_1^r,\ \cdots,\ a_n^r)dx=1,$

the identity (8) can be directly obtained from (9). Hence, the proof of Theorem 1 is completed.

Theorem 2. *As a function with two real variables* $(r,\ t)$, $S(r,\ t)$ *is continuous in the whole plane, including the infinite point, except the following four points*

$$(0,\ +\infty),\ (0,\ -\infty),\ (+\infty,\ 1-n),\ (-\infty,\ 1-n).$$

And the values of $S(r,\ t)$ at the infinite is additionally defined by the following limit values:

$$\lim_{|r|\to\infty}S(r,\ t)=\begin{cases}M, \text{ if } r>0 \text{ and } t>1-n,\\ m, \text{ if } r<0 \text{ and } t>1-n,\\ (a_{j_1}\cdots a_{j_{n-1}})^{-\frac{1}{t}}a_{j_n}^{1+\frac{n-1}{t}},\ r>0 \text{ and } t<1-n,\\ (a_{j_n}\cdots a_{j_2})^{-\frac{1}{t}}a_{j_1}^{1+\frac{n-1}{t}},\ r<0 \text{ and } t<1-n,\end{cases}$$

where $a_{j_1}\geq a_{j_2}\geq\cdots\geq a_{j_n}$ and

$$\lim_{|t|\to\infty}S(r,\ t)=\begin{cases}M, \text{ if } rt>0,\\ m, \text{ if } rt<0.\end{cases} \tag{10}$$

Besides, on some special directions the limit values at the above four discontinuous infinite points are

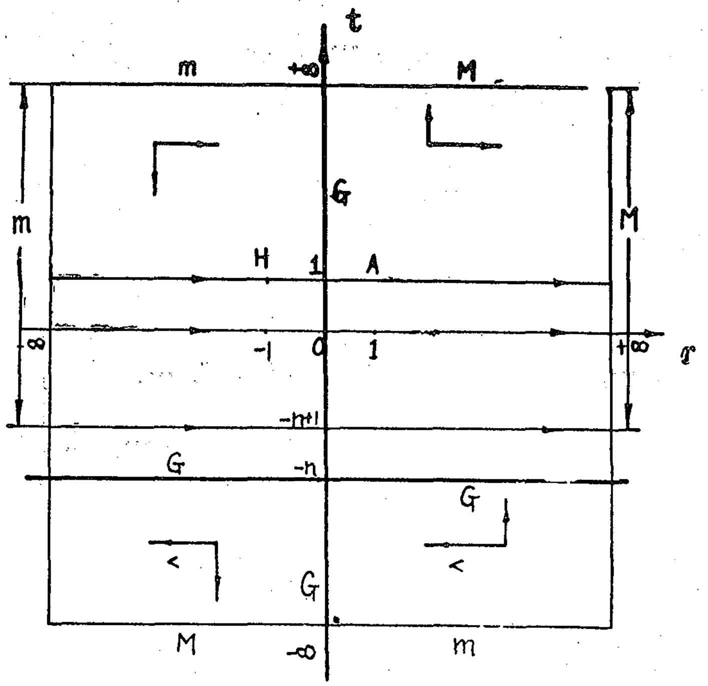
$$\lim_{r\to 0+}\lim_{t\to+\infty}S(r,\ t)=M,\ \lim_{r\to 0-}\lim_{t\to+\infty}S(r,\ t)=m;$$

$$\lim_{r\to 0+}\lim_{t\to-\infty}S(r,\ t)=m,\ \lim_{r\to 0-}\lim_{t\to-\infty}S(r,\ t)=M;$$

$$\lim_{r\to+\infty}\lim_{t+n\to 1+}S(r,\ t)=M,\ \lim_{r\to+\infty}\lim_{t+n\to 1-}S(r,\ t)=(a_{j_1}\cdots a_{j_{n-1}})^{\frac{1}{n-1}};$$

$$\lim_{r\to-\infty}\lim_{t+n\to 1+}S(r,\ t)=m,\ \lim_{r\to-\infty}\lim_{t+n\to 1-}S(r,\ t)=(a_{j_n}\cdots a_{j_2})^{\frac{1}{n-1}}. \tag{11}$$

The detailed proof is omitted.



Notations:
m—Minimum Value
M—Maximum Value
A—Arithmetic Means
G—Geometric Means
H—Harmonic Means

Fig. The generalized Means and their directions of inequalities on (r, t) plane.

§ 2. Inequalities for generalized Mean Values

In this section we extend some inequalities of the classical Means to the generalized Means.

As a first step, we consider k to be a natural number in $S_{r,k}$.

Denote
$$S_{r,k}=\{P_{r,k}\}^{\frac{1}{rk}},$$
where
$$P_{r,k}=\frac{k!\,(n-1)!}{(k+n-1)!}\sum_{j_\nu\in\sigma}\prod_{i=1}^{k}a_{j_i}^{r}. \tag{12}$$

Since $\log S_{r,k}=\frac{1}{rk}\log P_{r,k}$, differentiating the both sides with respect to r we obtain
$$\frac{r^2kP_{r,k}}{S_{r,k}}\frac{d}{dr}S_{r,k}=\frac{k!\,(n-1)!}{(k+n-1)!}\sum_{j_\nu\in\sigma}\left(\prod_{i=1}^{k}a_{j_i}^{r}\right)\left(\log\prod_{i=1}^{k}a_{j_i}^{r}\right)$$
$$-\left\{\frac{k!\,(n-1)!}{(k+n-1)!}\sum_{j_\nu\in\sigma}\prod_{i=1}^{k}a_{j_i}^{r}\right\}\log\left\{\frac{k!\,(n-1)!}{(k+n-1)!}\sum_{j_\nu\in\sigma}\prod_{i=1}^{k}a_{j_i}^{r}\right\}.$$

Using the convexity of function $f(x)=x\log x$ (i. e. $f''>0$), let
$$a_{r,k}=\prod_{i=1}^{k}a_{j_i}^{r},\quad \bar{a}_{r,k}=P_{r,k},$$
then
$$\overline{f(a_{rk})}\geqslant f(\bar{a}_{rk}). \tag{13}$$

Therefore, it leads to the following theorem.

Theorem 3. *For any natural number k*

$$\frac{d}{dr} S_{r,k} \geqslant 0, \tag{14}$$

where the equality holds if and only if $m=M$.

Proof We only need to point out that in our case

$$\overline{f(a_{rk})} = \frac{k!\,(n-1)!}{(k+n-1)!} \sum_{j_\nu \in \sigma} \left(\prod_{i=1}^{k} a_{j_i}^r\right) \left(\log \prod_{i=1}^{k} a_{j_i}^r\right),$$

hence, from (13), the inequality (14) follows.

If the equality holds in (14), it implies that the equality also holds in (13). Hence, the sufficient and necessary condition is all a_j^r to be the same. Q. E. D.

Fixing r, for different k, there is another inequality.

Theorem 4. *For* $k=1, 2, \cdots$

$$S_{r,k} \begin{cases} \leqslant \\ \geqslant \end{cases} S_{r,k+1} \quad \begin{matrix} , \text{ if } r>0, \\ , \text{ if } r<0, \end{matrix} \tag{15}$$

where the equality holds if and only if $m=M$.

Remark. If $r=0$, then $S_{0,k} \equiv G$, hence, this case is an exception.

Proof Applying Lemmas 1, 2 described in section 1, (12) may be expressed by

$$P_{rk} = \int_{a(r)}^{b(r)} x^k M_{n-1}(x;\ a_1^r, \cdots, a_n^r)\, dx. \tag{16}$$

Let $M(x) \equiv M_{n-1}(x;\ a_1^r, \cdots, a_n^r)$. Using Hölder inequality, we have

$$P_{rk} = \int_{a(r)}^{b(r)} x^k \{M(x)\}^{\frac{k}{k+1}} \cdot \{M(x)\}^{\frac{1}{k+1}} dx$$

$$\leqslant \left\{\int_{a(r)}^{b(r)} [x^k (M(x))^{\frac{k}{k+1}}]^{\frac{k+1}{k}} dx\right\}^{\frac{k}{k+1}} \left\{\int_{a(r)}^{b(r)} M(x)\, dx\right\}^{\frac{1}{k+1}} \leqslant \{P_{r,k+1}\}^{\frac{k}{k+1}},$$

because of the non-negativity and the normalization properties of B-splines. Hence

$$P_{rk}^{\frac{1}{k}} \leqslant P_{r,k+1}^{\frac{1}{k+1}}. \tag{17}$$

According to the condition with which the equality holds in Hölder inequality, it follows that the sufficient and necessary condition with which the equality in (17) holds is $m=M$. Using the definition in (12), we complete the proof of Theorem 4.

Now we extend the above two theorems to the more generalized Means defined in Definition 2.

Theorem 5 (t-direction inequality of the generalized Means).

$$\frac{\partial}{\partial t} S(r, t) \begin{cases} \geqslant 0 \text{ if } r>0, \\ \leqslant 0 \text{ if } r<0, \end{cases} \tag{18}$$

where the equality holds if and only if $m=M$.

Proof Let

$$S(r, t) = \{P(r, t)\}^{\frac{1}{rt}} \tag{19}$$

where

$$P(r, t)=\int_{a(r)}^{b(r)} x^t M(x)\,dx. \tag{20}$$

Hence

$$\frac{rt^2P(r, t)}{S(r, t)}\frac{\partial}{\partial t}S(r, t)=t\frac{\partial}{\partial t}P(r, t)-P(r, t)\log P(r,t). \tag{21}$$

Set $\bar{x}=P(r, t)$, $f(x)=x\log x$, then

$$\overline{f(x)}=\overline{x\log x}=\int_{a(r)}^{b(r)}(x^t\log x^t)M(x)\,dx=t\frac{\partial}{\partial t}p(r, t)$$

and
$$f(\bar{x})=P(r, t)\log P(r, t).$$

Due to the convexity of $f(x)$, $\overline{f(x)}\geqslant f(\bar{x})$, hence, the right hand side of (21) is non-negative. If $t\neq 0$, a sufficient and necessary condition under which the left hand side of (21) equals to zero is that the upper limit and the lower limit of the integral (20) are the same, i. e. $m=M$. If $t=0$, we may obtain the same conclusion by a limit process. In fact, from (20)—(21), by using l'Hospitale rule twice, as t tends to zero, a straightforward computation yields

$$\lim_{t\to 0}\frac{rP}{S}\frac{\partial S}{\partial t}=\lim_{t\to 0}\left\{\frac{\partial^2 P}{\partial t^2}+\left(\frac{\partial P}{\partial t}\right)^2\right\}\geqslant 0$$

because $\lim\limits_{t\to 0}P(r, t)=1$ and

$$\lim_{t\to 0}\frac{\partial^2 P}{\partial t^2}=\int_{a(r)}^{b(r)}(\log x)^2M(x)\,dx\geqslant 0.$$

Q. E. D.

Theorem 6 (r-direction inequality of the generalized Means).

$$\frac{\partial}{\partial r}S(r, t)\begin{cases}\geqslant 0 \text{ if } t+n>0,\\ \leqslant 0 \text{ if } t+n<0,\end{cases} \tag{22}$$

where the equality holds if and only if $m=M$.

Remark. If $t=-n$, then $S(r, -n)\equiv G$.

Proof For simplicity, here we only give a proof in the case of $n=2$ in detail.

From Definition 2, we denote

$$S\equiv S(r, t)=\{P(r, t)\}^{\frac{1}{rt}},$$

where
$$P\equiv P(r, t)=\frac{1}{1+t}\cdot\frac{a_2^{r(1+t)}-a_1^{r(1+t)}}{a_2^r-a_1^r}.$$

Hence

$$\frac{tr^2P}{S}\frac{\partial}{\partial r}S=r\frac{\partial}{\partial r}P-P\log P, \tag{23}$$

but

$$r\frac{\partial P}{\partial r}=(1+t)\frac{\partial P}{\partial t}-rP\log S(r, 0). \tag{24}$$

Therefore, on one hand, (23) may be written as

$$\frac{Pr^2}{S}\frac{\partial S}{\partial r}=\frac{1+t}{t^2}\left(t\frac{\partial P}{\partial t}-P\log P\right)+\frac{Pr}{t}(\log S-\log S(r, 0)). \tag{25}$$

From (18), (20), there are two inequalities

$$t\frac{\partial P}{\partial t}-P\log P\geqslant 0 \text{ and } rt(\log S-\log S(r,\ 0))\geqslant 0,$$

and both equalities hold if and only if $m=M$. Hence, in particular, we get the conclusion

$$\frac{\partial S(r,\ t)}{\partial r}\geqslant 0 \text{ if } t\geqslant -1 \tag{26}$$

with the equality holding if and only if $m=M$.

Strictly, we proved (26) with an assumption $t\neq 0$. For $t=0$, we need a limit procedure.

On another hand, (23) may also be written as

$$\frac{Pr^2}{S}\frac{\partial S}{\partial r}=\frac{2+t}{t^2}\left(t\frac{\partial P}{\partial t}-P\log P\right)+\frac{rP}{t}\left\{\log S-\log S(r,\ 0)-\frac{t}{S}\frac{\partial S}{\partial t}\right\}.$$

Similar to the above analysis, we obtain

$$\frac{\partial S(r,\ t)}{\partial r}\leqslant 0 \text{ if } t\leqslant -2. \tag{27}$$

Now the left interval is $-2<t<-1$. We introduce a concept of "conjugate points" on the $(r,\ t)$ plane. A real pair $(r^*,\ t^*)$ is called the conjugate point of another real pair $(r,\ t)$, if there exist the following two relationships

$$\begin{aligned}&(1+t)(1+t^*)=1,\\ &tr=-t^*r^*.\end{aligned} \tag{28}$$

It is easy to verify that

$$S(r,\ t)=S(r^*,\ t^*). \tag{29}$$

Hence

$$\frac{\partial S(r,\ t)}{\partial r}=\frac{\partial S(r^*,\ t^*)}{\partial r^*}\left(-\frac{t}{t^*}\right). \tag{30}$$

For $-2<t<-1$, $t^*<-2$. From (27), we have $\frac{\partial S(r^*,\ t^*)}{\partial r^*}\leqslant 0$. But in this case $-\frac{t}{t^*}<0$, Hence, finally we get the required result

$$\frac{\partial S(r,\ t)}{\partial r}\geqslant 0 \text{ if } -2<t<-1. \tag{31}$$

Both inequalities of (27) and (31) hold if and only if $m=M$. Combining (26), (27) and (31) leads to the conclusion (22) in the case of $n=2$.

For the cases of a general natural number n, the main idea of the proof is similar to the above. The derivation is a little bit complicated. We omitt the details.

Theorem 5 and Theorem 6 are two main inequalities. Some other inequalities about the classical Means may also be extended to the above generalized Means.

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