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THE EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF SINGULARLY PERTURBED SYSTEMS

HUANG YUANSHI (黄元石)*

Abstract

First the author considers the system (1) $\frac{dx}{dt}=f(t,x,y,\varepsilon)$, $\varepsilon\frac{dy}{dt}=g(t,x,y,\varepsilon)$ and its degenerate system (2) $\frac{dx}{dt}=f(t, x, y, 0)$, $g(t, x, y, 0)=0$. In both noncritical and critical cases, sufficient conditions are established for the existence of almost periodic solutions of system (1) near the given solutions of system (2). The main method of proof is that, by performing suitable transformation, the author establishes exponential dichotomies, and then applies the theory of integral manifolds. Secondly, for the autonomous system (3) $\frac{dx}{dt}=f(x, y, \varepsilon)$, $\varepsilon\frac{dy}{dt}=g(x, y, \varepsilon)$, analogous results are obtained by performing the generalized normal coordinate transformation.

§ 1. Introduction

In this paper, at first, we consider the singularly perturbed nonautonomous system

$$\begin{aligned}\frac{dx}{dt}&=f(t, x, y, \varepsilon),\\ \varepsilon\frac{dy}{dt}&=g(t, x, y, \varepsilon),\end{aligned}\tag{1.1}$$

where ε is a small real parameter, x, y are respectively real n and m dimensional vectors, f, g are respectively real n and m dimensional vector functions. When $\varepsilon=0$, we get the degenerate system

$$\begin{aligned}\frac{dx}{dt}&=f(t, x, y, 0),\\ g(t, x, y, 0)&=0.\end{aligned}\tag{1.2}$$

The question is under what conditions system (1.1) has a solution which converges to a given solution of (1.2). The investigation on this problem with respect to periodic solutions have been done by many authors, for instance, L. Flatto–N. Levinson [1], E. R. Reng[2] and so on, that with respect to almost periodic solutions

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* Departmene of Mathematics, Fuzhou University, Fuzhou, China.

was given by J. K. Hale–G. Seifert[3] in 1961. Later, using exponential dichotomies, K. W. Chang[4] obtained a better result in 1968. These results have many applications, especially, in the field of nonlinear mechanics. All these results, however, were not concerned in the critical case. For example, L. Flatto–N. Levinson supposed that the first variation system of (1.2) had no nontrivial periodic solution, J. K. Hale–G. Seifert supposed that the trivial solution of the first variation system was exponentially asymptotical stable, and K. W. Chang then supposed that the first variation system admitted an exponential dichotomy. In the critical case, i. e., when the first variation system of (1.2) has zero characteristic exponents, to solve this problem is obviously much difficult because all these conditions mentioned above do not hold now.

In this paper we discuss both noncritical and critical cases, and establih sufficient conditisns for the existence of almost periodic solutions of system (1.1) near the given solutions of system (1.2). The main method of proof is that, by performing suitable transformations, we establish exponential dichotomies, and then apply the theory of integral manifolds.

Secondly, we consider the singularly perturbed autonomous system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, \varepsilon), \\ \varepsilon \frac{dy}{dt} &= g(x, y, \varepsilon) \end{aligned} \tag{1.3}$$

and its degenerate system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, 0), \\ g(x, y, 0) &= 0. \end{aligned} \tag{1.4}$$

The problem on the existence of periodic solutions can be found in W. Wasow[5] and K. W. Chang[6]. In this paper, by performing the generalized normal coordinate transformation, we change our problem into that of the nonautonomous systems and obtain satisfactory results.

Evidently, as a corollary of our results, the same problems on periodic or quasi-periodic solutions are also solved. In particular, for the critical case, it will be significant. Moreover, our results can be also used to investigate the initial value problems and the boundary value problems in the singular perturbation theory.

In this paper we shall use the definition of "characteristic exponenents in the extensive sense" introduced by Lin Zhensheng[7]. Moreover, we define the number of zero characteristic exponents in the extensive sense for a linear system as follows.

Let $x(t)$ be a nontrivial solution of system

$$\frac{dx}{dt} = A(t)x. \tag{1.5}$$

Put

$$N = \{x(t):\ \bar{\lambda}(x(t)) < 0\},$$

$$P=\{x(t):\underline{\lambda}(x(t))>0\}.$$

Denote the maximum number of linearly independent elements in N and P by r and s, respectively. Obviously, if $A(t)$ is bounded and system (1.5) has zero characteristic exponents in the extensive sense, we have $0\leqslant r+s<n$. In this case we say that system (1.5) has $n-(r+s)$ zero characteristic exponents in the extensive sense.

§ 2. The results for nonautonomous systems

Suppose that system (1.2) has a family of almost periodic solutions

$$x=u(t,\ \alpha),\ y=v(t,\ \alpha), \tag{2.1}$$

where $u(t,\ \alpha)$ and $v(t,\ \alpha)$ are almost periodic in t, $\alpha=\mathrm{Col.}\ (\alpha_1,\ \cdots,\ \alpha_k)$, and α_1, α_2, $\cdots$, α_k are k independent parameters, i. e., the rank of $n\times k$ matrix

$$\left(\frac{\partial u(t,\ \alpha)}{\partial \alpha_1},\ \cdots,\ \frac{\partial u(t,\ \alpha)}{\partial \alpha_k}\right)$$

is equal to k. Then the first variation system of (1.2) with respect to (2.1) has the form

$$\frac{dz}{dt}=A(t,\ \alpha)z, \tag{2.2}$$

where $A(t,\ \alpha)=f_x(t,\ \alpha)-f_y(t,\ \alpha)g_y^{-1}(t,\ \alpha)g_x(t,\ \alpha)$. Here $f_x(t,\ \alpha)$ denote $\frac{\partial f}{\partial x}(t,\ u(t,\ \alpha),\ v(t,\ \alpha),\ 0)$ and similar meanings are attached to $f_y(t,\ \alpha)$, $f_\varepsilon(t,\ \alpha)$, $g_x(t,\ \alpha)$, $g_y(t,\ \alpha)$ and $g_\varepsilon(t,\ \alpha)$. Meanwhile, we assume that f and g are almost periodic in t, uniformly continuous and of $C^{(2)}$ in all arguments; the Jacobian matrices f_x, f_y, f_ε, g_x, g_y, g_ε are all almost periodic in t, uniformly continuous in all arguments; the inverse matrix $g_y^{-1}(t,\ \alpha)$ exists and $g_y^{-1}(t,\ \alpha)g_x(t,\ \alpha)$ has continuous and bounded first derivatives.

It is easy to verify that each $\frac{\partial u(t,\ \alpha)}{\partial \alpha_j}(j=1,\ 2,\ \cdots,\ k)$ is a nontrivial solution of system (2.2), and hence system (2.2) has at least k zero characteristic exponents.

We obtain the following theorem.

Theorem 1. *Suppose that*

(I) *system* (1.2) *has a family of almost periodic solutions* (2.1) *with k independent parameters, and $u(t,\ \alpha)$, $v(t,\ \alpha)$ are also almost periodic in each $\alpha_j(j=1,\ 2,\ \cdots,\ k)$*;

(II) *the first variation system* (2.2) *has k zero characteristic exponents in the extensive sense*;

(III) *every eigenvalue of $g_y(t,\ \alpha)$ has nonzero real part for all t and α.*

Then, for ε sufficiently small, system (1.1) *has a unique family of almost periodic solutions*

$$x=x(t,\ \alpha,\ \varepsilon),\ y=y(t,\ \alpha,\ \varepsilon) \tag{2.3}$$

satisfying

$$\|x(t, \alpha, \varepsilon)-u(t, \alpha)\|+\|y(t, \alpha, \varepsilon)-v(t, \alpha)\| \to 0 \text{ as } \varepsilon \to 0.$$

Remark 1. If we remove the assumption that $u(t, \alpha)$ and $v(t, \alpha)$ are almost periodic in each $\alpha_j (j=1, 2, \cdots, k)$ in hypothesis (I), then we can also come to the conclution of Theorem 1 by means of the averaging method and other suitable conditions.

In the absence of α in Theorem 1, as a special case of Theorem 1, we obtain immediately the result for the noncritical case as follows.

Theorem 2. *Suppose that*

(I) *system* (1.2) *has a nonconstant almost periodic solution*

$$x=u(t), \quad y=v(t); \tag{2.4}$$

(II) *the first variation system of system* (1.2) *with respect to* (2.4) *has no zero characteristic exponent in the extensive sense*;

(III) *every eigenvalue of $g_y(t)$ has nonzero real part for all t.*

Then, for ε sufficiently small, system (1.1) *has a unique almost periodic solution*

$$x=x(t, \varepsilon), \quad y=y(t, \varepsilon) \tag{2.5}$$

satisfying

$$\|x(t, \varepsilon)-u(t)\|+\|y(t, \varepsilon)-v(t)\| \to 0 \text{ as } \varepsilon \to 0.$$

Remark 2. Theorem 2 can also follow directly from the result of K. W. Chang[4] and Theorem 3.1 of Lin Zhen-sheng[7].

§ 3. Some lemmas

We need the following lemmas.

Lemma 1. *Let $x_1(t), \cdots, x_k(t)$ be k independent almost periodic solutions of the linear system*

$$\frac{dx}{dt}=A(t)x, \tag{3.1}$$

where $A(t)$ is an almost periodic $n\times n$ matrix function. Then there is an almost periodic $n\times(n-k)$ matrix function $S(t)$ such that $Q(t)=(x_1(t), \cdots, (x_k(t), S(t))$ is a regular matrix function.

Proof Put $x_j(t)=\text{Col.}\ (x_{j1}(t), \cdots, x_{jn}(t))$ for $j=1, \cdots, k$. By Theorem 5.7 of A. M. Fink ([8 p. 85]), we have $\inf\limits_{t\in R}\|x_j(t)\|=\delta_0>0$ because $x_j(t)$ is a nontrivial almost periodic solution of system (3.1). Choose $\delta=\delta_0/4n$ and let $L(\delta)$ be the inclusion length of almost periodic vector function $x_j(t)$ belonging to δ.

We first show that there is an almost periodic $n\times n$ elementary matrix function $J(t)$ such that, for $\bar{x}_j(t)=J(t)x_j(t)$ and $\bar{x}_j(t)=\text{Col.}\ (\bar{x}_{j1}(t), \cdots, \bar{x}_{jn}(t))$, we have $\inf\limits_{0\le t\le L(\delta)}(\bar{x}_{j1}(t))=2\delta>0$, and hence $\inf\limits_{t\in R}(\bar{x}_{j1}(t))=\delta>0$.

In fact, if the conclution is false, we can find a sequence of $J^{(i)}(t)$ such that, for $x_j^{(i)}(t)=J^{(i)}(t)x_j(t)$ and the maximum interval $[0, L_i]$ with $x_{j1}^{(i)}(t)\geqslant 2\delta$, one has $L_i\to L_0$ as $i\to\infty$, but $L_i\leqslant L_0<L(\delta)$. Now let μ be a sufficiently large integer and put $\bar{x}_j^{(\mu+1)}(t)=J^{(\mu+1)}(t)\bar{x}_j^{(\mu)}(t)$, where

$$J^{(\mu+1)}(t)=\begin{pmatrix} 1 & \dfrac{\bar{x}_{j2}^{(\mu)}(t)\,|\bar{x}_{j2}^{(\mu)}(t)|^{\lambda}}{\bar{x}_{j2}^{(\mu)}(t)} & \cdots & \dfrac{\bar{x}_{jn}^{(\mu)}(t)\,|\bar{x}_{jn}^{(\mu)}(t)|^{\lambda}}{\bar{x}_{jn}^{(\mu)}(t)} \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and $0<\lambda<1$. Then in the interval $[0, L_{\mu+1}]$, we have

$$\bar{x}_{j1}^{(\mu+1)}(t)=\bar{x}_{j1}^{(\mu)}(t)+\sum_{\nu=2}^{n}|\bar{x}_{j\nu}^{(\mu)}(t)|^{1+\lambda}\geqslant 2\delta,$$

and $L_{\mu+1}>L_0$, which is contrary to the assumption.

By the almost periodicity of $\bar{x}_j(t)$, for any $t\in R$, there is a real number τ such that $0\leqslant t+\tau\leqslant L(\delta)$, and

$$|\bar{x}_{j1}(t+\tau)-\bar{x}_{j1}(t)|<\delta.$$

Combining it with $\inf\limits_{0\leqslant t\leqslant L(\delta)}(\bar{x}_{j1}(t))=2\delta>0$, we obtain

$$\inf_{t\in R}(\bar{x}_{j1}(t))=\delta>0.$$

Then, by using the induction in k, it is not difficult to show that there are almost periodic $n\times n$ elementary matrix function $J_1(t)$ and almost periodic $k\times k$ elementary matrix function $J_2(t)$ such that, for $(\bar{x}_1(t), \cdots, \bar{x}_k(t))=J_1(t)(x_1(t), \cdots, x_k(t))J_2(t)$, the value of the k-order determinant consisting of its first k row is greater than zero for $t\in R$. Moreover, the inverse matrices $J_1^{-1}(t)$ and $J_2^{-1}(t)$ are also almost periodic in t.

Now take $S_0=\begin{pmatrix} 0_1 \\ I_{n-k} \end{pmatrix}$, where S_0 is the $n\times(n-k)$ matrix and I_{n-k} is the $(n-k)\times(n-k)$ unit matrix. Then the matrix function $(\bar{x}_1(t), \cdots, \bar{x}_k(t), S_0)$ is an almost periodic regular matrix function for $t\in R$. Let $J_3=\begin{pmatrix} J_2^{-1}(t) & 0 \\ 0 & I_{n-k} \end{pmatrix}$ and let

$$J_1^{-1}(t)(\bar{x}_1(t), \cdots, \bar{x}_k(t), S_0)J_3(t)=(x_1(t), \cdots, x_k(t), S(t)).$$

Then the matrix function $S(t)$ is what we required, and it is almost periodic in t because of its construction.

Lemma 2. *Consider the system*

$$\varepsilon\frac{dx}{dt}=A(t)x, \tag{3.2}$$

where ε is a real small parameter, $A(t)$ is an almost periodic $n\times n$ matrix function. If every eigenvalue of $A(t)$ has real part different from zero for $t\in R$, then system (3.2) *admits an exponential dichotomy for ε sufficiently small.*

For the proof, please refer to K.W. Chang[4] and W. A. Coppel[9].

Lemma 3. *Consider the almost periodic linear system* (3.1). *If it has no zero characteristic exponent in the extensive sense, then it admits an exponential dichotomy.*

This is Theorem 3.1 of Lin Zhensheng[7].

Lemma 4. *Consider the system*

$$\begin{aligned}\frac{d\alpha}{dt}&=F(t,\ \xi,\ \eta,\ \alpha,\ \varepsilon),\\ \frac{d\xi}{dt}&=B(t,\ \alpha)\xi+B_1(t,\ \alpha)\eta+G(t,\ \xi,\ \eta,\ \alpha,\ \varepsilon),\\ \varepsilon\frac{d\eta}{dt}&=C(t,\ \alpha)\eta+H(t,\ \xi,\ \eta,\ \alpha,\ \varepsilon),\end{aligned}\tag{3.3}$$

where ε is a real small parameter, α, ξ, η are k, $n-k$, m dimension vectors, respectively. Suppose that the matrix functions B, B_1, C are all almost periodic in t and each component α_j of α; both system $\frac{d\xi}{dt}=B(t,\alpha)\xi$ and system $\varepsilon\frac{d\eta}{dt}=C(t,\alpha)\eta$ admit exponential dichotomies; the vector functions F, G, H are continuously differentiable in all arguments, almost periodic in t and each α_j uniformly in other arguments, $F=O(|\xi|+|\eta|+|\varepsilon|)$, G, $H=O(|\xi|^2+|\eta|^2+|\varepsilon|)$, and for any t, α, there is a constant $M>0$ such that

$$\|F(t,\ 0,\ 0,\ \alpha,\ \varepsilon)\|+\|G(t,\ 0,\ 0,\ \alpha,\ \varepsilon)\|+\|H(t,\ 0,\ 0,\ \alpha,\ \varepsilon)\|<M\varepsilon.$$

Then system (3.3) has the center integral manifold.

For the proof of this lemma please refer to Lin Zhensheng[7] and J. K. Hale[10].

§ 4. The proof of Theorem 1

Since $\frac{\partial u(t,\ \alpha)}{\partial\alpha_1},\ \cdots,\ \frac{\partial u(t,\ \alpha)}{\partial\alpha_k}$ are k independent almost periodic solutions of system (2.2), by Lemma 1, there is an $n\times(n-k)$ matrix function $S(t,\ \alpha)$ which is almost periodic in t and each α_j such that $Q(t,\ \alpha)=\left(\frac{\partial u}{\partial\alpha_1},\ \cdots,\ \frac{\partial u}{\partial\alpha_k},\ S(t,\ \alpha)\right)$ is a regular matrix function. By our assumptions, we see that $\frac{dS(t,\ \alpha)}{dt}$ and $\frac{\partial S(t,\ \alpha)}{\partial\alpha_j}(j=1,\ \cdots,\ k)$ exist, which are also almost periodic in t and each α_j. For convenience, write $\frac{\partial u(t,\ \alpha)}{\partial\alpha}=\left(\frac{\partial u}{\partial\alpha_1},\ \cdots,\ \frac{\partial u}{\partial\alpha_k}\right)$, $\frac{\partial S(t,\ \alpha)}{\partial\alpha}=\left(\frac{\partial S}{\partial\alpha_1},\ \cdots,\ \frac{\partial S}{\partial\alpha_k}\right)$ and let

$$Q^{-1}(t,\ \alpha)=P(t,\ \alpha)=\begin{pmatrix}P_1(t,\ \alpha)\\ P_2(t,\ \alpha)\end{pmatrix},$$

where $P_1(t,\ \alpha)$ is a $k\times n$ matrix function, P_2 is a $(n-k)\times n$ matrix function. It is easy to see that $P_1(t,\ \alpha)$ and $P_2(t,\ \alpha)$ are also almost periodic in t and each α_j, and we have

$$PQ=\begin{pmatrix}P_1\\ P_2\end{pmatrix}\left(\frac{\partial u}{\partial\alpha},\ S\right)=\begin{pmatrix}P_1\left(\frac{\partial u}{\partial\alpha}\right) & P_1S\\ P_2\left(\frac{\partial u}{\partial\alpha}\right) & P_2S\end{pmatrix}=\begin{pmatrix}I_k & 0\\ 0 & I_{n-k}\end{pmatrix}.\tag{4.1}$$

In order to emphasize the special role that the given solutions (2.1) play, for system (1.1) we are going to perform the following transformation

$$\begin{aligned} x &= u(t, \alpha) + S(t, \alpha)\xi, \\ y &= v(t, \alpha) - g_y^{-1}(t, \alpha) g_x(t, \alpha) S(t, \alpha)\xi + \eta, \end{aligned} \tag{4.2}$$

where ξ, η are real $n-k$ dimension and m dimension vectors respectively. The Jacobian of this transformation on (2.1) satisfies the inequality

$$\left|\det.\begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{pmatrix}\right| = \left|\det.\begin{pmatrix} \dfrac{\partial u}{\partial x} & S & 0 \\ * & * & I_m \end{pmatrix}\right| > 0.$$

Since $\xi=0$, $\eta=0$ when $\varepsilon=0$, by continuity, it follows that transformation (4.2) can be satisfactorily performed at least in a neighberhood of (2.1).

Now, by (4.2), the derivative of x can be represented as

$$\frac{dx}{dt} = \frac{du}{dt} + \frac{\partial u}{\partial \alpha} \cdot \frac{d\alpha}{dt} + \frac{dS}{dt}\xi + \left(\frac{\partial S}{\partial \alpha_1}\xi_1, \cdots, \frac{\partial S}{\partial \alpha_k}\xi_k\right)\frac{d\alpha}{dt} + S\frac{d\xi}{dt}. \tag{4.3}$$

On the other hand, by (1.1), the Taylor expansion of $f(t, x, y, \varepsilon)$ at point $(t, u, v, 0)$ gives

$$\begin{aligned} \frac{dx}{dt} &= f(t, u+S\xi, v-g_y^{-1}g_x S\xi+\eta, \varepsilon) \\ &= f(t, u, v, 0) + AS\xi + f_y\eta + f_\varepsilon \varepsilon + G_1(t, \xi, \eta, \alpha, \varepsilon), \end{aligned} \tag{4.4}$$

where $G_1 = O(|\xi|^2 + |\eta|^2 + |\varepsilon|)$. Combining (4.3) and (4.4), we get

$$\begin{aligned} &\left(\frac{\partial u}{\partial \alpha_1} + \frac{\partial S}{\partial \alpha_1}\xi_1, \cdots, \frac{\partial u}{\partial \alpha_k} + \frac{\partial S}{\partial \alpha_k}\xi_k\right)\frac{d\alpha}{dt} + S\frac{d\xi}{dt} \\ &= \left(AS - \frac{dS}{dt}\right)\xi + f_y\eta + G_2(t, \xi, \eta, \alpha, \varepsilon), \end{aligned} \tag{4.5}$$

where $G_2 = f_\varepsilon \varepsilon + G_1 = 0(|\xi|^2 + |\eta|^2 + |\varepsilon|)$. Since ξ, η are sufficiently small for ε sufficiently small and $Q(t, \alpha) = \left(\frac{\partial u}{\partial \alpha}, S\right)$ is regular, by the continuity, we see that the determinant of the coefficient matrix of the linear equation system with respect to $\frac{d\alpha}{dt}$ and $\frac{d\xi}{dt}$ is different from zero. Therefore, we can determine $\frac{d\alpha}{dt}$ and $\frac{d\xi}{dt}$ by (4.5). Now let us make the further estimation. Let

$$\frac{d\alpha}{dt} = F(t, \xi, \eta, \alpha, \varepsilon). \tag{4.6}$$

From the assumptions, it follows that $F = O(|\xi| + |\eta| + |\varepsilon|)$ is continuously differentiable in all arguments, almost periodic in t and each α_j uniformly in other arguments, and for any $\varepsilon > 0$ and any t, there is a constant $M_1 > 0$ such that

$$\|F(t, 0, 0, \alpha, \varepsilon)\| < M_1 \varepsilon.$$

Now we substitute (4.6) into (4.5) and then multipy both of the two sides from the left by $P_2(t, \alpha)$. On account of (4.1), it follows that

$$\frac{d\xi}{dt}=B(t,\ \alpha)\xi+B_1(t,\ \alpha)\eta+G(t,\ \xi,\ \eta,\ \alpha,\ \varepsilon), \tag{4.7}$$

where $B(t,\ \alpha)=P_2\left(AS-\frac{dS}{dt}\right)$, $B_1(t,\ \alpha)=P_2f_y$, $G=P_2G_2-P_2\left(\frac{\partial S}{\partial\alpha_1}\xi_1,\ \cdots,\ \frac{\partial S}{\partial\alpha_k}\xi_k\right)$. Moreover, it is easy to see that $G=O(|\xi|^2+|\eta|^2+|\varepsilon|)$ is continuously differentiable in all arguments, almost periodic in t and each α_j uniformly in other arguments, and for any $\varepsilon>0$ and any t, α, there is a constant $M_2>0$ such that

$$\|G(t,\ 0,\ 0,\ \alpha,\ \varepsilon)\|<M_2\varepsilon.$$

Then, by (1.1), (1.2) and (4.2), it follows that

$$\begin{aligned}\varepsilon\frac{d\eta}{dt}&=\varepsilon\frac{dy}{dt}-\varepsilon\frac{d}{dt}(v-g_y^{-1}g_xS\xi)\\&=g(t,\ u+S\xi,\ v-g_y^{-1}g_xS\xi+\eta,\ \varepsilon)-\varepsilon\frac{d}{dt}(v-g_y^{-1}g_xS\xi)\\&=g_xS\xi+g_y(-g_y^{-1}g_xS\xi+\eta)+g_\varepsilon\varepsilon+H_1(t,\ \xi,\ \eta,\ \alpha,\ \varepsilon)-\varepsilon\frac{d}{dt}(v-g_y^{-1}g_xS\xi),\end{aligned}$$

where $H_1=O(|\xi|^2+|\eta|^2+|\varepsilon|)$. Therefore, we have

$$\varepsilon\frac{d\eta}{dt}=C(t,\ \alpha)\eta+H(t,\ \xi,\ \eta,\ \alpha,\ \varepsilon), \tag{4.8}$$

where $C(t,\ \alpha)=g_y(t,\ \alpha)$, $H=g_\varepsilon\varepsilon+H_1-\varepsilon\frac{d}{dt}(v-g_y^{-1}g_xS\xi)$. Moreover, $H=O(|\xi|^2+|\eta|^2+|\varepsilon|)$ is continuously differentiable in all arguments, almost periodic in t and α_j uniformly in other arguments, and for any $\varepsilon>0$ and any t, α, there is a constant $M_3>0$ such that

$$\|H(t,\ 0,\ 0,\ \alpha,\ \varepsilon)\|<M_3\varepsilon.$$

Now we show that the system

$$\frac{d\xi}{dt}=B(t,\ \alpha)\xi \tag{4.9}$$

admits an exponential dichotomy.

In fact, by assumption (II), system (2.2) has k zero characteristic exponents in the extensive sense. For (2.2), we perform the regular transformation

$$z=Q(t,\ \alpha)h.$$

Since

$$\begin{aligned}\frac{dz}{dt}&=\frac{d}{dt}\left(\frac{\partial u}{d\alpha},\ S\right)h+Q\frac{dh}{dt}\\&=\left(A\frac{\partial u}{\partial\alpha},\ \frac{dS}{dt}\right)h+Q\frac{dh}{dt},\end{aligned}$$

and meanwhile,

$$\frac{dz}{dt}=A(t,\ \alpha)z=AQh=\left(A\frac{\partial u}{\partial\alpha},\ AS\right)h,$$

we have

$$Q\frac{dh}{dt}=\left(0,\ AS-\frac{dS}{dt}\right)h.$$

Let both the two sides be multiplied from the left by $P(t,\ \alpha)$. On account of (4.1), it follows that

$$\frac{dh}{dt}=\begin{pmatrix} 0 & P_1\left(AS-\frac{dS}{dt}\right) \\ 0 & P_2\left(AS-\frac{dS}{dt}\right) \end{pmatrix} h,$$

or

$$\frac{dh}{dt}=\begin{pmatrix} 0 & * \\ 0 & B(t, \alpha) \end{pmatrix} h.$$

Hence, system (4.9) has no zero characteristic exponent in the extensive sense. Otherwise, system (2.2) has more than k zero characteristic exponents in the extensive sense which contradicts the assumption. Thus, by Lemma 3, we see that system (4.9) admits an exponential dichotomy.

Meanwhile, for ε sufficiently small, by Lemma 2 and assumption (III), we see that the system

$$\varepsilon \frac{d\eta}{dt}=C(t, \alpha)\eta$$

admits an exponential dichotomy, too.

Thus, by performing transformation (4.2), system (1.1) is carried into system

$$\begin{aligned} &\frac{d\alpha}{dt}=F(t, \xi, \eta, \alpha, \varepsilon), \\ &\frac{d\xi}{dt}=B(t, \alpha)\xi+B_1(t, \alpha)\eta+G(t, \xi, \eta, \alpha, \varepsilon), \\ &\varepsilon \frac{d\eta}{dt}=C(t, \alpha)\eta+H(t, \xi, \eta, \alpha, \varepsilon), \end{aligned} \tag{4.10}$$

which satisfies the conditions of Lemma 4 for ε sufficiently small. Therefore, system (4.10) has a center integral manifold for ε sufficiently small. The proof of Theorem 1 is completed.

§ 5. The results for autonomous systems

We suppose still that system (1.4) has a family of solutions (2.1), and the rank of the $n\times(k+1)$ matrix

$$\left(\frac{du(t, \alpha)}{dt}, \frac{\partial u(t, \alpha)}{\partial \alpha_1}, \cdots, \frac{\partial u(t, \alpha)}{\partial \alpha_k}\right)$$

is equal to $k+1$. Then the first variation system of (1.4) with respect to (2.1) has the same form as (2.2). Certainly, in this case, the assumption that f, g and their Jacobian matrices are all almost periodic in t must be removed. However, apart from $\frac{\partial u(t, \alpha)}{\partial \alpha_j}$ $(j=1, \cdots, k)$, $\frac{du(t, \alpha)}{dt}$ is also an almost periodic solution of system (2.2), and hence system (2.2) has at least $(k+1)$ characteristic exponents equal to zero.

We obtained the following theorem.

Theorem 3. *Suppose that*

(I) *system* (1.4) *has a family of almost periodic solutions* (2.1) *with* k *independent parameters, and* $u(t, \alpha)$, $v(t, \alpha)$ *are also almost periodic in each* $\alpha_j (j=1, \cdots, k)$;

(II) *the first variation system* (2.2) *has* $(k+1)$ *zero characteristic exponents in the extensive sense*;

(III) *every eigenvalue of* $g_y(t, \alpha)$ *has nonzero real part for all* t *and* α.

Then, for ε *sufficiently small, system* (1.3) *has a unique family of almost periodic solutions as* (2.3).

Corresponding to Theorem 2, we have the result for the noncritical case as follows.

Theorem 4. *Suppose that*

(I) *system* (1.4) *has a nonconstant almost periodic solution* (2.4);

(II) *the first variation system of* (1.4) *with respect to* (2.4) *has one zero characteristic exponent in the extensive sense*;

(III) *every eigenvalue of* $g_y(t)$ *has real part different from zero for all* t.

Then, for ε *sufficiently small, system* (1.3) *has a unique almost periodic solution as* (2.5).

Remark 3 A direct proof of Theorem 4 has been given in Huang Yuanshi[11].

§ 6. The proof of Theorem 3

By Lemma 1, there is a $n\times(n-k-1)$ matrix fnnction $S(t, \alpha)$ which is almost periodic in t and each $\alpha_j (j=1, \cdots, k)$ such that $Q(t, \alpha)=\left(\frac{du}{dt}, \frac{\partial u}{\partial \alpha_1}, \cdots, \frac{\partial u}{\partial \alpha_k}, S\right)$ is a regular matrix function. Moreover, $\frac{dS}{dt}$ and $\frac{\partial S}{\partial \alpha_j}$ $(j=1, \cdots, k)$ exist, which are also almost periodic in t and each α_j. We write

$$Q^{-1}(t, \alpha)=P(t, \alpha)=\begin{pmatrix} P_1(t, \alpha) \\ P_2(t, \alpha) \\ P_3(t, \alpha) \end{pmatrix},$$

where P_1, P_2, P_3 are $1\times n$, $k\times n$, $(n-k-1)\times n$ matrix functions, respectively. They are all almost periodic in t and each α_j, meanwhile

$$PQ=\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}\left(\frac{du}{dt}, \frac{\partial u}{\partial \alpha}, S\right)=\begin{pmatrix} P_1\left(\frac{du}{dt}\right) & P_1\left(\frac{\partial u}{\partial \alpha}\right) & P_1 S \\ P_2\left(\frac{du}{dt}\right) & P_2\left(\frac{\partial u}{\partial \alpha}\right) & P_2 S \\ P_3\left(\frac{du}{dt}\right) & P_3\left(\frac{\partial u}{\partial \alpha}\right) & P_3 S \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}. \tag{6.1}$$

For the autonomous system (1.3), we are going to perform the following transformation

$$\begin{aligned} x&=u(\vartheta, \alpha)+S(\vartheta, \alpha)\xi, \\ y&=v(\vartheta, \alpha)-g_y^{-1}(\vartheta, \alpha)g_x(\vartheta, \alpha)S(\vartheta, \alpha)\xi+\eta, \end{aligned} \tag{6.2}$$

where ϑ is a real parameter, ξ, η are real $(n-k-1)$ and m dimension vectors respectively. We call this transformation "the generalized normal coordinate transformation". Evidently, this transformation can be performed at least in a neighberhood of (2.1). Furthermore, when $\varepsilon=0$, we have $\vartheta=0$, $\xi=0$, $\eta=0$. Therefore, we can assume that

$$\frac{d\vartheta}{dt}=1+\Theta(\vartheta, \xi, \eta, \alpha, \varepsilon),$$

where $\Theta(\vartheta, \xi, \eta, \alpha, \varepsilon)=0(|\xi|+|\eta|+|\varepsilon|)$ is continuously differentiable in all arguments, almost periodic in ϑ and each α_j uniformly in other arguments, and for any $\varepsilon>0$ and any ϑ, α, there is a constant $M>0$ such that

$$|\Theta(\vartheta, 0, 0, \alpha, \varepsilon)|<M\varepsilon.$$

As in the proof of Theorem 1, by performing transformation (6.2), system (1.3) is carried into system

$$\begin{aligned} \frac{d\vartheta}{dt}&=1+\Theta(\vartheta, \xi, \eta, \alpha, \varepsilon), \\ \frac{d\alpha}{dt}&=F(\vartheta, \xi, \eta, \alpha, \varepsilon), \\ \frac{d\xi}{dt}&=B(\vartheta, \alpha)\xi+B_1(\vartheta, \alpha)\eta+G(\vartheta, \xi, \eta, \alpha, \varepsilon), \\ \varepsilon\frac{d\eta}{dt}&=C(\vartheta, \alpha)\eta+H(\vartheta, \xi, \eta, \alpha, \varepsilon). \end{aligned}$$

From this, it follows that

$$\begin{aligned} \frac{d\alpha}{d\vartheta}&=\bar{F}(\vartheta, \xi, \eta, \alpha, \varepsilon), \\ \frac{d\xi}{d\vartheta}&=B(\vartheta, \alpha)\xi+B_1(\vartheta, \alpha)\eta+\bar{G}(\vartheta, \xi, \eta, \alpha, \varepsilon), \\ \varepsilon\frac{d\eta}{d\vartheta}&=C(\vartheta, \alpha)\eta+\bar{H}(\vartheta, \xi, \eta, \alpha, \varepsilon), \end{aligned}$$

which satisfies the conditions of Lemma 4 for ε sufficiently small. Theorem 3 is proved completely.

§ 7. Applications

In [12] and [13], S. Cerneau first consider some problems on the singularly perturbed system of the form

$$\frac{dx}{dt}=f(t,\ x,\ y,\ \varepsilon),\qquad \Omega\frac{dy}{dt}=g(t,\ x,\ y,\ \varepsilon),\tag{7.1}$$

where ε is a small parameter, x, y are respectively n, m dimension vectors, f, g are respectively n, m dimension vector functions, and $\Omega=\text{diag.}\ \{\varepsilon^{h_1},\ \cdots,\ \varepsilon^{h_m}\}$, $h_1,\ \cdots,\ h_m$ are integers, $0<h_1\leqslant\cdots\leqslant h_m$. Then F. A. Howes[14] considered the existence and asymptotic behavior of periodic, almost periodic and bounded solutions of system (7.1) when$\varepsilon\to 0$. He rlso considered the corresponding autonomous system

$$\frac{dx}{dt}=f(x,\ y,\ \varepsilon),\quad \Omega\frac{dy}{dt}=g(x,\ y,\ \varepsilon).$$

His principl tool is a famous iemma of M. Nagumo. However, in the nonautonomous case, his hypotheses for the first variation system resemble closely the hypotheses of L. Flatto–N. Levinson[1] for periodic solutions, and the hypotheses of K. W. Chang[4] for almost periodic solutions. And in the autonomous case, his hypotheses for the first variation system is similar to K. W. Chang [6]. It is easy to see that our results can be applied to systems (7.1) and (7.2), and the results of F. A. Howes[14] can be generalized immediately to the critical case.

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