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A THEOREM ON THE CONTINUITIES OF STOCHASTIC PROCESSES

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Abstract

If a stochastic process X_t is continuous in probability, then the points of discontinuity in r-th mean constitute a set of 1st category.

Suppose $\{\Omega, F, P\}$ is a probability space, and $\{X_t, t\in T\}$ a stochastic process on it. When T is an interval, we can define various continuities of X_t, such as continuity in probability, a. s. continuity, continuity in r-th mean, a. s. sample continuity etc.

There are many well-kown relations among these continuities. For instance, a classical result of Kolmogorov asserts as follows. If X_t is separable, and there exists $r>0$, $\varepsilon>0$, such that

$$E|X_{t+\Delta t}-X_t|^r=O(|\Delta t|^{1+\varepsilon}),$$

then X_t is a. s. sample continuous.

Here separability is important. Without this restriction, it is easy to construct a simple example such that for all t, ω, $X_{t'}(\omega)$ has no limit as $t'\to t$.

Nevertheless, on the contrary, starting from a. s. sample continuity, or merely from continuity in probability, we can arrive at a certain conclusion provided that the r-th mean exists.

Theorem. *Suppose $\{X_t, t\in[a, b]\}$ is continuous in probability on $[a, b]$, and $E|X_t|^r<\infty$, then the points of discontinuity in r-th mean constitute a set of 1st category, and the points of continuity in r-th mean constitute a set of 2nd category.*

For a proof of this theorem, we need only to establish the following lemma.

Lemma. *Under the conditions of the theorem, the set*

$$D_\varepsilon=\{t:\limsup_{t'\to t} E|X_{t'}-X_t|^r>\varepsilon\}$$

is nowhere dense on $[a, b]$, where ε is an arbitrary positive number.

Proof Assuming the conclusion is not true, D_ε must be dense in some nondegenerate interval J_0. By definition of D_ε and so called C_r-inequality

$$E|X_{t'}-X_t|^r\leqslant C_rE|X_{t'}|^r+C_rE|X_t|^r,$$

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where
$$C_r=\begin{cases}1, & \text{for } 0<r\leqslant 1,\\ 2^{r-1}, & \text{for } r>1,\end{cases}$$

We can deduce that J_0 contains an interior point t_0 such that
$$E|X_{t_0}|^r>\frac{\varepsilon_0}{2C_r}=\delta.$$

Put
$$X_t^{(n)}=\begin{cases}X_t, & \text{for } |X_t|\leqslant n,\\ n, & \text{for } X_t>n,\\ -n, & \text{for } X_t<-n.\end{cases}$$

Choose a sufficient large n such that
$$E|X_{t_0}^{(n)}|^r>\delta.$$

Let $t\to t_0$, we have
$$X_t^{(n)}\xrightarrow{P}X_{t_0}^{(n)},$$

Consequently, according to Lebesgue's bounded convergence theorem
$$E|X_t^{(n)}|^r\to E|X_{t_0}^{(n)}|^r>\delta.$$

Thus, we can find a closed interval $J_1\subset J_0$, such that J_1 includes t_0 as an interior point and for all $t\in J_1$ we have
$$E|X_t^{(n)}|^r>\delta,$$

a fortiori
$$E|X_t|^r>\delta.$$

Now, let's assume that we have taken a nest of k nondegenerate closed intervals
$$J_1\supset J_2\supset\cdots\supset J_k,$$
such that for all $t\in J_j$, $j=1, 2, \cdots, k$, we have
$$E|X_t|^r>j\delta. \tag{1}$$

Let $\tau\in D_s$ be an interior point of J_k, and I_t be the indicator of the set
$$A_t=\{\omega:\ |X_t-X_\tau|<1\}.$$

Then, as $t\to\tau$, we have
$$I_tX_t\xrightarrow{P}X_\tau.$$

Choose a sufficientlylarge n such that
$$\int_B|X_\tau|^r\,dp>k\delta, \tag{2}$$

where
$$B=\{\omega: |X_\tau|<n\}.$$

By Lebesgue's bounded convergence theorem, as $t\to\tau$, we have
$$\int_B|I_tX_t-X_\tau|^r\,dP\to 0, \tag{3}$$

therefore
$$\int_B|I_tX_t|^r\,dP\to\int_B|X_\tau|^r\,dP>k\delta. \tag{4}$$

From (3) we can deduce that as $t\to\tau$
$$\int_{A_t\cap B}|X_t-X_\tau|^r\,dP\to 0, \tag{5}$$

and
$$\int_{B-A_t} |X_\tau|^r dP \to 0. \tag{6}$$
For $\tau \in D_\varepsilon$, (5) implies that there exist interior points τ_j of J_k such that $\tau_j \to \tau$ and
$$\int_{\Omega-(A\tau_j \cap B)} |X_{\tau_j} - X_\tau|^r dP > \varepsilon.$$
Thus, at least one of the following two inequalities
$$\int_{\Omega-(A\tau_j \cap B)} |X_{\tau_j}|^r dP > \delta, \tag{7}$$
$$\int_{\Omega-(A\tau_j \cap B)} |X_\tau|^r dP > \delta \tag{8}$$
is valid.

If there are infinitely many j which make (7) valid, then from (4) we can deduce that there exists a j such that
$$\int_{A\tau_j \cap B} |X_{\tau_j}|^r dP > k\delta.$$

Adding this inequality to (7), we obtain
$$E|X_{\tau_j}|^r > (k+1)\delta \tag{9}$$

If there are at most finitely many j which make (7) valid, then from some j on, (8) is always valid. Hence, combining (8) with (6), it gives
$$\int_{\Omega-B} |X_\tau|^r dP \geqslant \delta.$$
Adding this inequality to (2), we obtain
$$E|X_\tau|^r > (k+1)\delta. \tag{10}$$

(9) and (10) show that there must be an interior point t_k of J_k, such that
$$E|X_{t_k}|^r > (k+1)\delta. \tag{11}$$

In a similar way as in the primary deduction of the existence of J_1 from $E|X_{t_0}|^r > \delta$, we can deduce from (11) that there exists a nondegenerate closed interval $J_{k+1} \subset J_k$, such that for all $t \in J_{k+1}$, we have
$$E|X_t|^r > (k+1)\delta$$

Hence we have inductively proved that there exists a nest of nondegenerate closed intervals $J_1 \supset J_2 \supset \cdots$, such that for all $t \in J_j$, $j = 1, 2, \cdots$, we have
$$E|X_t|^r > j\delta.$$

Thus, taking $t_\infty \in \bigcap_{j=1}^{\infty} J_j$, we obtain $E|X_{t_\infty}|^r = \infty$, which contradicts the assumption. This completes the proof.

Proof of the theorem Because the set of points of discontinuity in r-th mean is $\bigcup_{n=1}^{\infty} D_{\frac{1}{n}}$, hence it is of 1st category. Accordingly, by Baire's category theorem, the set of points of continuity in r-th mean is of 2nd category. The proof is completed.

By the way, we point out that there exists such a process $\{X_t, t \in [a, b]\}$ that it is sample continuous, and possesses the finite mean, yet the set of points of discontinuity in r-th mean is dense in $[a, b]$ and has measure $q(b-a)$, where q is an arbitrary number on $[0, 1]$.

It can be constructed as follows.

Suppose $T=[a, b]$, $\Omega=[c, d)$, $P=\frac{\mu}{d-c}$, where μ is Lebesgue measure. First, consider the case $q\in[0, 1)$. Take $a_n>0$ such that

$$\sum_{n=1}^{\infty} 2^{n-1}a_n=(1-q)(b-a).$$

Imitating the construction of Cantor set, excise a concentric openinterval J_{11} of length a_1 from T, and then excise concentric open intervals J_{21}, J_{22}, each of length a_2, from the remaining two closed intervals respectively, $\cdots$. Let the centre of T be m. On every rectangle

$$J_{ni}\times\left[c+\sum_{k=1}^{n-1}\frac{m-c}{2^k},\ c+\sum_{k=1}^{n}\frac{m-c}{2^k}\right),\quad i=1, 2, \cdots, 2^{n-1}, n=1, 2, \cdots,$$

let $X(t, \omega)$ be a quadrangular pyramid based on it and of height 2^n. On the other place of $T\times[c, m)$, let $X(t, \omega)=0$. Furthermore, denote all rationals on T by r_1, r_2, $\cdots$, and on each rectangle

$$T\times\left[m+\sum_{k=1}^{n-1}\frac{m-c}{2^k},\ m+\sum_{k=1}^{n}\frac{m-c}{2^k}\right), n=1, 2, \cdots,$$

let $X(t, \omega)=f_n(t, \omega)$, where $f_n(t, \omega)$ is a continuous function defined on this rectangle and possessing the following properties

$$\int|f_n(t, \omega)|d\omega<2^{-n}, \text{ for all } t\in T,$$

$$\int|f_n(t', \omega)-f_n(t, \omega)|d\omega\to 0, \text{ as } t'\to t, \text{ for all } t\neq r_n,$$

$$\int|f_n(t', \omega)-f_n(r_n, \omega)|d\omega \text{ diverges as } t'\to r_n,$$

where the integration domains are all the same interval

$$\left[m+\sum_{k=1}^{n-1}\frac{m-c}{2^k},\ m+\sum_{k=1}^{n}\frac{m-c}{2^k}\right).$$

Now, we have definid $X(t, \omega)$on $R=T\times\Omega$. It is easy to see that $X(t, \omega)$ possesses the desired properties.

Denote the process so constructed by $X_{R,q}$. Divide R into denumerablely many rectangles

$$R_n=T\times\left[c+\sum_{k=1}^{n-1}\frac{b-a}{2^k},\ c+\sum_{k=1}^{n}\frac{b-a}{2^k}\right),\quad n=1, 2, \cdots.$$

Take positive sequence $q_n\uparrow 1$, and on each R_n define

$$X(t, \omega)=\frac{1}{2^n}X_{R_n,q_n}(t, \omega).$$

It is easy to see that this gives the desired example for the case $q=1$.