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BIVARIATE CUBIC *B*-SPLINES RELATIVE TO CROSS-CUT TRIANGULATIONS

CHARLES K. CHUI* (崔锦泰) WANG RENHONG (王仁宏)
(*Texas A&M University USA*) (*Jilin University*)

Abstract

In this paper we prove that there are no locally supported bivariate C^{k-1} spline functions of degree k on cross-cut grid partitioned regions with no more than three lines meeting at a common vertex. We also give explicit expressions of bivariate C^1 cubic B-spline with smallest local support on cross-cut triangular grid partitioned regions where each vertex is the intersection of three lines. Properly normalized, these B-splines are proved to be uniquely determined and form a partition of unity. Furthermore, the corresponding variation diminishing bivariate spline operators are proved to preserve all linear polynomials of two variables. These facts enable us to give error estimates for approximation by bivariate C^1 cubic splines for functions of class C, C^1 and C^2.

1. Introduction

Let $\mathscr{D}$ be a domain in $\mathbf{R}^2$. A line or line segment is called a cross-cut of $\mathscr{D}$ if it divides $\mathscr{D}$ into two subdomains which are called cells, such that both of its endpoints lie on the boundary $\partial\mathscr{D}$ of $\mathscr{D}$. If $\mathscr{D}$ is unbounded, then the point at infinity is also considered as a boundary point of $\mathscr{D}$. Let Δ be a grid partition of $\mathscr{D}$ that consists of a finite number (or countable number if $\mathscr{D}$ is unbounded) of cross-cuts of $\mathscr{D}$. Then Δ divides $\mathscr{D}$ into a finite number (or countable number, resp.) of cells. Δ will be called a cross-cut grid partition of $\mathscr{D}$. The points of intersection of the cross-cuts are called grid-points (or vertices) and the straight line segments separated by the grid points are called grid-segments (or edges) of the partition Δ. A cell of this partition is called an interior cell if its boundary intersects $\partial\mathscr{D}$ at no more than a finite number of points, and a grid-point is called an interior grid-point if it is the common vertex of interior cells only.

Let $\mathbf{P}_k$ be the space of all polynomials in two real variables of total degree k. A function $s(x, y)$ in $C^\mu(\mathscr{D})$, where μ is a nonnegative integer, is called a bivariate spline function of (total) degree k, belonging to the smoothness class C^μ, and having

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† This author is a visiting scholar at Texas A&M University during 1981~82.

the grid partition Δ, if the restriction of $s(x, y)$ to each cell of this partition is in $\mathbf{P}_k$. The collection of all such bivariate spline functions will be denoted by $S_k^\mu = S_k^\mu(\Delta) = S_k^\mu(\Delta, \mathscr{D})$. A bivariate spline function $s(x, y)$ in S_k^μ is said to be locally supported if it identically vanishes outside a simple closed polygonal curve which is made up of certain edges of the cells of the partition Δ, such that the interior of this supporting Jordan curve is a simply-connected subdomain of $\mathscr{D}$. Such a locally supported bivariate spline function will be called a B-spline in S_k^μ if it is strictly positive at each point inside the supporting Jordan curve.

We will first show that for $k \geqslant 2$ when no more than three cross-cuts of the partition Δ meet at a common grid-point (or vertex), then $S_k^{k-1}(\Delta, \mathscr{D})$ has no nontrivial locally supported bivariate spline functions. Hence, to look for the smoothest possible bivariate B-splines, we must work with $S_k^{k-2}(\Delta, \mathscr{D})$. We will only consider the case $k=3$ so that $\mu=k-2=1$ and the bivariate spline functions are bivariate C^1 cubic splines. The partition Δ of $\mathscr{D}$ will consist of cross-cuts that are parallel to one of the three distinct lines: $a_1x+b_1y=0$, $a_2x+b_2y=0$, and $a_3x+b_3y=0$, where each grid-point (or vertex) of Δ on $\bar{\mathscr{D}}=\mathscr{D}\cup\partial\mathscr{D}$ is the intersection of three cross-cuts of this partition. Such a partition will be called a cross-cut triangular grid partition (or cross-cut triangulation) of the first kind of $\mathscr{D}$. It is clear that each interior cell of a cross-cut triangulation of the first kind is a triangular region. The following result also shows that it is the "simplest" cross-cut triangulation.

Lemma 1.1. *Let Δ be a cross-cut grid partition of a domain $\mathscr{D}$ that consists of lines parallel to one of the three distinct lines: $a_1x+b_1y=0$, $a_2x+b_2y=0$, and $a_3x+b_3y=0$. If each interior cell of this partition is a triangular region, then each interior grid-point of Δ is the intersection of three cross-cuts.*

Hence, to obtain a cross-cut grid partition of $\mathscr{D}$ so that each interior cell is a triangular region and that at least one interior grid-point is the intersection of only two cross-cuts, we need at least four mutually non-parallel sets of parallel cross-cuts. The proof of the above lemma is trivial. Less trivial but quite elementary is the following result.

Lemma 1.2. *Let Δ be a cross-cut triangulation of the first kind of a domain $\mathscr{D}$ such that each cross-cut is parallel to one of the three distinct lines $a_1x+b_1y=0$, $a_2x+b_2y=0$, and $a_3x+b_3y=0$. Then there exist a positive number η_1 and a point (x_0, y_0) in $\mathscr{D}$ such that each cross-cut lies on one of the following lines:*

$$\begin{cases} a_1(x-x_0)+b_1(y-y_0)+j\eta_1=0, \\ a_2(x-x_0)+b_2(y-y_0)+j\eta_2=0, \\ a_3(x-x_0)+b_3(y-y_0)+j\eta_3=0, \end{cases} \tag{1.1}$$

$j=\cdots, -1, 0, 1, \cdots$, where η_2 and η_3 satisfy

$$(a_2b_3-a_3b_2)\eta_1=(a_1b_3-a_3b_1)\eta_2=(a_1b_2-a_2b_1)\eta_3. \tag{1.2}$$

Hence, any cross-cut triangulation of the first kind can be linearly transformed into a cross-cut triangulation where each interior cell is a regular triangular region by setting

$$\begin{cases} a_1(x-x_0)+b_1(y-y_0)=-\dfrac{1}{2}\eta_1 x'-\dfrac{\sqrt{3}}{6}\eta_1 y', \\ a_2(x-x_0)+b_2(y-y_0)=-\dfrac{1}{2}\eta_2 x'+\dfrac{\sqrt{3}}{6}\eta_2 y'. \end{cases} \tag{1.3}$$

This transformation is non-singular since $\eta_1\eta_2\neq 0$, and by using the relationship (1.2), it maps the lines in (1.1) onto the lines

$$\begin{cases} \sqrt{3}\,x'+y'-2\sqrt{3}\,j=0, \\ \sqrt{3}\,x'-y'-2\sqrt{3}\,j=0, \\ y'+\sqrt{3}\,j=0, \end{cases} \tag{1.4}$$

$j=\cdots, -1, 0, 1, \cdots$ consecutively. The grid partition Δ' consisting of the lines (1.4) divides $\mathbf{R}^2$ into regular triangular cells. We will construct and study the approximation properties of bivariate C^1 cubic B-splines in $S_3^1(\Delta', \mathbf{R}^2)$. By the linear polynomial transformation (1.3), we can obtain bivariate C^1 cubic B-spline functions on any cross-cut triangulation of the first kind. The corresponding approximation properties can also be obtained.

Let $A_i=(\alpha_i, \beta_i)$ be a grid point (or vertex) of Δ'. We will show that there is a unique bivariate B-spline $B_i(x, y)$ in $S_3^1(\Delta')$ whose support is a regular hexagonal region centered at A_i and consisting of 24 regular triangular cells of Δ' such that $B_i(\alpha_i, \beta_i)=1/3$ and that $B_i(x, y)$ is symmetric about all the three diagonals of the hexagonal support. This hexagonal support is the smallest, and the collection of all these bivariate B-splines $B_i(x, y)$, where A_i runs over all grid-points of Δ', will be shown to form a partition of unity. In fact, if we define the variation diminishing bivariate spline operator V by $(Vf)(x, y)=\Sigma_i f(\alpha_i, \beta_i)B_i(x, y)$, then we will see that V preserves all linear polynomials namely: $V_p=p$ for all $p\in\mathbf{P}_1$. By using this important property we are able to obtain efficient bivariate C^1 cubic spline approximations to functions in C, C^1, and C^2 that give optimal orders of approximation for C and C^1 and order $O(\delta^2)$ for C^2 where δ is the mesh size of the grid partition. In particular, the closure in the topology of uniform convergence on compact subsets of a domain $\mathscr{D}$ of the union of $S_3^1(\bar{\Delta}, \mathscr{D})$, where $\bar{\Delta}$ runs over all cross-cut triangulations of the first kind of $\mathscr{D}$ with all cross-cuts parallel to one of the lines (1.4), is all of $C(\mathscr{D})$. This result is somewhat surprising since when simple cross-cut partitions are considered, the corresponding closure in the approximation from S_k^μ is all of $C(\mathscr{D})$ if and only if $\mu\leqslant(k-2)/2$. Note that if $k=3$ then μ must be zero and that cross-cut triangulations of the first kind are limits of simple cross-cut grid partitions. This result on simple cross-cut grid partitions was announced in [2] and proved in [3].

It seems appropriate to state a main result there for comparison. To be more precise, we need the following notation.

A cross-cut grid partition of $\mathscr{D}$ is said to be simple if no more than two cross-cuts meet at a grid-point in $\mathscr{D}$. Let $(a_1, b_1), \cdots, (a_N, b_N)$ be pairwise linearly independent ordered pairs, $\underset{\sim}{c}=[c_{i\rho}]$ a matrix of numbers and $\Gamma_{i\rho}$: $a_ix+b_iy+c_{i\rho}=0$ be a collection of lines. Let $\Delta_N=\Delta_N(\underset{\sim}{c})$ be a grid partition of $\mathscr{D}$ consisting of simple cross-cuts $l_{ij}=l_{i\rho j}$, $j=1, \cdots, t_i$ and $i=1, \cdots, N$, where each $l_{i\rho j}$ is a segment of the line $\Gamma_{i\rho}$. Note that if $\mathscr{D}$ is not convex, then there may be more than one $l_{i\rho j}$ lying on $\Gamma_{i\rho}$. We define the index set $\Omega(h, r)=\{(m, s): l_{hm}$ and l_{rs} have a common vertex in $\mathscr{D}\}$. Fix a point (x_0, y_0) in $\mathscr{D}\backslash\Delta_N$. Each cross-cut $l_{i\rho j}$ divides $\mathscr{D}$ into two cells. The one not containing (x_0, y_0) is denoted by $\mathscr{D}_{i\rho j}$ and the other by $D'_{i\rho j}$. In [2, 3], we introduced the function $(l_{ij})_\#$ defined by

$$(l_{ij})_\#(x, y)=(l_{i\rho j})_\#(x, y)=\begin{cases} a_ix+b_iy+c_{i\rho} & \text{if}(x, y)\in D_{i\rho j}, \\ 0 & \text{if}(x, y)\in D'_{i\rho j}\cup l_{i\rho j}, \end{cases}$$

and let $(l_{ij})_\#^{\mu+1}(x, y)=[(l_{ij})_\#(x, y)]^{\mu+1}$. The following result was proved in [3].

Theorem 1.1. *Let $\mathscr{D}$ be a simply-connected domain in $\mathbf{R}^2$, $N\geqslant 2$ and $0\leqslant\mu\leqslant k-1$. Then the linearly independent set*

$$B=\{x^ay^b,\ x^cy^d(l_{ij})_\#^{\mu+1},\ x^uy^v(l_{hm})_\#^{\mu+1}(l_{rs})_\#^{\mu+1}:$$
$$0\leqslant a+b\leqslant k,\ 0\leqslant c+d\leqslant k-\mu-1,\ 0\leqslant u+v\leqslant k-2\mu-2,$$
$$h\neq r \text{ and } (m, s)\in\Omega(h, r)\}$$

is a basis of $S_k^\mu(\Delta_N, \mathscr{D})$.

Here, if $k<2\mu+2$, then the functions $x^uy^v(l_{hm})_\#^{\mu+1}(l_{rs})_\#^{\mu+1}$ are to be deleted. Hence, it follows that the closure in the topology of uniform convergence on compact subsets of $\mathscr{D}$ of the union of $S_k^\mu(\Delta_N, \mathscr{D})$ over all c is all of $C(\mathscr{D})$ if and only if $\mu\leqslant(k-2)/2$. If $\mu>(k-2)/2$, then this closure consists of all functions of the form $f(x, y)=P_k(x, y)+q_1(x, y)f_1(a_1x+b_1y)+\cdots+q_N(x, y)f_N(a_Nx+b_Ny)$, where $P_k\in\mathbf{P}_k$ and $q_1, \cdots, q_N\in\mathbf{P}_{k-\mu-1}$. In this paper, we prove that when the cross-cuts are not simple, then the closures may be larger although non-simple cross-cuts are limits of simple ones.

We note, in addition, that although $\mathscr{B}$ is a basis of S_k^μ, none of the functions in $\mathscr{B}$ are locally supported. It is therefore, quite tempting to construct locally supported ones, or even $\mathscr{B}$-splines, by taking linear combinations of the basis elements in $\mathscr{B}$. For rectangular grid partitions, this has been done in [3]. However, it has also been proved in [3] that when simple cross-cuts are considered, there are no locally supported spline functions if $\mu>(k-2)/2$. Hence, to obtain fairly smooth $\mathscr{B}$-splines, we must allow non-simple cross-cut partitions.

The notion of multivariate $\mathscr{B}$-splines was introduced by C. de Boor[1] and studied in detail by C. A. Micchelli (cf. [5]). Their $\mathscr{B}$-splines are determined by a given set of "knots" instead of grid-segments. In fact, the "knots" determine certain simplices which in turn give the grid-segments that separate the polynomial pieces. Hence, in general, several grid-segments share a common grid-point. W. Dahmen[4] also provided truncated power representations of these B-splines

2. Smooth B-splines

We first discuss the basic property, which will be called the conformality condition, that a bivariate spline function must satisfy. Let $\mathscr{D}$ be a domain in $\mathbf{R}^2$ and Δ a grid partition of $\mathscr{D}$ consisting of algebraic curves (or segments of algebraic curves), and let A be a grid-point (or vertex) of this grid partition. Also, let $\Gamma_1, \cdots, \Gamma_N$ be the grid-segments (or edges) with A as a common end-point ordered in the counter-clockwise direction, such that Γ_1 separates a cell D_2 from a cell D_1, Γ_2 separates a cell D_3 from D_2, $\cdots$, and Γ_N separates the first cell D_1 from a cell D_N. Also, let $l_1(x, y), \cdots, l_N(x, y)$ be irreducible algebraic polynomials such that $\Gamma_1{:}l_1(x, y)=0, \cdots, \Gamma_N{:}l_N(x, y)=0$. Then, if $s(x, y)$ is a bivariate spline function in $S_k^\mu(\Delta, \mathscr{D})$, where $0\leqslant\mu\leqslant k-1$ and $P_1(x, y), \cdots, P_N(x, y)$ are polynomials in $\mathbf{P}_k$ which are restrictions of $s(x, y)$ on $D_1, \cdots, D_N$ respectively, it has been proved in [3] and [6] that $P_2-P_1=q_{1,2}(l_1)^{\mu+1}, \cdots, p_N-p_{N-1}=q_{N-1,N}(l_{N-1})^{\mu+1}$, and $p_1-p_N=q_{N,1}(l_N)^{\mu+1}$ for some polynomials $q_{1,1}, \cdots, q_{N-1,N}$ and $q_{N,1}$ in $\mathbf{P}_{k-\mu-1}$. These polynomials $q_{1,2}, \cdots, q_{N-1,N}, q_{N,1}$ are called the smoothing cofactors of $s(x, y)$ across the grid-segments $\Gamma_1, \cdots, \Gamma_{N-1}, \Gamma_N$ respectively. It follows that these smoothing cofactors satisfy the identity

$$\sum_{i=1}^{N} q_{i,i+1}(l_i)^{\mu+1}=0 \tag{2.1}$$

(where $q_{N,N+1}:=q_{N,1}$). This identity will be called the conformality condition of the bivariate spline function $s(x, y)$ at the grid-point A (cf. [3, 6]). Every bivariate spline function in $S_k^\mu(\Delta, \mathscr{D})$ must satisfy the conformality conditions at all grid-points in $\mathscr{D}$ (cf. [6]). Conversely, from the conformality conditions, one can find the smoothing cofactors of the bivariate spline functions. The following result also follows by using the conformality conditions.

Theorem 2.1. *Let Δ be a cross-cut grid partition of a domain $\mathscr{D}$ in $\mathbf{R}^2$ such that no more than three cross-cuts meet at a common grid-point in $\mathscr{D}$. Then for $k\geqslant 2$, $S_k^{k-1}(\Delta, \mathscr{D})$ does not contain any non-trivial locally supported bivariate spline function.*

It is clear that this theorem does not hold for $k=1$. Let $k\geqslant 2$, and suppose that there is a non-trivial locally supported bivariate spline function $s(x, y)$ in $S_k^{k-1}(\Delta, \mathscr{D})$. Then there exists a grid-point $A=(x_0, y_0)$ which is on the simple closed polygonal curve that defines the local support of $s(x, y)$ such that at worst three smoothing cofactors of $s(x, y)$ at A are not all identically zero. Since $\mu=k-1$, these smoothing cofactors are constants, say, d_1, d_2 and d_3, and we have assumed that $d_1^2+d_2^2+d_3^2>0$. Let Γ_1, Γ_2 and Γ_3 be the corresponding grid-segments that share the common grid-point A. Then we may represent the lines containing these grid-segments by $a_1(x-x_0)+b_1(y-y_0)=0$, $a_2(x-x_0)+b_2(y-y_0)=0$ and $a_3(x-x_0)+b_3(y-y_0)=0$ respectively, where (a_1, b_1), (a_2, b_2) and (a_3, b_3) are pairwise linearly independent ordered pairs. By the conformality condition (2.1) at A, we have the identity

$$d_1[a_1(x-x_0)+b_1(y-y_0)]^k+d_2[a_2(x-x_0)+b_2(y-y_0)]^k +d_3[a_3(x-x_0)+b_3(y-y_0)]^k=0. \tag{2.2}$$

By equating the coefficients of $(x-x_0)^i(y-y_0)^{k-i}$ in (2.2), we obtain the linear system

$$\sum_{j=1}^{3} a_j^i b_j^{k-i} d_j=0, \quad i=0, \cdots, k. \tag{2.3}$$

Assume, for the time being, that $a_1, a_2, a_3\neq 0$. Then by (2.3) for $i=k-2$, $k-1$ and k, the determinant of the coefficient matrix is

$$a_1^k a_2^k a_3^k\left(\frac{b_3}{a_3}-\frac{b_2}{a_2}\right)\left(\frac{b_3}{a_3}-\frac{b_1}{a_1}\right)\left(\frac{b_2}{a_2}-\frac{b_1}{a_1}\right)$$
$$=a_1^{k-2}a_2^{k-2}a_3^{k-2}(a_2b_3-a_3b_2)(a_1b_3-a_3b_1)(a_1b_2-a_2b_1)$$

which is nonzero since (a_1, b_1), (a_2, b_2) and (a_3, b_3) are pairwise linearly independent. If $b_1, b_2, b_3\neq 0$, the determinant of the coefficient matrix (2.3) for $i=0, 1, 2$ is again nonzero by the same argument. Hence, we may assume that at least one of the a_1, a_2, a_3 and one of the b_1, b_2, b_3 are zero. Without loss of generality, let $a_1=0$ and $b_2=0$. Then $a_2, a_3, b_1, b_3\neq 0$ since the three ordered pairs are pairwise linearly independent. But by using (2.3) for $i=0$, $k-1$, and k we have $a_2^k d_2+a_3^k d_3=0$, $b_1^k d_1+b_3^k d_3=0$, and $a_3^{k-1}b_3d_3=0$. Hence $d_1=d_2=d_3=0$, which is again a contradiction. Note that in the above proof, we need $k\geqslant 2$.

One of the most important grid partitions of a domain $\mathscr{D}$ in applications is a cross-cut triangulation of the first kind which we denote by Δ. Here, every interior cell of Δ is a triangular region, each grid-segment (or edge) lies on a cross-cut, and each vertex of Δ is a point of intersection of exactly three cross-cuts. By the above theorem, we know that $S_k^{k-1}(\Delta, \mathscr{D})$ does not contain any non-trivial locally supported bivariate spline function. We will now construct a bivariate B-spline in $S_3^1(\Delta, \mathscr{D})$. This spline function is unique when it is properly normalized and its support which consists of 24 triangular regions is the smallest. By the transformation (1.3) it is sufficient to consider a cross-cut grid-partition Δ' which is determined by the lines (1.4), namely:

$$\begin{cases} l_{1,j}: \sqrt{3}\,x+y-2\sqrt{3}\,j=0, \\ l_{2,j}: \sqrt{3}\,x-y-2\sqrt{3}\,j=0, \\ l_{3,j}: y+\sqrt{3}\,j=0, \quad j=\cdots, -1, 0, 1, \cdots. \end{cases} \tag{2.4}$$

These lines divide $\mathbf{R}^2$ into regular triangular cells. We will construct a bivariate B-spline in $S_3^1(\Delta', \mathbf{R}^2)$ whose support is centered at the origin O such that it is symmetric about the lines $l_{1,0}$, $l_{2,0}$ and $l_{3,0}$. The other bivariate B-spliens in $S_3^1(\Delta', \mathbf{R}^2)$ are simply translations of this one. We need the following notation.

Let $A_1=(-2, 2\sqrt{3})$, $A_2=(0, 2\sqrt{3})$, $A_3=(2, 2\sqrt{3})$, $A_4=(-3, \sqrt{3})$, $A_5=(-1, \sqrt{3})$, $A_6=(1, \sqrt{3})$, $A_7=(3, \sqrt{3})$, $A_8=(-4, 0)$, $A_9=(-2, 0)$, $A_{10}=(0, 0)$, $A_{11}=(2, 0)$, $A_{12}=(4, 0)$, $A_{13}=(-3, -\sqrt{3})$, $A_{14}=(-1, -\sqrt{3})$, $A_{15}=(1, -\sqrt{3})$, $A_{16}=(3, -\sqrt{3})$, $A_{17}=(-2, -2\sqrt{3})$, $A_{18}=(0, -2\sqrt{3})$, and $A_{19}=(2, -2\sqrt{3})$.

A triangular cell with vertices at A_i, A_j, A_k will be denoted by $[i, j, k]:=[A_i, A_j, A_k]$. Hence, the vertices $A_1, \cdots, A_{19}$ determine 24 regular triangular cells $D_1, \cdots, D_{24}$, namely:

$D_1=[1, 4, 5]$, $D_2=[1, 5, 2]$, $D_3=[2, 5, 6]$, $D_4=[2, 6, 3]$, $D_5=[3, 6, 7]$, $D_6=[4, 8, 9]$, $D_7=[4, 9, 5]$, $D_8=[5, 9, 10]$, $D_9=[5, 10, 6]$, $D_{10}=[6, 10, 11]$, $D_{11}=[6, 11, 7]$, $D_{12}=[7, 11, 12]$, $D_{13}=[8, 13, 9]$, $D_{14}=[9, 13, 14]$, $D_{15}=[9, 14, 10]$, $D_{16}=[10, 14, 15]$, $D_{17}=[10, 15, 11]$, $D_{18}=[11, 15, 16]$, $D_{19}=[11, 16, 12]$, $D_{20}=[13, 17, 14]$, $D_{21}=[14, 17, 18]$, $D_{22}=[14, 18, 15]$, $D_{23}=[15, 18, 19]$, and $D_{24}=[15, 19, 16]$.

Let $K=\operatorname{clos}(D_1\cup\cdots\cup D_{24})$ be the closure of the union of these cells. Then the boundary ∂K of K is a simple closed hexagonal curve. We will construct a bivariate B-spline in $S_3^1(\Delta')$ supported on K It is easy to see that there is no non-trivial locally supported bivariate spline in $S_3^1(\Delta')$ supported on

$$K_0=\operatorname{clos}(D_8\cup D_9\cup D_{10}\cup D_{15}\cup D_{16}\cup D_{17}). \tag{2.5}$$

Hence, K is the smallest compact support that is symmetric with respect to $l_{1,0}$, $l_{2,0}$ and $l_{3,0}$. We have the following result.

Theorem 2.2. *There is a unique bivariate spline function $B(x, y)$ in $S_3^1(\Delta')$ which is symmetric with respect to $l_{1,0}$, $l_{2,0}$ and $l_{3,0}$, such that $B(0, 0)=1/3$ and that $B(x, y)=0$ for all (x, y) outside K. This bivariate spline $B(x, y)$ is strictly positive in the interior of ∂K and its restriction on $K_0\cup\operatorname{clos}(D_1\cup D_2\cup D_6\cup D_{13}\cup D_{20}\cup D_{21}\cup D_{23}\cup D_{24}\cup D_{19}\cup D_{12}\cup D_5\cup D_4)$ is of smoothness class C^2. Let $P_i(x, y)\in\mathbf{P}_3$ denote the restriction of $B(x, y)$ on D_i, $i=1, \cdots, 24$. Then*

$$P_{10}(x, y)=\left(\frac{1}{3}-\frac{1}{12}x^2+\frac{1}{72}x^3\right)+\left(-\frac{1}{12}+\frac{1}{72}x\right)y^2+\frac{\sqrt{3}}{162}y^3,$$

$$P_{11}(x, y)=\left(\frac{5}{9}-\frac{1}{4}x+\frac{1}{144}x^3\right)+\left(-\frac{\sqrt{3}}{12}+\frac{\sqrt{3}}{18}x-\frac{\sqrt{3}}{144}x^2\right)y$$
$$+\left(-\frac{1}{18}+\frac{1}{144}x\right)y^2+\frac{7\sqrt{3}}{1296}y^3,$$

$$P_{12}(x,y)=\left(\frac{8}{9}-\frac{2}{3}x+\frac{1}{6}x^2-\frac{1}{72}x^3\right)+\left(-\frac{1}{18}+\frac{1}{72}x\right)y^2+\frac{\sqrt{3}}{324}y^3,$$

$$P_5(x,y)=\left(\frac{8}{9}-\frac{1}{3}x+\frac{1}{144}x^3\right)+\left(-\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{9}x-\frac{\sqrt{3}}{144}x^2\right)y$$
$$+\left(\frac{1}{9}-\frac{1}{48}x\right)y^2-\frac{5\sqrt{3}}{1296}y^3,$$

$$P_4(x,y)=\left(\frac{8}{9}-\frac{1}{3}x\right)+\left(-\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{9}x\right)y+\left(\frac{1}{9}-\frac{1}{36}x\right)y^2-\frac{\sqrt{3}}{324}y^3,$$

$$P_3(x,y)=\left(\frac{5}{9}-\frac{1}{12}x^2\right)+\left(-\frac{\sqrt{3}}{6}+\frac{\sqrt{3}}{72}x^2\right)y+\frac{1}{36}y^2+\frac{\sqrt{3}}{648}y^3,$$

$$P_9(x,y)=\left(\frac{1}{3}-\frac{1}{12}x^2\right)+\frac{\sqrt{3}}{72}x^2y-\frac{1}{12}y^2+\frac{5\sqrt{3}}{648}y^3,$$

$P_1(x,y)=P_5(-x,y)$, $P_2(x,y)=P_4(-x,y)$, $P_6(x,y)=P_{12}(-x,y)$, $P_7(x,y)=P_{11}(-x,y)$, $P_8(x,y)=P_{10}(-x,y)$, $P_{13}(x,y)=P_{12}(-x,-y)$, $P_{14}(x,y)=P_{11}(-x,-y)$, $P_{15}(x,y)=P_{10}(-x,-y)$, $P_{16}(x,y)=P_9(x,-y)$, $P_{17}(x,y)=P_{10}(x,-y)$, $P_{18}(x,y)=P_{11}(x,-y)$, $P_{19}(x,y)=P_{12}(x,-y)$, $P_{20}(x,y)=P_5(-x,-y)$, $P_{21}(x,y)=P_4(-x,-y)$, $P_{22}(x,y)=P_3(x,-y)$, $P_{23}(x,y)=P_4(x,-y)$, *and* $P_{24}(x,y)=P_5(x,-y)$.

To prove this theorem, we first construct the polynomial $P_{10}(x,y)$. This polynomial is uniquely determined by the ten interpolation conditions:

$$P_{10}(A_{10})=\frac{2}{9}(1+h),\quad P_{10}(A_6)=P_{10}(A_{11})=\frac{2}{9}h,$$

$$P_{10}((A_{10}+A_6+A_{11})/3)=\frac{2}{9}(t+h),\quad \frac{\partial}{\partial x}P_{10}(A_{10})=\frac{\partial}{\partial y}P_{10}(A_{10})=0,$$

the tangential derivatives of $P_{10}(x,y)$ along $l_{1,1}$ from A_{11} to A_6 at A_{11} and from A_6 to A_{11} at A_6 are equal to v, and the outer normal derivatives of $P_{10}(x,y)$ with respect to the line $l_{1,1}$ at A_6 and A_{11} are equal to u. Here, h, t, u and v are parameters. Since $B(x,y)$ is symmetric with respect to $l_{3,0}$, $P_{17}(x,y)=P_{10}(x,-y)$ and since

$$P_{17}(x,y)-P_{10}(x,y)=C_{10,17}(x,y)\,[l_{3,0}(x,y)]^2,$$

where $C_{10,17}(x,y)\in\mathbf{P}_1$ is the smoothing cofactor of $B(x,y)$ across the grid-segment $\overline{A_{10}A_{11}}$, it can be shown that

$$t=\frac{30-16\sqrt{3}u}{81}\ \text{and}\ v=-\frac{\sqrt{3}}{3}u.$$

Next, we use the conformality conditions of $B(x,y)$ at the grid-points A_{11} and A_{12} simultaneously. These algebraic equations determine h and u uniquely, namely: $h=1/2$ and $u=-3\sqrt{3}/8$. Hence all the parameters of $P_{10}(x,y)$ are determined. The polynomials $P_{11}(x,y)$, $P_{12}(x,y)$, $P_5(x,y)$, $P_4(x,y)$, $P_3(x,y)$ and $P_9(x,y)$ are determined (uniquely) by first finding the smoothing cofactors. They are:

$$C_{10,11}(x, y) = -\frac{\sqrt{3}}{1296}(\sqrt{3}x+y-8\sqrt{3}),$$

$$C_{11,12}(x, y) = -\frac{\sqrt{3}}{432}(\sqrt{3}x+y-4\sqrt{3}),$$

$$C_{11,5}(x, y) = -\frac{\sqrt{3}}{108}(\sqrt{3}x+y-4\sqrt{3}),$$

$$C_{5,4}(x, y) = -\frac{\sqrt{3}}{1296}(\sqrt{3}x-y),$$

$$C_{4,3}(x, y) = \frac{\sqrt{3}}{216}(y-2\sqrt{3}), \text{ and}$$

$$C_{3,9}(x, y) = \frac{\sqrt{3}}{162}(y-4\sqrt{3}).$$

The other polynomial pieces are obtained by symmetry. In the above construction of $B(x,y)$, we have set $B(x,y)=0$ for all (x,y) outside K. It is clear that $B(x,y)>0$ in the interior of ∂K, and the smoothness conditions can easily be verified since $B(x, y)$ is obtained from $P_{10}(x, , y)$ by using the smoothing cofactors.

3. Approximation properties of the B-splines

Let $l_{i,j}$, $i=1, 2, 3$, $j=\cdots, -1, 0, 1, \cdots$, be the lines in (2.4) that define a cross-cut triangulation of the first kind Δ' of $\mathbf{R}^2$, and $B(x, y)$ be the bivariate C^1 cubic B-spline function obtained in Theorem 2.2. If $G_i=(x_i, y_i)$ is a grid-point of the partition Δ', we define

$$B_i(x, y) = B(x-x_i, y-y_i).$$

Here, we need $(x_0, y_0)=(0, 0)$ and note that $B_0(x, y)=B(x, y)$. Each $B_i(x, y)$ is a bivariate B-spline function in $S_3^1(\Delta')$ whose support is enclosed by the simple closed (regular) hexagonal curve joining the points (x_i+4, y_i), $(x_i+2, y_i+2\sqrt{3})$, $(x_i-2, y_i+2\sqrt{3})$, (x_i-4, y_i), $(x_i-2, y_i-2\sqrt{3})$, $(x_i+2, y_i-2\sqrt{3})$ and (x_i+4, y_i) consecutively. The interior of this curve will be denoted by H_i and its closure by $\overline{H}_i$. Let $\mathscr{D}$ be a domain in $\mathbf{R}^2$ and $\Omega=\Omega(\mathscr{D})=\{i\colon H_i\cap\mathscr{D}\neq\emptyset\}$. We will now show that the bivariate B-splines $B_i(x, y)$ form a partition of unity, as in the following:

Theorem 3.1. *For all* $(x, y)\in\mathscr{D}$

$$\sum_{i\in\Omega(\mathscr{D})} B_i(x, y)=1. \tag{3.1}$$

Since $B_i(x, y)>0$ for all $(x, y)\in H_i$ and $B_i(x, y)=0$ for all $(x, y)\in\mathscr{D}\backslash H_i$, the identity (3.1) implies that $\{B_i\colon i\in\Omega(\mathscr{D})\}$ is a partition of unity for $\mathscr{D}$. To prove (3.1), let

$$F(x, y) = \sum_{i\in\Omega(\mathscr{D})} B_i(x, y).$$

We first note that $B(A_{10})=1/3$, $B(A_6)=B(A_{11})=1/9$, $B(A_2)=B(A_3)=B(A_7)=B(A_{12})=0$, $B((A_2+A_5+A_6)/3)=B((A_6+A_{11}+A_7)/3)=17/243$, $B((A_2+A_6+A_3)/3)=B((A_3+A_6+A_7)/3)=B((A_7+A_{11}+A_{12})/3)=5/486$, and $B((A_5+A_{10}+A_6)/3)=B((A_6+A_{10}+A_{11})/3)=59/243$. The rest of the values of $B(x,y)$ at the grid-points and mid-points of the cells can be determined from these values by symmetry. Consider any cell D of the grid partition Δ' such that $D\cap\mathscr{D}\neq\emptyset$. It is clear that $\Omega(D)$ has cardinality 12. The value of $F(x, y)$ at each of the three vertices of D is

$$5(0)+1\left(\frac{1}{3}\right)+6\left(\frac{1}{9}\right)=1,$$

and is

$$3\left(\frac{17}{243}\right)+3\left(\frac{59}{243}\right)+6\left(\frac{5}{486}\right)=1$$

at the mid-point of D. Also, it is easy to verify that $\partial F/\partial x$ and $\partial F/\partial y$ are both equal to O at the three vertices. Hence, $F(x, y)=1$ for all $(x, y)\in D$. Since D is any cell and $F(x, y)$ is continuous, this completes the proof of the theorem.

The above result can be extended. To do this, we define the operator

$$V: C(\mathscr{D})\to S_3^1(\Delta', \mathscr{D})$$

by

$$(Vf)(x, y)=\sum_{i\in\Omega(\mathscr{D})} f(x_i, y_i)B_i(x, y) \tag{3.2}$$

for all $f\in C(\mathscr{D})$. This is a generalization of the variation diminishing spline operator in univariate spline function theory. Hence, we will also call it a variation diminishing bivariate cubic spline operator with respect to the grid partition Δ'. We have the following result.

Theorem 3.2. *$Vf=f$ for all $f\in \mathbf{P}_1$.*

To prove this result, we first note that V is a (positive) linear operator on $C(\mathscr{D})$. Hence, by a linear transformation and using Theorem 3.1, it is sufficient to prove that $(Vf)(x, y)=f(x, y)$ for $f(x, y)=x$ and $f(x, y)=y$ and $(x, y)\in K_0$, where K_0 is given in (2.5). By symmetry it is sufficient to prove the identity on D_{10} and for $f(x, y)=x$. Let $f(x, y)=x$. By using the computation analogous to the proof of Theorem 3.1, we see that Vf and f agree at the vertices and the mid-point of the cell D_{10}. It can also be verified that $\partial(Vf)/\partial x=1$ and $\partial(Vf)/\partial y=0$ at the three vertices of D_{10}. Hence, $Vf=f$ on D_{10}.

Next, we will apply this result to study the approximation properties of the variation diminishing bivariate spline operator V. of course, we have to refine the grid partition Δ'. To do this, let $\delta>0$ and consider the cross-cut triangulation Δ'_δ of $\mathbf{R}^2$ consisting of lines $L_{i,j}=L_{i,j}(\delta)$ defined by:

$$\begin{cases} L_{1,j}\colon \sqrt{3}\,x+y-\delta\sqrt{3}\,j=0, \\ L_{2,j}\colon \sqrt{3}\,x-y-\delta\sqrt{3}\,j=0, \quad j=\cdots, -1, 0, 1, \cdots. \\ L_{3,j}\colon y+\delta\sqrt{3}\,j/2=0, \end{cases} \tag{3.3}$$

This grid partition divides $\mathbf{R}^2$ into regular triangular cells where the length of each of the three sides of the triangle is δ. If $B_i(x, y)$ is the bivariate cubic B-spline function in $S_3^1(\Delta')$ defined above, we define our (normalized) bivariate cubic B-splines $N_{i,\delta}(x, y)$ in $S_3^1(\Delta'_\delta)$ by

$$N_{i,\delta}(x, y) = B_i(2x/\delta, 2y/\delta). \tag{3.4}$$

Hence, $N_{i,\delta}(x, y)$ has support on a regular hexagonal region $H_{i,\delta}$ consisting of 24 regular triangular cells of Δ'_δ and centered at $G_{i,\delta} := (\delta x_i/2, \delta y_i/2)$, where $G_i = (x_i, y_i)$ are the grid-points of Δ'. Again, $N_{i,\delta}(x, y) > 0$ for all $(x, y) \in H_{i,\delta}$ and $N_{i,\delta}(x, y) = 0$ for $(x, y) \in \mathbf{R}^2 \backslash H_{i,\delta}$. Let F be a compact set, $\mathscr{D}$ a bounded domain containing F, and

$$\Omega_\delta = \Omega_\delta(F) = \{i : H_{i,\delta} \cap F \neq \emptyset\}.$$

Hence, there is a $\delta_0 > 0$ such that $G_{i,\delta} \in \mathscr{D}$ whenever $i \in \Omega_\delta$ and $0 < \delta \leqslant \delta_0$. Also, let $C(\overline{\mathscr{D}})$ be the Banach space of functions continuous on $\overline{\mathscr{D}}$ with the supremum norm $\|\cdot\|_{\overline{\mathscr{D}}}$, and let $C^n(\overline{\mathscr{D}})$ be those functions in $C(\overline{\mathscr{D}})$ which are n-times continuously differentiable on $\overline{\mathscr{D}}$ relative to $\overline{\mathscr{D}}$. If $f \in C^1(\overline{\mathscr{D}})$, then for each (x_0, y_0) in $\overline{\mathscr{D}}$, the derivative of f at (x_0, y_0), denoted by $Df(x_0, y_0)$, is the linear functional on $\mathbf{R}^2$ defined by

$$Df(x_0, y_0)(x, y) = f_1(x_0, y_0)x + f_2(x_0, y_0)y.$$

Here and in the remainder of the paper, we use the standard notions:

$$f_1 = \frac{\partial f}{\partial x}, \quad f_2 = \frac{\partial f}{\partial y}, \quad f_{11} = \frac{\partial^2 f}{\partial x^2},$$

$$f_{12} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{21} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{22} = \frac{\partial^2 f}{\partial y^2},$$

where $f = f(x, y)$. If $f \in C^2(\overline{\mathscr{D}})$, then its second derivative $D^2 f(x_0, y_0)$ at a point $(x_0, y_0) \in \overline{\mathscr{D}}$ is the linear transformation from $\mathbf{R}^2 \times \mathbf{R}^2$ to $\mathbf{R}$ defined by

$$D^2 f(x_0, y_0)((x, y), (u, v)) = f_{11}(x_0, y_0)xu + f_{12}(x_0, y_0)xv + f_{21}(x_0, y_0)yu + f_{22}(x_0, y_0)yv.$$

Also, denote $\quad \|D^2 f\| = \max\{\|D^2 f(x, y)\| : (x, y) \in \overline{\mathscr{D}}\}$,

where $\|D^2 f(x, y)\|$ is the norm of the linear transformation $D^2 f(x, y)$.

Let V_δ, $\delta > 0$, be the variation diminishing bivariate cubic spline operator defined by

$$(V_\delta f)(x, y) = \sum_{i \in \Omega_\delta} f(x_i, y_i) N_{i,\delta}(x, y). \tag{3.5}$$

Hence, by Theorem 3.2, we have, on F

$$V_\delta f = f, \ f \in \mathbf{P}_1. \tag{3.6}$$

We next discuss the error analysis for the approximation of $f \in C(\overline{\mathscr{D}})$ by $Vf \in S_3^1(\Delta'_\delta, \mathscr{D})$. We use the standard notation

$$\omega(f, \delta) = \max_{(x,y),(u,v) \in \overline{\mathscr{D}}} \{|f(x, y) - f(u, v)| : \|(x, y) - (u, v\| \leqslant \delta\}. \tag{3.7}$$

The following result is obtained.

Theorem 3.3. *Let $f\in C(\overline{\mathscr{D}})$. Then for* $0<\delta\leqslant\delta_0$

$$\|f-V_\delta f\|_F\leqslant\omega(f,\ 2\delta). \tag{3.8}$$

If, in addition, $f\in C^1(\overline{\mathscr{D}})$, *then*

$$\|f-V_\delta f\|_F\leqslant\frac{3+\sqrt{3}}{3}\,\delta\cdot\max\left[\omega\left(f_1,\ \frac{\sqrt{3}\,\delta}{3}\right),\ \omega\left(f_2,\ \frac{\sqrt{3}\,\delta}{3}\right)\right], \tag{3.9}$$

and if $f\in C^2(\overline{\mathscr{D}})$, *then*

$$\|f-V_\delta f\|_F\leqslant\frac{1}{3}\,\delta^2\|D^2f\|. \tag{3.10}$$

The proof of (3.8) is trivial by using the property of partition of unity of $\{N_{i,\delta}\}$, namely

$$\begin{aligned}|(f-V_\delta f)(x,\ y)|&=\Big|\sum_{i\in\Omega_\delta}(f(x,\ y)-f(x_i,\ y_i))N_{i,\delta}(x,\ y)\Big|\\&\leqslant\sum_{i\in\Omega_\delta}|f(x,\ y)-f(x_i,\ y_i)|N_{i,\delta}(x,\ y)|,\end{aligned}$$

since $N_{i,\delta}(x,\ y)\geqslant0$. By using the support property of $N_{i,\delta}$ and that $\sum_{i\in\Omega_\delta}N_{i,\delta}\equiv1$, we have the required estimate (3.8). Note that 2δ is the radius of $H_{i,\delta}$. Next let $f\in C^1(\overline{\mathscr{D}})$ and let $D_{\delta,i}$ be a cell of Δ'_δ such that

$$\|f-V_\delta f\|_F=\|f-V_\delta f\|_{D_{\delta,i}\cap F}. \tag{3.11}$$

Also, let $(\bar{x}_0,\ \bar{y}_0)$ be the center of $D_{\delta,i}$. Then by the Mean Value Theorem, we have

$$f(x,\ y)=f(\bar{x}_0,\ \bar{y}_0)+f_1(u,\ v)(x-\bar{x}_0)+f_2(u,\ v)(y-\bar{y}_0)$$

for some $(u,\ v)=t(x,\ y)+(1-t)(\bar{x}_0,\ \bar{y}_0)$, where $0\leqslant t\leqslant1$. Let

$$p(x,\ y)=f(\bar{x}_0,\ \bar{y}_0)+f_1(\bar{x}_0,\ \bar{y}_0)(x-\bar{x}_0)+f_2(\bar{x}_0,\ \bar{y}_0)(y-\bar{y}_0). \tag{3.12}$$

Then we have

$$\begin{aligned}f(x,\ y)-p(x,\ y)&=(f_1(u,\ v)-f_1(\bar{x}_0,\ \bar{y}_0))(x-\bar{x}_0)\\&\quad+(f_2(u,\ v)-f_2(\bar{x}_0,\ \bar{y}_0))(y-\bar{y}_0).\end{aligned}$$

Hence by (3.6), we have, from (3.11)

$$\begin{aligned}\|f-V_\delta f\|_F&\leqslant\|f-p\|_{D_{\delta,i}}+\|V_\delta(f-p)\|_{D_{\delta,i}}\\&\leqslant(1+\|V_\delta\|)\|f-p\|_{D_{\delta,i}}\\&\leqslant(1+\|V_\delta\|)\max\left[\omega\left(f_1,\ \frac{\sqrt{3}\,\delta}{3}\right),\ \omega\left(f_2,\ \frac{\sqrt{3}\,\delta}{3}\right)\right]g(\delta),\end{aligned}$$

where $g(\delta):=\max\{|x-\bar{x}_0|+|y-\bar{y}_0|:\ (x,\ y)\in\overline{D}_{\delta,i}\}=(3+\sqrt{3})\delta/6$. Since it is clear that $\|V_\delta\|=1$, the inequality (3.9) follows. Now let $f\in C^2(\overline{\mathscr{D}})$. Then by Taylor's Theorem, we have

$$f(x,\ y)=p(x,\ y)+\frac{1}{2}D^2f(u,\ v)((x-\bar{x}_0,\ y-\bar{y}_0),\ (x-\bar{x}_0,\ y-\bar{y}_0))$$

for some $(u,\ v)=t(x,\ y)+(1-t)(\bar{x}_0,\ \bar{y}_0)$, $t\in[0,\ 1]$, where $p(x,\ y)$ is the linear polynomial given in (3.12). Hence by an argument similar to that given above, we have

$$\|f-V_\delta f\|_F=\|f-V_\delta f\|_{D_{i,\delta}\cap F}\leqslant2\|f-p\|_{D_{i,\delta}}\leqslant\frac{1}{3}\,\delta^2\|D^2f\|.$$

This completes the proof of the theorem.

The above result can be extended to the most general cross-cut triangulation of the first kind. From Lemma 1.2, we know that any cross-cut triangulation Δ of the first kind of $\mathbf{R}^2$ is made up of lines

$$\begin{cases} M_{1,j}: a_1(x-x_0)+b_1(y-y_0)+j\eta_1=0, \\ M_{2,j}: a_2(x-x_0)+b_2(y-y_0)+j\eta_2=0, \\ M_{3,j}: a_3(x-x_0)+b_3(y-y_0)+j\eta_3=0, \end{cases} \tag{3.13}$$

where $j=\cdots, -1, 0, 1, \cdots,$ and (a_1, b_1), (a_2, b_2), (a_3, b_3) are pairwise linearly independent ordered pairs (x_0, y_0) is some grid-point of Δ, η_1 is some positive number, and η_2, η_3 must satisfy the relationship

$$(a_2b_3-a_3b_2)\eta_1=(a_1b_3-a_3b_1)\eta_2=(a_1b_2-a_2b_1)\eta_3. \tag{3.14}$$

Write $\eta_1=\sqrt{3}\,\delta$ where $\delta>0$. We will call δ the mesh-size of the gridpartition (or triangulation) Δ. Note that if $(x_0, y_0)=0$ and $(a_1, b_1)=(\sqrt{3}, 1)$, $(a_2, b_2)=(\sqrt{3}, -1)$, $(a_3, b_3)=(0, 1)$, then the grid partition Δ coincides with the grid portition Δ' in (3.3) with this relationship between η_1 and δ. Recall also that δ is the length of each grid-segment in Δ'_δ. Let

$$\Delta(\delta)=\Delta(\delta;\ (a_1, b_1),\ (a_2, b_2),\ (a_3, b_3);\ (x_0, y_0))$$

denote the grid portition Δ determined by the lines (3.13) with $\eta_1=\sqrt{3}\,\delta$, and let D_i be any cell of $\Delta(\delta)$. It can be proved that the lengths of the three sides of the triangle D_i are

$$\begin{cases} \dfrac{\sqrt{3(a_3^2+b_3^2)}}{|a_1b_3-a_3b_1|}\,\delta, \\ \dfrac{\sqrt{3(a_2^2+b_2^2)}}{|a_2b_1-a_1b_1|}\,\delta \text{ and} \\ \dfrac{\sqrt{3(a_1^2+b_1^2)}\,|a_2b_3-b_2a_3|}{|(a_1b_3-a_3b_1)(a_2b_1-a_1b_2)|}\,\delta, \end{cases} \tag{3.15}$$

and the distances from the center of D_i to the three vertices are

$$\begin{cases} \dfrac{\sqrt{3}\,[(2a_1b_2b_3-a_2b_1b_3-a_3b_1b_2)^2+(2a_2a_3b_1-a_1a_2b_3-a_1a_3b_2)^2]^{1/2}}{3|(a_2b_1-a_1b_2)(a_1b_3-a_3b_1)|}\,\delta, \\ \dfrac{\sqrt{3}\,[(2a_2b_1b_3-a_1b_2b_3-a_3b_1b_2)^2+(2a_1a_3b_2-a_1a_2b_3-a_2a_3b_1)^2]^{1/2}}{3|(a_2b_1-a_1b_2)(a_1b_3-a_3b_1)|}\,\delta, \\ \dfrac{\sqrt{3}\,[(2a_3b_1b_2-a_1b_2b_3-a_2b_1b_3)^2+(2a_1a_2b_3-a_1a_3b_2-a_2a_3b_1)^2]^{1/2}}{3|(a_2b_1-a_1b_2)(a_1b_3-a_3b_1)|}\,\delta. \end{cases} \tag{3.16}$$

Let α, β, γ be three positive numbers difined as follows: $\alpha\cdot\delta$ is the maximum of the lengths of three sides of the triangular cells of $\Delta(\delta)$ given in (3.15), $\beta\cdot\delta$ is the maximum of the distances from the vertices of a cell to its center given in (3.16), and γ is the maximum of the three numbers

$$\begin{cases} \dfrac{\sqrt{3}\,[|2a_1b_2b_3-a_2b_1b_3-a_3b_1b_2|+|2a_2a_3b_1-a_1a_2b_3-a_1a_3b_2|]}{3|(a_2b_1-a_1b_2)(a_1b_3-a_3b_1)|}, \\ \dfrac{\sqrt{3}\,[|2a_2b_1b_3-a_1b_2b_3-a_3b_1b_2|+|2a_1a_3b_2-a_1a_2b_3-a_2a_3b_1|]}{3|(a_2b_1-a_1b_2)(a_1b_3-a_3b_1)|}, \\ \dfrac{\sqrt{3}\,[|2a_3b_1b_2-a_1b_2b_3-a_2b_1b_3|+|2a_1a_2b_3-a_1a_3b_2-a_2a_3b_1|]}{3|(a_2b_1-a_1b_2)(a_1b_3-a_3b_1)|}. \end{cases} \tag{3.17}$$

In the special case when $\Delta(\delta)$ is the cross-cut regular triangulation Δ'_δ of $\mathbf{R}^2$, we have $\alpha=1$, $\beta=\sqrt{3}/3$ and $\gamma=(3+\sqrt{3})/6$.

Let $N_{i,\delta}(x', y')$ be the bivariate cubic B-splines in $S_3^1(\Delta'_\delta)$ defined in (3.4). Consider the (non-singular) linear transformation

$$\begin{cases} a_1(x-x_0)+b_1(y-y_0)=-\dfrac{\eta_1}{\delta}x'-\dfrac{\sqrt{3}\,\eta_1}{3\delta}y', \\ a_2(x-x_0)+b_2(y-y_0)=-\dfrac{\eta_2}{\delta}x'+\dfrac{\sqrt{3}\,\eta_2}{3\delta}y' \end{cases} \tag{3.18}$$

between the grid-partition Δ'_δ and $\Delta(\delta)$, and define the (normalized) bivariate cubic B-splines $\tilde{N}_{i,\delta}(x, y)$ in $S_3^1(\Delta(\delta))$ by

$$\tilde{N}_{i,\delta}(x, y)=N_{i,\delta}(x', y'), \tag{3.19}$$

where (x, y) and (x', y') satisfied (3.18). Then $\tilde{N}_{i,\delta}$ has support on a hexagonal region $F_{i,\delta}$ centered at a grid point E_i of $\Delta(\delta)$ and consisting of 24 triangular cells of $\Delta(\delta)$. As before, let F be o compact set, $\mathscr{D}$ a bounded domain containing F, and define the index set

$$\tilde{\Omega}_\delta=\tilde{\Omega}_\delta(F)=\{i: F_{i,\delta}\cap F\neq\emptyset\}.$$

Let $\delta_0>0$ be chosen such that $E_i\in\mathscr{D}$ whenever $i\in\tilde{\Omega}_\delta$ and $0<\delta\leqslant\delta_0$. Also, define the variation diminishing bivariate cubic spline operator $\tilde{V}_\delta$: $C(\overline{\mathscr{D}})\to S_3^1(\Delta(\delta))$ by

$$(\tilde{V}_\delta f)(x, y)=\sum_{i\in\tilde{\Omega}_\delta} f(E_i)\tilde{N}_{i,\delta}(x, y).$$

Then by a proof similar to that of Theorem 3.3, we have the following result:

Theorem 3. 4. *Let $f\in C(\overline{\mathscr{D}})$. Then for $0<\delta\leqslant\delta_0$,*

$$\|f-\tilde{V}_\delta f\|_F\leqslant\omega(f, 2\alpha\delta).$$

If, in addition, $f\in C^1(\overline{\mathscr{D}})$, then

$$\|f-\tilde{V}_\delta f\|_F\leqslant 2\gamma\delta\max[\omega(f_1, \beta\delta), \omega(f_2, \beta\delta)],$$

and if $f\in C^2(\overline{\mathscr{D}})$, then

$$\|f-V_\delta f\|_F\leqslant\beta^2\delta^2\|D^2f\|.$$

4. Final Remarks

The bivariate B-spline series

$$\sum_{i\in\tilde{\Omega}_\delta} d^i\tilde{N}_{i,\delta}(x, y) \tag{4.1}$$

can be used to study the approximation and interpolation of functions on $\mathscr{D}$ or $\overline{\mathscr{D}}$ by bivariate spline functions in $S_3^1(\Delta(\delta))$. It can be proved, however, that $\{\tilde{N}_{i,\delta}: i\in\tilde{\Omega}_\delta\}$ is only a basis of a proper subspace of $S_3^1(\Delta(\delta))$. Therefore, it is quite fortunate that we have the Jackson order in uniform approximation of functions in $C(\overline{\mathscr{D}})$ and $C^1(\overline{\mathscr{D}})$. Also, by using the variation diminishing bivariate spline operators, the order $O(\delta^2)$ is the best we can hope for, even in univariate spline approximation. In interpolation, if the sample points are taken at the grid-points of the partition $\Delta(\delta)$, then by an appropriate ordering of the grid-points, the coefficient matrix in determining the B-spline series (4.1) is a symmetric Toepli z matrix which has at most seven non-trivial diagonals.

References

[1] de Boor, C., Splines as linear combination of B-splines. In: *Approximation Theory II*, ed. by Lorentz, G. G., Chui, C. K. and Schumaker, L. L., Academic Press, N. Y. 1976, 1—47.

[2] Chui, C. K. and Wang, R. H., Bases of bivariate spline spaces with cross-cut grid partitions, *J. Math. Research and Exposition*, 1 (1982), 1—4.

[3] Chui, C. K. and Wang, R. H., On smooth multivariate spline functions, Submitted for publication.

[4] Dahmen, W., On multivariate B-splines, *SIAM J. Num. Analysis*, **17** (1980), 179—190.

[5] Micchelli, C. A., On numerically efficient method for computing multivariate B-splines. In: *Multivariate Approximation Theory*, ed. by Schempp, W. and Zeller, K., ISNM 51, Birkhauser Verlag, Basel, 1979.

[6] Wang, R. H., The structural characterization and interpolation for multivariate splines, *Acta Math. Sinica*, **18** (1975), 91—106.