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MULTIPLICITY OF SOLUTIONS TO NONLINEAR BOUNDARY VALUE PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract

In this paper the author considers the following nonlinear boundary value problem with nonlocal boundary conditions

$$\begin{cases} Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f(x, u, t) \\ u|_{\Gamma} = \text{const}, \ -\int_{\Gamma} \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i)ds = 0. \end{cases}$$

Under suitable assumptions on f it is proved that there exists $t_0 \in R$, $-\infty < t_0 < +\infty$ such that the problem has no solution as $t > t_0$, at least one solution at $t = t_0$, at least two solutions as $t < t_0$.

§ 1. Introduction

In this paper we consider the following nonlinear boundary value problem with nonlocal boundary conditions

$$(\mathrm{P}_t) \begin{cases} Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f(x, u, t), \text{ in } \Omega, \\ u|_{\Gamma} = \text{const}, \text{(unknown)}, \ -\int_{\Gamma} \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i)ds = 0, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ is a bounded domain with $C^{2+\mu}$ boundary Γ, $a_{ij}(x) \in C^{1+\mu}(\bar{\Omega})$,

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant \alpha \sum_{i=1}^{n} \xi_i^2, \ \alpha > 0,$$

f: $\bar{\Omega} \times R \times R \longrightarrow R$ is a C^1 function with parameter t.

In the recent years wide attention has been paid to such kinds of nonlinear boundary value problem with nonlocal boundary conditions which arise in plasma physics and other physical problems.

In [2] we apply the supersolution and subsolution method, different from the methods used by other authors, to this problem and obtain the existence of solutions. In this paper by means of the supersolution and subsolution method combined with

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the technique in [7] we obtain the multiplicity results for (1.1).

We make the following assumptions on the problem (1.1):

(i) $f: \bar{\Omega}\times R\times R \longrightarrow R$, $f\in C^1$.

(ii) For each $m\in R$ there exists a function $h(x)\in C(\bar{\Omega})$ such that

$$f_t(x, \xi, t)\geqslant h(x)>0, \ \forall x\in\Omega, \ \xi\geqslant m, \ t\in R.$$

(iii) For each $x\in\bar{\Omega}$, $t\in R$,

$$(1) \ \limsup_{\xi\to-\infty} \frac{f(x, \xi, t)}{\xi}<0; \tag{1.2}$$

$$(2) \ \liminf_{\xi\to+\infty} \frac{f(x, \xi, t)}{\xi}>0. \tag{1.3}$$

$$(\text{iv}) \ \limsup_{\xi\to+\infty} \frac{f(x, \xi, t)}{\xi}<+\infty. \tag{1.4}$$

The above limits are assumed to be uniform with respect to $x\in\bar{\Omega}$ and t in any finite intervals.

We now have

Main Theorem. *Under the above assumptions* (i)—(iv) *there exists* $t_0\in R$, $-\infty<t_0<+\infty$ *such that the problem* (P_t) *has no solution as* $t>t_0$, *at least one solution at* $t=t_0$, *at least two solutions as* $t<t_0$.

§ 2. Some Lemmas

Lemma 1. *Let* $c(x)\in C^\mu(\bar{\Omega})$, $c(x)\geqslant 0$ in $\bar{\Omega}$, $c(x)\not\equiv 0$. *If* $u\in C^2$ *satisfies*

$$\begin{cases} Lu+cu\geqslant 0(>0), \text{ in } \Omega, \\ u|_\Gamma=\text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u}{\partial x_j}\cos(n, x_i)ds\leqslant 0, \end{cases} \tag{2.1}$$

then $u\geqslant 0$ (>0) *in* $\bar{\Omega}$.

Proof See [2].

Define the supersolution u_+ and the subsolution u_- of (P_t)as follows:

$u_+, u_-\in C^2$,

$$\begin{cases} Lu_+\geqslant f(x, u_+, t), \\ u_+|_\Gamma=\text{const}, \ -\sum_{i,j=1}^n \int_\Gamma a_{ij}\dfrac{\partial u_+}{\partial x_j}\cos(n, x_i)ds\leqslant 0, \end{cases} \tag{2.2}$$

and

$$Lu_-\leqslant f(x, u_-, t),$$

$$u_-|_\Gamma=\text{const}, \ -\sum_{i,j=1}^n \int_\Gamma a_{ij}\frac{\partial u_-}{\partial x_j}\cos(n, x_i)ds\geqslant 0, \tag{2.3}$$

respectively.

We now have

Theorem 1. *Let the assumptions*(i)—(iv) *be satisfied, then there exists* t_0, $-\infty$

$<t_0<+\infty$ *such that the problem* (P_t) *has no solution as* $t>t_0$, *and at least one solution as* $t<t_0$.

Proof Let the problem (P_t) be solvable at $t=c$ and denote by u_c the solution. By the assumption (ii), $f_t\geqslant 0$, we have

$$\begin{cases} Lu_c=f(x,\ u_c,\ c)\geqslant f(x,\ u_c,\ t), \\ u_c|_\Gamma=\text{const},\ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u_c}{\partial x_j}\cos(n,\ x_i)ds=0. \end{cases} \tag{2.4}$$

Thus u_c is the supersolution of (P_t) for $t<c$.

On the other hand, the assumption (iii) implies that there exists a sufficiently negative subsolution for $t<c$. In fact, it follows from (1.2) that there exists $s_-<0$ such that when $\xi<s_-<0$

$$\frac{f(x,\ \xi,\ t)}{\xi}<0, \tag{2.5}$$

hence

$$f(x,\ \xi,\ t)>0. \tag{2.6}$$

Taking $u_-=\text{constant}\ \ s<s_-$, we have

$$\begin{cases} Lu_-=0<f(x,\ u_-,\ t), \\ u_-|_\Gamma=\text{const.},\ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u_-}{\partial x_j}\cos(n,\ x_i)ds=0. \end{cases} \tag{2.7}$$

This means that we have the supersolution u_+ and the subsolution u_- with $u_-<u_+$ in $\bar{\Omega}$. Thus it follows from Theorem 2 in [2] that (P_t) is solvable for $t<c$. The above argument means that if (P_t) is solvable at $t=c$, then so is (P_t) for any $t>c$.

Let $t_0=\sup c$. For our purpose it suffices to prove $-\infty<t_0<+\infty$.

It is easy to see that (P_t) is equivalent to

$$\begin{cases} \tilde{L}u\triangleq Lu+\omega u=f(x,\ u,\ t)+\omega u\triangleq F(x,\ u,\ t), \\ u|_\Gamma=\text{const},\ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u}{\partial x_j}\cos(n,\ x_i)ds=0, \end{cases} \tag{2.8}$$

where ω is any fixed positive number.

We first prove $t_0>-\infty$. Let $s_0,\ s_1,\ 0<s_0<s_1$ be two constants. If $s_0<s<s_1$, then by (ii) there exists $h(x)>0$ in Ω such that for $t<0$ we have

$$\frac{F(x,\ s,\ t)-F(x,\ s,\ 0)}{t}=F_t(x,\ s,\ \theta t)\geqslant h(x). \tag{2.9}$$

Therefore

$$F(x,\ s,\ t)-F(x,\ s,\ 0)\leqslant th(x), \tag{2.10}$$

$$F(x,\ s,\ t)-t\cdot h(x)\leqslant F(x,\ s,\ 0)\triangleq F_0(x,\ s). \tag{2.11}$$

Since we have the sufficiently negative subsolution as stated before, it suffices to find a supersolution u_+ satisfying $s_0\leqslant u_+\leqslant s_1$ and

$$\begin{cases} \tilde{L}u_+\geqslant th+F_0(x,\ u_+)\geqslant F(x,\ u_+,\ t), \\ u_+|_\Gamma=\text{const},\ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u_+}{\partial x_j}\cos(-n,\ x_i)ds\leqslant 0. \end{cases} \tag{2.12}$$

Let

$$m=\max_{x\in\Omega, s\leqslant s\leqslant s_1} F_0(x, s). \tag{2.13}$$

If $m\leqslant 0$, then $u_+=s_1$ is a supersolution for $t=0$. So we only need to investigate the case $m>0$.

Consider the linear boundary problem

$$\begin{cases} \widetilde{L}w=g \\ w|_\Gamma=\text{const.}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial w}{\partial x_i}\cos(n, x_i)ds=0, \end{cases} \tag{2.14}$$

where $g\in L^p(\Omega)$, $p>n$. By the results in [2], the above problem has a unique solution $w\in W^{2,p}(\Omega)\subset C^1(\overline{\Omega})$. Moreover,

$$\|w\|_{c^1}\leqslant C_1\|w\|_{w^{2,p}}\leqslant C_2\|g\|_p. \tag{2.15}$$

Let Ω_1, Ω_2 be the subdomains of Ω such that $\Omega_1\subset\subset\Omega_2\subset\subset\Omega$,

$$\operatorname{vol}(\Omega-\Omega_1)^{1/p}<\frac{(s_1-s_0)}{mC_2}, \tag{2.16}$$

and $H(x)\in C^\infty$,

$$H(x)=\begin{cases} 0, & \Omega_1, \\ m, & \Omega\setminus\Omega_2 \end{cases} \tag{2.17}$$

with $0\leqslant H(x)\leqslant m$. As is well known, such $H(x)$ exists. Since $h(x)>0$ in Ω_2, we can take t sufficiently negative so that $th(x)+m<0$ in Ω_2. For such t, we have

$$H(x)\geqslant th(x)+m \tag{2.18}$$

in $\overline{\Omega}$.

Let v be the solution of

$$\begin{cases} \widetilde{L}v=H(x), \\ v|_\Gamma=\text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ii}\dfrac{\partial v}{\partial x_j}\cos(n, x_i)ds=0. \end{cases} \tag{2.19}$$

Thus we have $v>0$ and

$$\|v\|_{c^1}\leqslant C_2\|H\|_p\leqslant C_2\, m\operatorname{vol}(\Omega-\Omega_1)^{1/p}<s_1-s_0. \tag{2.20}$$

Let

$$u_+=s_0+v. \tag{2.21}$$

Thus we have

$$s_0<u_+<s_1, \tag{2.22}$$

$$F_0(x, u_+)\leqslant m, \tag{2.23}$$

$$\begin{cases} \widetilde{L}u_+=\widetilde{L}v+\widetilde{L}s_0=H+\omega s_0\geqslant H\geqslant th+m\geqslant th+F_0(x, u_+)\geqslant F(x, u_+, t), \\ u_+|_\Gamma=\text{const.}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u_+}{\partial x_j}\cos(n, x_i)ds=0, \end{cases} \tag{2.24}$$

i. e, u_+ is a supersolution. This means for sufficiently negative t, (P_t) is solvable. This completes the proof of $t_0>-\infty$.

We now turn to the proof of $t_0<+\infty$.

By (1.3) of the assumption (iii) and (i) there exist t_2, s_+, $t_2>-\infty$, $s_+>-\infty$ such that when $u>s_+$, $t>t_+$, $f(x, u, t)>0$.

It follows from(1.2), (1.3) that there exist positive constants $\omega>0$, $s_2>0$ such that when $\xi>s_2>0$

$$\frac{f(x, \xi, t)}{\xi} > -\omega. \tag{2.25}$$

Hence

$$f(x, \xi, t) + \omega\xi > 0, \text{ as } \xi > s_2. \tag{2.26}$$

There also exists a constant $s_- < 0$ such that when $\xi < s_- < 0$

$$\frac{f(x, \xi, t)}{\xi} < -\omega. \tag{2.27}$$

Hence

$$f(x, \xi, t) + \omega\xi > 0, \text{ as } \xi < s_- < 0. \tag{2.28}$$

Thus there exists a constant K such that for every $x \in \bar{\Omega}$, $\xi \in R$ and $t \geqslant 0$ we have

$$f(x, \xi, t) + \omega\xi \geqslant K. \tag{2.29}$$

Let z be the solution of

$$\begin{cases} \tilde{L}z = K, \\ z|_\Gamma = \text{const.}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \dfrac{\partial z}{\partial x_j} \cos(n, x_i) ds = 0. \end{cases} \tag{2.30}$$

If u_+ is the solution of (P_t) $(t \geqslant 0)$, then

$$\begin{cases} \tilde{L}u_+ = f(x, u_+, t) + \omega u_+, \\ u_+|_\Gamma = \text{const.}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \dfrac{\partial u_+}{\partial x_j} \cos(n, x_i) ds = 0. \end{cases} \tag{2.31}$$

Therefore,

$$\begin{cases} \tilde{L}(u-z) \geqslant 0, \\ (u-z)|_\Gamma = \text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \dfrac{\partial (u-z)}{\partial x_j} \cos(n, x_i) ds = 0. \end{cases} \tag{2.32}$$

It follows from Lemma 1 that

$$u - z \geqslant 0, \text{ in } \bar{\Omega}. \tag{2.33}$$

Therefore

$$u \geqslant z \geqslant \min_{x \in \bar{\Omega}} z \triangleq \tilde{m}. \tag{2.34}$$

For $t \geqslant 0$ we have

$$f(x, \xi, t) \geqslant f(x, \xi, 0) + t\, h(x). \tag{2.35}$$

Let v be the solution of

$$\begin{cases} \tilde{L}v = h \geqslant 0, \\ v|_\Gamma = \text{const.}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \dfrac{\partial v}{\partial x_j} \cos(n, x_i) ds = 0. \end{cases} \tag{2.36}$$

By Lemma 1

$$v > 0, \text{ in } \bar{\Omega}. \tag{2.37}$$

Thus we have

$$\begin{cases} \tilde{L}(u-z-tv) = f(x, u, t) + \omega u - K - t\, h \geqslant f(x, u, 0) + \omega u - K \geqslant 0, \\ (u-z-tv)|_\Gamma = \text{const.}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij} \dfrac{\partial (u-z-tv)}{\partial x_j} \cos(n, x_i) ds = 0. \end{cases} \tag{2.38}$$

It follows from Lemma 1 that

$$u - z - tv > 0, \text{ in } \bar{\Omega}. \tag{2.39}$$

Since by (2.37)

$$\min_{x\in\bar{\Omega}} v>0, \tag{2.40}$$

there exists a sufficiently large number $t_2^*>0$ such that when $t\geqslant t_2^*$,

$$u>s_+. \tag{2.41}$$

Let $t_1=\max(t_2^*, t_2)$, then the problem (P_{t_1}) have no solution. In fact in this case we have

$$f(x, u, t_1)>0. \tag{2.42}$$

On the other hand, it is easy to see that the necessary condition for solvability of (P_t) is

$$\int_\Omega f(x, u, t)dx=0, \tag{2.43}$$

which contradicts (2.42) for $t=t_1$. This implies (P_{t_1}) is not solvable, i. e, $t_0<+\infty$. Thus the proof is completed.

Lemma 2. *The linear nonlocal boundary problem*

$$\begin{cases} Lu+\omega u=g(x)\in C(\bar{\Omega}), \\ u|_\Gamma=\text{const.}, \quad -\displaystyle\int_\Gamma \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i)ds=0 \end{cases} \tag{2.44}$$

with $\omega>0$ is uniquely solvable. If we denote by $u=Kg$ the solution, then K is a compact operator mapping from $C(\bar{\Omega})$ into $C^1(\bar{\Omega})$. Moreover, K is strongly increasing, i. e, if $g_1(x)\geqslant g_2(x)$, then $u_1=Kg_1>u_2=Kg_2$ in $\bar{\Omega}$.

Proof It follows from [2] that for any $g(x)\in C(\bar{\Omega})\in L^p(\Omega)$, $p>n$, the problem (2.44) admits a unique solution $u\in W^{2,p}(\Omega)$. Moreover,

$$\|u\|_{C^1}\leqslant C_1\|u\|_{W^{2,p}}\leqslant C_2\|g\|_p. \tag{2.45}$$

Since the imbedding operator from $W^{2,p}(\Omega)$ into $C^1(\bar{\Omega})$ is compact, K is compact, too. By Lemma 1, K is strongly increasing. Thus the proof is completed.

§ 3. Proof of Main Theorem

Let

$$E=\left\{u|u\in C^1(\bar{\Omega}),\ u|_\Gamma=\text{const},\ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i)ds=0\right\} \tag{3.1}$$

equipped with C^1 norm. It is easy to see that E is a closed ordered subspace of $C^1(\bar{\Omega})$ with natural order. Thus the problem (P_t) is equivalent to fixed point equation

$$u=KF(x, u, t)\triangleq KF(u, t). \tag{3.2}$$

We have proved in Theorem 1 that there exists t_0, $-\infty<t_0<+\infty$ such that (P_t) has no solution as $t>t_0$, at least one solution as $t<t_0$.

Let t_0^* be any fixed number $t_0^*<t_0$, We first prove that $(P_{t_0^*})$ has at least two solutions.

For any $\tau\in(t_0^*, t_0)$, (P_τ) is solvable and the solution u_τ is a supersolution for $(P_{t_0^*})$ as stated before.

On the other hand, as proved before, the problem $(P_{t_0^*})$ always has the sufficiently negative subsolution $\underline{u}$. Thus the problem (P_τ) has the supersolution $\bar{u}=u_\tau$ and the subsolution $\underline{u}$ with

$$\underline{u}<\bar{u}, \text{ in } \bar{\Omega}. \tag{3.3}$$

Let the ordered interval in E be

$$X=[\underline{u}, \bar{u}]=\{u\in E, \underline{u}\leqslant u\leqslant\bar{u}\}. \tag{3.4}$$

Since X is bounded in $C(\bar{\Omega})$, it follows from Lemma 2 that $KF(X, t_0^*)$ is compact in E. Let

$$G=KF(\cdot, t_0^*),$$

where we have set

$$\omega=\max_{\min \underline{u}\leqslant s\leqslant\max\bar{u}}|f_s(x, s, t_0^*)|+1. \tag{3.5}$$

Thus by Lemma 1

$$G(X)\subset X \tag{3.6}$$

and $u_1=G\underline{u}>\underline{u}$, $u_2=G\bar{u}<\bar{u}$. By the property of monotone operator G has a fixed point u_0 in X. Moreover,

$$\underline{u}<u_0<\bar{u}, \text{ in } \bar{\Omega}. \tag{3.7}$$

Suppose now u_0 is the unique fixed point of G in X, otherwise we have completed the proof. Thus there exists $\varepsilon>0$ such that

$$u_0+\varepsilon B\subset X, \tag{3.8}$$

where B is the open unit ball in E. This means that for any $v\in B$ we have

$$\underline{u}<u_0<\varepsilon v<\bar{u}. \tag{3.9}$$

Thus the Leray-Schauder degree

$$\deg(I-G, u+\varepsilon B, 0) \tag{3.10}$$

is well defined.

By the well known definition and properties of the Leray-Schauder degree and index[7] we have

$$\begin{aligned}\deg(I-G, u_0+\varepsilon B, 0)&=i(G, u_0+\varepsilon B, E)=i(G, u_0+\varepsilon B, X)\\&=i(G, X, X)=1.\end{aligned} \tag{3.11}$$

We claim that

$$\begin{cases}\text{there exists } \rho>0 \text{ such that } u_0+\varepsilon B\subset\rho B \text{ and}\\ \forall t\in I=[t_0^*, t_0+1], \forall u\in E, \|u\|_E=\rho,\\ KF(u, t)\neq u.\end{cases} \tag{3.12}$$

If (3.12) is valid, then by the homotopy invariance of the Leray-Schauder degree,

$$\deg(I-G, \rho B, 0)=\deg(I-KF(\cdot, t_0+1), \rho B, 0)=0, \tag{3.13}$$

since, according to the definition of t_0, $KF(\cdot, t_0+1)$ has no fixed point at all in E. Thus by (3.11) we have

$$\begin{aligned}\deg(I-G, \rho B\backslash(u_0+\varepsilon B), 0)&=\deg(I-G, \rho B, 0)\\&-\deg(I-G, u_0+\varepsilon B, 0)=-1,\end{aligned} \tag{3.14}$$

which implies that there is a fixed point of G in $\rho B\backslash(u_0+sB)$. Therefore, the existence of at least two solutions of the problem $(P_{t_0^*})$ is proved, provided we verify (3.12).

We now prove (3.12) by contradiction argument. Suppose now (3.12) is not true. Then we find sequences $t_j\in I=[t_0^*, t_0+1]$ and $u_j=u_{t_j}$ in E such that $\|u_j\|_E\to\infty$ and

$$u_j=KF(u_j, t_j). \tag{3.15}$$

Let

$$v_j=\frac{u_j}{\|u_j\|_E}. \tag{3.16}$$

The assumption (iii) implies that there exist $\mu<\omega$ and $k\geqslant 0$ such that

$$F(x, u, t)\geqslant\mu u-k \tag{3.17}$$

for all $u(x)$: $\bar{\Omega}\to R$ and $t\in[t_0^*, t_0+1]$.

Let w be the solution of

$$\begin{cases} Lw+(\omega-\mu)w=-k, \\ w|_\Gamma=\text{const}, \quad -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial w}{\partial x_j}\cos(n, x_i)ds=0. \end{cases} \tag{3.18}$$

For each fixed point u_t of $KF(\cdot, t)$, $t\in I$, we have

$$\begin{cases} \tilde{L}u_t=F(u_t, t)\geqslant\mu u_t-k, \\ u_t|_\Gamma=\text{const.}, \quad -\int_\Gamma \sum_{i,j=1}^n a_{ij}\dfrac{\partial u_t}{\partial x_j}\cos(n, x_i)ds=0. \end{cases} \tag{3.19}$$

Thus

$$\begin{cases} L(u_t-w)+(\omega-\mu)(u_t-w)\geqslant 0, \\ u_t-w|_\Gamma=\text{const.}, \quad -\int_\Gamma \sum_{,j=1}^n a_{ij}\dfrac{\partial(u_t-w)}{\partial x_j}\cos(n, x_i)ds=0. \end{cases} \tag{3.20}$$

By Lemma 1 we have $\forall t\in I$

$$u_t-w\geqslant 0, \text{ in } \bar{\Omega}. \tag{3.21}$$

On the other hand, by the assumption (iv)

$$\limsup_{\xi\to+\infty}\frac{f(x, \xi, t)}{\xi}<+\infty, \ \forall t\in I, \tag{3.22}$$

there exist constants $\tilde{s}_+$, M_1, M_2 such that when $\xi>\tilde{s}_+$, $t\in I$,

$$|f(x, \xi, t)|\leqslant M_1|\xi| \tag{3.23}$$

and when $x\in\bar{\Omega}$, $\xi\in[\min_{\bar{\Omega}} w, \tilde{s}_+]$, $t\in I$,

$$f(x, \xi, t)\leqslant M_2. \tag{3.24}$$

Thus

$$\frac{F(u_j, t_j)}{\|u_j\|_E} \tag{3.25}$$

is bounded in $C(\bar{\Omega})$.

Since

$$v_j=K\frac{F(u_j, t_j)}{\|u_j\|_E} \tag{3.26}$$

and K is a compact operator from $C(\bar{\Omega})$ into E, we can choose a subsequence, denoted still by v_j, such that

$$v_j \xrightarrow{E} v \tag{3.27}$$

and

$$v_j \geqslant \frac{w}{\|u_j\|_E} \longrightarrow 0, \text{ in } \bar{\Omega}, \tag{3.28}$$

which implies

$$v \geqslant 0, \text{ in } \bar{\Omega}. \tag{3.29}$$

On the other hand, the assumption (iii) implies that there exist $\alpha>0$, $\beta\geqslant 0$ such that

$$F(u, t) \geqslant (\omega+\alpha)u-\beta, \ \forall u: \bar{\Omega}\to R, \ t\in I. \tag{3.30}$$

Thus

$$\begin{aligned} v_j &= K\frac{F(u_j, t_j)}{\|u_j\|_E} \geqslant K\frac{(\omega+\alpha)u_j}{\|u_j\|_E} - K\frac{\beta}{\|u_j\|_E} \\ &= (\omega+\alpha)Kv_j - K\frac{\beta}{\|u_j\|_E}. \end{aligned} \tag{3.31}$$

Taking limit in (3.31), we obtain

$$v \geqslant (\omega+\alpha)Kv \triangleq z. \tag{3.32}$$

By the difinition of K we obtain

$$\begin{cases} \tilde{L}v-(\omega+\alpha)v\geqslant 0, \\ v|_\Gamma=\text{const}, \ -\int_\Gamma \sum_{i,j=1}^n a_{ij}\frac{\partial v}{\partial x_j}\cos(n, x_i)ds=0. \end{cases} \tag{3.33}$$

Hence

$$\begin{cases} Lv\geqslant \alpha v, \\ v|_\Gamma=\text{const}, \ -\int_\Gamma \sum_{i,j=1}^u a_{ij}\frac{\partial v}{\partial x_j}\cos(n, x_i)ds=0. \end{cases} \tag{3.34}$$

Thus it follows from $v\geqslant 0$, $\alpha>0$ and

$$0=\int_\Omega Lv\,dx \geqslant \int_\Omega \alpha v\,dx \tag{3.35}$$

that

$$v\equiv 0, \text{ in } \bar{\Omega}, \tag{3.36}$$

which contradicts $\|v\|_E=1$. This completely proves that when $t<t_0$, (P_t) has at least two solutions.

When $t=t_0$, there exist $t_j<t_0$ such that $t_j\to t_0$. Let u_j satisfy

$$u_j=KF(u_j, t_j). \tag{3.37}$$

It can be seen from the previous argument that u_j is bounded in E, and is also bounded in $C(\bar{\Omega})$, too. Hence, $F(u_j, t_j)$ is bounded in $C(\bar{\Omega})$. By the compactness of K from $C(\bar{\Omega})$ into E we can choose a subsequence, denoted still by u_j, such that $u_j \xrightarrow{E} u$. Thus by taking limit in (3.37) we obtain

$$u=KF(u, t_0), \tag{3.38}$$

which means that u is a solution of (P_{t_0}). Thus the proof of Main Theorem is completed.

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