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NONUNITAL POSITIVE LINEAR MAPS ON C^*-ALGEBRAS

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Abstract

In many cases, studying positive linear maps on C^*-algebras, we always assume that it is unital(i. e. it carries identity to identity). In this paper, the author discusses nonunital positive linear maps on C^*-algebras. First, similar to positive functionals on A, if Φ is a positive linear map from C^*-algebra A into B, then

$$\|\Phi\|=\sup\{\|\Phi(a)\| \mid a\in A_+,\ \|a\|\leqslant 1\}=\lim_l\|\Phi(v_l)\|=\lim_l\|\Phi(v_l^2)\|$$

where $\{v_l\}$ is an approximate identity of A. Then the author proves that if Φ is an n-positive linear map from A into B, and $\|\Phi\|=1$, then Φ can be extended to an unital n-positive linear map $\overline{\Phi}$ from $A\dotplus I_A$ into $B\dotplus I_B$(or into B, if B has identity). This result can also be used to generalize some results about unital positive maps.

Let A, B be two C^*-algebras (not necessary with identity), Φ be a linear map from A into B. Φ is called positive if $\Phi(A_+)\subset B_+$. It is known[1] that if Φ is positive, then Φ is bounded. Let $M_n(A)=A\otimes M_n$, $M_n(B)=B\otimes M_n$, and $\Phi_n=\Phi\otimes I_n$, then Φ_n is a linear map from $M_n(A)$ into $M_n(B)$. Φ is called n-positive if Φ_n is a positive map from C^*-algebra $M_n(A)$ into C^*-algebra $M_n(B)$. It is also known that if Φ is n-positwe, then Φ_k is also positive from $M_k(A)$ into $M_k(B)$, $\forall 1\leqslant k\leqslant n$. Φ is called completely positive, if Φ_n is positive for all positive integer n.

Proposition 1. *Let Φ: $A\to B$ be positive, then*

$$\|\Phi\|=\sup\{\|\Phi(a)\| \mid a\in A,\ \|a\|\leqslant 1\}=\lim_l\Phi(v_l)\|=\lim_l\|\Phi(v_l^2)\|$$

where $\{v_l\}$ is an approximate identity of A.

Proof Because Φ is bounded, we have

$$\alpha=\sup\{\|\Phi(a)\| \mid a\in A_+,\ \|a\|\leqslant 1\}\leqslant\|\Phi\|<+\infty.$$

Now we define a linear map $\overline{\Phi}$: $A\dotplus I_A\to B\dotplus I_B$ as follows

$$\overline{\Phi}(a+\lambda I_A)=\Phi(a)+\lambda\alpha I_B\quad \forall a\in A,\ \lambda\in\mathbb{C}.$$

We say that $\overline{\Phi}$ is positive. In fact, let $h+\lambda I_A\in(A\dotplus I_A)_+$, then $h=h^*\in A$, and $\lambda\geqslant 0$. Let $C(h)$ be the C^*-subalgebra of A generated by h. Because $C(h)$ is commutative, by [1], Φ: $C(h)\to B$ is completely positive.

Let $B\dotplus I_B\subset B(H)$ and $I_B=I_H$ (the identity operator in Hilbert space H). By

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Stinespring Theorem[1], there exists a nondegenerate representation $\{\pi, K\}$ of $C(h)$ and a bounded linear operator $V: H\to K$ such that

$$\Phi(c)=V^*\pi(c)V,\ \forall c\in C(h).$$

If we define that $\pi(I_A)=I_k$, then $c\to V^*\pi(c)V$ is a completely positive map from $(C(h)\dotplus I_A$ into $B(H)$. In particular,

$$V^*\pi(h+\lambda I_A)V=\Phi(h)+\lambda V^*V$$

is a positive operator in H.

If $\{u_t\}$ is an approximate identity of $C(h)$, by [1], we have

$$V^*V\xi=\lim_t\Phi(u_t)\xi,\ \forall\xi\in H,$$

so that $\|V^*V\|\leqslant\overline{\lim_t}\|\Phi(v_t)\|\leqslant\alpha$, and $0\leqslant V^*V\leqslant\alpha I_H=\alpha I_B$. Now by $\lambda\geqslant 0$,

$$\overline{\Phi}(h+\lambda I_A)=\Phi(h)+\lambda\alpha I_B\geqslant\Phi(h)+\lambda V^*V\geqslant 0$$

and $\overline{\Phi}$: $A\dotplus I_A\longrightarrow B\dotplus I_B$ is positive.

Now by [2]

$$\alpha\leqslant\|\Phi\|\leqslant\|\overline{\Phi}\|=\|\overline{\Phi}(I_A)\|=\alpha,$$

so that

$$\|\Phi\|=\sup\{\|\Phi(a)\|\,|a\in A_+,\ \|a\|\leqslant 1\}.$$

Finally, let $\{v_l\}$ be an approximate identity of A, $a\in A_+$, $\|a\|\leqslant 1$. Then

$$\|\Phi(a)\|=\lim_l\|\Phi(v_lav_l)\|\leqslant\lim_l\|\Phi(v_l^2)\|\leqslant\lim_l\|\Phi(v_l)\|\leqslant\|\Phi\|,$$

so we also have

$$\|\Phi\|=\lim_l\|\Phi(v_l)\|=\lim_l\|\Phi(v_l^2)\|.$$

Thus the proof is complete.

Remark 2. From the proof of Proposition 1, we have also seen that if Φ: $A\to B$ is positive, then

$$\overline{\Phi}(a+\lambda I_A)=\Phi(a)+\lambda\|\Phi\|I_B \tag{1}$$

(or if B has identity e_B, we can define $\overline{\Phi}(a+\lambda I_A)=\Phi(a)+\lambda\|\Phi\|e_B$) is a positive map from $A\dotplus I_A$ into $B\dotplus I_B$ (or into B, if B has identity). However, if $\|\Phi\|=1$, $\overline{\Phi}$ will be unital, and it is also clear that $\overline{\Phi}$ is the minimal positive extention of Φ.

Corollary 3. *If $\Phi: A\to B$ is n-positive, then $\|\Phi_n\|=\|\Phi\|$.*

In fact, if $\{v_l\}$ is an approximate identity of A, then $\{V_l\otimes I_n\}$ is an approximate identity of $M_n(A)=A\otimes M_n$. So by proposition 1

$$\|\Phi_n\|=\lim_l\|\Phi_n(v_l\otimes I_n)\|=\lim_l\|\Phi(v_l)\otimes I_n\|=\lim_l\|\Phi(v_l)\|=\|\Phi\|.$$

Lemma 4. *Let C be a C^*-algebra. There exists a completely positive linear map of norm one*

$$\Lambda:\ (C\otimes M_n)^*\longrightarrow((C\dotplus I_c)\otimes M_n)^*$$

such that for each state ϕ on $C\otimes M_n$, $\hat{\phi}$ will be a state on $(C\dotplus I_c)\otimes M_n$ and

$$(\hat{\phi}(x+\lambda I_c)\otimes M)=\phi(x\otimes M)+\lim_t\phi(\lambda r_t\otimes M)$$

$\forall x\in C$, $M\in M_n$, $\lambda\in\mathbf{C}$, where $\{r_t\}$ is an approximate identity of C.

Proof See [3].

Proposition 5. *Let Φ: $A \to B$ be n-positive. Then, $\overline{\Phi}$: $A \dot{+} I_A \to B \dot{+} I_B$* (See (1)) *is also n-positive.*

Proof We can assume that $\|\Phi\| = 1$, and consider the following maps

$$\begin{array}{ccc} (B \otimes M_n)^* & \xrightarrow{\Phi_n^*} & (A \otimes M_n)^* \\ \Lambda \downarrow & & \downarrow \Lambda \\ ((B \dot{+} I_B) \otimes M_n)^* & \xrightarrow{\overline{\Phi}_n^*} & ((A \dot{+} I_A) \otimes M_n)^* \end{array}$$

(the diagram is not commutative) and let $f \in ((B \dot{+} I_B) \otimes M_n)^*_+$.

If $g = f|_{B \otimes M_n} \in (B \otimes M_n)^*_+$, by Lemma 4, in $((B \dot{+} I_B) \otimes M_n)^*$ we shall have

$$\hat{g} \leqslant f.$$

Let $h + I_A \otimes M \geqslant 0$ in $(A \dot{+} I_A) \otimes M_n = M_n(A) \dot{+} I_A \otimes M_n$, where $h \in M_n(A)$, $M \in M_n$. Then $h = h^*$ in $M_n(A)$, $M \geqslant 0$ in M_n. So

$$\overline{\Phi}_n^*(f - \hat{g})(h + I_A \otimes M) = (f - \hat{g})(\Phi_n(h) + I_B \otimes M) = (f - \hat{g})(I_B \otimes M) \geqslant 0.$$

This means that

$$\overline{\Phi}_n^* f \geqslant \overline{\Phi}_n^* \hat{g}.$$

Now let $\{u_s\}$ be an approximate identity of B, and $\{v_l\}$ be an approximate identity of A. Then $0 \leqslant \Phi(v_l) \leqslant I_B$. By Lemma 4 and $M \geqslant 0$ in M_n,

$$\begin{aligned} \overline{\Phi}_n^* \hat{g}(h + I_A \otimes M) &= \hat{g}(\Phi_n(h) + I_B \otimes M) = g \circ \Phi_n(h) + \hat{g}(I_B \otimes M) \\ &\geqslant g \circ \Phi_n(h) + \lim_l \hat{g}(\Phi(v_l) \otimes M) = \lim_l g \circ \Phi_n(h + v_l \otimes M) \\ &= \widehat{\Phi_n^* g}(h + I_A \otimes M). \end{aligned}$$

Because Φ_n is positive, Φ_n^* is also positive, so that

$$\widehat{\Phi_n^* g}(h + I_A \otimes M) \geqslant 0.$$

Therefore

$$\overline{\Phi}_n^* f \geqslant \overline{\Phi}_n^* \hat{g} \geqslant \widehat{\Phi_n^* g} \geqslant 0,$$

and $\overline{\Phi}_n^*$ is positive. Hence $\overline{\Phi}$ is n-positive[1] and the proof is complete.

Remark 6. By [4] and Proposition 5, we have the following result. If Φ: $A \to B$ positive, and let t, $a \in A$ such that $t \geqslant a^* a$, $ta = at$, then

$$\|\Phi\| \Phi(t) \geqslant \Phi(a)^* \Phi(a), \quad \|\Phi\| \Phi(t) \geqslant \Phi(a) \Phi(a)^*.$$

Remark 7. Let $L = \{\Phi \mid \Phi$: $A \to B$ positive, and $\|\Phi\| \leqslant 1\}$ (this is a convex set), and Φ be an extreme point of L, a be a central element of A such that $\Phi(a)$ is also a central element of B. Then by [5] and Proposition 5, we have

$$\Phi(ab) = \Phi(a) \Phi(b), \ \forall b \in A.$$

Remark 8. By [6] and Proposition 5, we also have the following result. If Φ: $A \to B$ 2-positive, then for each $h \in A$,

$$\|\Phi\| \Phi(a^* a) \geqslant \Phi(a)^* \Phi(a).$$

References

[1] Takesaki, M., Theory of operator algebras I, Springer-Verlag, 1979.

[2] Russo, B. and Dye, H. D., A note on unitary operators in C^*-algebras, *Duke Math. J.*, **33** (1966), 413—416.
[3] Lance, C., Tensor products of nonunital C^*-algebras, *J. London Math. Soc.*, **12** (1976), 160—168.
[4] Choi, M. D., Some assorted inequalities for positive linear maps on C^*-algebras, Preprint.
[5] Stormer, E., Postive linear maps of C^*-algebras, *Lecture Notes in Physics*, **29** (1974), 85—106, Springer-Verlag.
[6] Choi, M. D., A Schwartz inequality for positive linear maps on C^*-algebras, *J. Math., Canada*, **18** (1974), 565—574.