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GENERALIZED SEMIGROUP ON (V, H, a)

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Abstract

Let V and H be two Hilbert spaces satisfying the imbedding relation $V\subsetneq H$. Let $-\mathscr{A}$: $V\to V'$ be the linear operator determined by $a(u, v)=\langle\mathscr{A}u, v\rangle$ for $u, v\in V$, where $a(u, v)$ is a continuous sesquilinear form on V satisfying

$$a(u, u)+\lambda|u|_H^2\geqslant c\|u\|_V^2$$

for $u\in V$ and some $\lambda\in R$ and $c>0$.

In this paper it is proved that $-\mathscr{A}$ is the generator of an analytic C_0-semigroup on V'. Furthermore, if $b(u, v)$ is a continuous sesquilinear form on $H\times V$ and $\mathscr{B}$: $H\to V$, the linear operator determined by $b(u, v)=\langle\mathscr{B}u, v\rangle$ for $u, v\in V$, then $-\mathscr{A}-\mathscr{B}$ is also the generator of c_0-semigroup on V'.

Also, similar results are proved on "inserted" spaces $V_\theta(\theta\geqslant -1)$ which are determined by the spectrum system of $\mathscr{A}$.

§ 1. $-\mathscr{A}$ is a Generator on V'

Let V be Hilbert space, H be the pivot space and $H=H'$ (H' is the dual space of H). We assume that V is dense and continuously imbedded in H, and $V\subsetneq H$. Hence, we have the inclusions

$$V\subsetneq H=H'\subsetneq V'. \tag{1.1}$$

Denote by $\|\cdot\|_V$ (resp. $|\cdot|_H$) the norm in V (resp. in H) and by $(.,.)_V$ (resp. $(.,.)_H$) the corresponding scalar product.

Let

$$a(u, v)=\text{continuous sesquilinear from on } V\times V \tag{1.2}$$

and assume that for some $c>0$

$$a(u, u)\geqslant c\|u\|_V^2, \ \forall u\in V \tag{1.3}$$

and that

$$a(u, v)=\overline{a(v, u)}, \ \forall u, v\in V. \tag{1.4}$$

By Lax-Milgram theorem, there exists a unique linear bounded operator $\mathscr{A}\in\mathscr{L}(V, V')$ with domain $D(\mathscr{A})=V$ dense in V' which is an isomorphism such that

$$a(u, v)=\langle\mathscr{A}u, v\rangle, \ \forall u, v\in V, \tag{1.5}$$

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where $\langle .,.\rangle$ denotes the scalar product between V' and V.

Let linear operator A with domain $D(A)=\mathscr{A}^{-1}H$ be the restriction of $\mathscr{A}$, that is

$$A=\mathscr{A}|_{D(A)}, \text{ where } D(A)=\mathscr{A}^{-1}H. \tag{1.6}$$

It is well known that $-A$ is the generator of an analytic semigroup e^{-At} on H. This result is very restrictive and does not allow us to consider unbounded control problem of practical importance. We are going to show that $-\mathscr{A}$ is a generator on V'.

Set $(u, v)_a=a(u, v)$ for $u, v\in V$. It is obvious, by hypotheses of (1.2), (1.3) and (1.4), that $(.,.)_a$ is a scalar product of V and it is equivalent to the original scalar product $(.,.)_V$. Hence $V_a \overset{\text{def.}}{=\!=} (V, (.,.)_a)$ is also a Hilbert space and $V'_a=V'$ (V'_a-the dual space of V_a). The dual norm of $\|\cdot\|_a$ in V'_a is defined by

$$|f|_{-a}=\sup_{v\in V-\{0\}}(|f(v)|/\|v\|_a), \ \forall f\in V'_a=V', \tag{1.7}$$

which is equivalent to the original dual norm $|f|_{V'}$ in V'. Denote by $(.,.)_{-a}$ the corresponding scalar product in V'_a. Obviously, T_t is a semigroup with generator $-\mathscr{A}$ on V' iff so is T_t on V'_a.

From $\langle\mathscr{A}u, v\rangle=a(u, v)=(u,v)_a$, $\forall u, v\in V$, it follows that

$$\mathscr{A} \text{ is Riesz map from } V_a \text{ to } V'_a. \tag{1.8}$$

Now we can show

$$(u, v)_H=(\mathscr{A}u, v)_{-a}, \ \forall u, v\in V. \tag{1.9}$$

In fact, for $u, v\in V$

$$\begin{aligned}(\mathscr{A}u, v)_{-a}&=(\mathscr{A}^{-1}\mathscr{A}u, \mathscr{A}^{-1}v)_a \quad (\mathscr{A} \text{ is Riesz map})\\ &=(u, \mathscr{A}^{-1}v)_a=\overline{a(\mathscr{A}^{-1}v, u)}\\ &=\overline{\langle\mathscr{A}\mathscr{A}^{-1}v, u\rangle}=\overline{\langle v, u\rangle}\\ &=(u, v)_H \quad (H \text{ is a pivot space}).\end{aligned}$$

Theorem 1. *The linear operator $-\mathscr{A}$: $V\to V'$ is the generator of an analytic semigroup (denote by $e^{-\mathscr{A}t}$) on V' which is strongly V'-continuous in t on the right half plane* $\mathrm{Re}\,t\geqslant 0$ *and strongly V'-analytic on* $\mathrm{Re}\,t>0$. *Moreover*

$$e^{-\mathscr{A}t}V'\subset D(\mathscr{A}^\infty)=\bigcap_{n=1}^{\infty}D(\mathscr{A}^n), \ \forall \ \mathrm{Re}\,t>0, \tag{1.10}$$

where $D(\mathscr{A}^n)$ is the domain of $\mathscr{A}^n$.

Proof From (1.1), $\cdots$, (1.4), by Lax-Milgram theorem and (1.9), it follows that $\mathscr{A}$, which is an operator from the subspace V of V'_a to V'_a itself, is dense definite, surjective, symmetric

$$(u, \mathscr{A}v)_{-a}=\overline{(\mathscr{A}v, u)_{-a}}=\overline{(v, u)_H}=(u, v)_H=(\mathscr{A}u, v)_{-a}$$

for $u, v\in D(\mathscr{A})(=V)$, and positive definite

$$(\mathscr{A}u, v)_{-a}=|u|^2_H\geqslant\delta_1|u|^2_{-a}, \ \forall u\in D(\mathscr{A})$$

for some $\delta_1>0$ (it follows from $H\subsetneqq V'_a$). Hence $\mathscr{A}$ is a positive definite self-adjoint operator from V'_a to itself.

By the spectral resolution of a self-adjoint operator, there exists $\{E_\lambda\}$ ($E_\lambda=0$, for $\lambda\leqslant\delta_1$), the resolution of the identity of $\mathscr{A}$ on V'_a such that

$$\mathscr{A}=(V'_a)\int_0^{+\infty}\lambda\,dE_\lambda, \tag{1.11}$$

where the integral is in the strongly sense and

$$V_a=D(\mathscr{A})=\left\{f\in V';\ \int_0^{+\infty}\lambda^2\,d|E_\lambda f|^2_{-a}<+\infty\right\}. \tag{1.12}$$

From (1.15) below, it is easy to show that

$$E_\lambda V'\subset D(\mathscr{A}^\infty)\text{ for }\lambda\in(-\infty,\ +\infty). \tag{1.13}$$

Setting

$$e^{-\mathscr{A}t}=(V'_a)\int_0^{+\infty}e^{-\lambda t}\,dE_\lambda \tag{1.14}$$

for Re $t\geqslant 0$, we may show that $e^{-\mathscr{A}t}$ (Re $t\geqslant 0$) is an analytic semigroup on V' by computation. Let us prove the generator of $e^{-\mathscr{A}t}$ is exactly $-\mathscr{A}$. First, we have for each $v\in D(\mathscr{A})=V$,

$$\left|\frac{e^{-\mathscr{A}t}v-v}{t}-(-\mathscr{A}v)\right|_{-a}$$

$$=\int_0^{+\infty}\left|\frac{e^{-\lambda t}-1}{\lambda t}+1\right|^2\lambda^2 d|E_\lambda v|^2_{-a}\to 0 \text{ as } t\to 0,\ \mathrm{Re}\,t\geqslant 0 \text{ (by Lebesgue theorem)}$$

and for each $f\in V'-V$ and each complex sequence $\{t_n\}$ ($t_n\to 0$ as $n\to+\infty$ and Ret_n $\geqslant 0$), we have

$$\varliminf_{n\to+\infty}\left|\frac{e^{-\mathscr{A}t_n}f-f}{t_n}\right|^2_{-a}$$

$$=\varliminf_{n\to+\infty}\int_0^{+\infty}\left|\frac{e^{-\lambda t_n}-1}{\lambda t_n}\right|^2\lambda^2 d|E_\lambda f|^2_{-a}$$

$$\geqslant\int_0^{+\infty}\varliminf_{n\to+\infty}\left|\frac{e^{-\lambda t_n}-1}{\lambda t_n}\right|^2\lambda^2 d|E_\lambda f|^2_{-a}\quad\text{(by Fatou theorem)}$$

$$=\int_0^{+\infty}\lambda^2 d|E_\lambda f|^2_{-a}=+\infty\quad(f\notin V).$$

Hence, by the definition of a generator, the generator of $e^{-\mathscr{A}t}$ is $-\mathscr{A}$.

Let us prove (1.10).

$$D(\mathscr{A}^n)=\left\{f\in V';\ \int_0^{+\infty}\lambda^{2n}d|E_\lambda f|^2_{-a}<+\infty\right\}\quad(n=1,\ 2,\ \cdots) \tag{1.15}$$

and For each $f\in V'$ and t: Re $t>0$ we have

$$\int_0^{+\infty}\lambda^{2n}d|E_\lambda e^{-\mathscr{A}t}f|^2_{-a}=\int_0^{+\infty}\lambda^{2n}|e^{-\lambda t}|^2 d|E_\lambda f|^2_{-a}$$

$$\leqslant\text{const}\int_0^{+\infty}d|E_\lambda f|^2_{-a}<+\infty.$$

Hence $e^{-\mathscr{A}t}f\in D(\mathscr{A}^n)$ and so $e^{-\mathscr{A}t}f\in D(\mathscr{A}^\infty)=\bigcup_{n=1}^{\infty}D(\mathscr{A}^n)$.

Remark 1. If for some $\lambda>0$ and $c>0$,

$$a(u,\ v)+\lambda|u|^2_H\geqslant c\|u\|^2_V,\ \forall u\in V, \tag{1.3$'$}$$

then Theorem 1 is still true.

§ 2. The "Inserted" Spaces V_θ

For each $\theta\geqslant -1$, let the subspace V_θ of V' be

$$V_\theta=\left\{f\in V';\ \int_0^{+\infty}\lambda^{\theta+1}d|E_\lambda f|_{-a}^2<+\infty\right\} \tag{2.1}$$

with the scalar product

$$(f,\ g)_\theta=\int_0^{+\infty}\lambda^{\theta+1}d(E_\lambda f,\ E_\lambda g)_{-a}. \tag{2.2}$$

Then V_θ are Hilbert spaces and $V_\theta\subsetneq V_{\theta'}$ for each pair $(\theta,\ \theta')$ which satisfies $\theta>\theta'\geqslant -1$. It is clear that $V_{-1}=V'$ and $(f,\ g)_{-1}=(f,\ g)_{-a}$ for $f,\ g\in V'$.

Without confusion, we may use the same sign E_λ to represent the operator $E_\lambda|_H$ which is E_λ restricted on H.

Lemma 1. *$\{E_\lambda\}$ is a resolution of the identity on H.*

Proof We need to show the following

1° $E_\lambda\in\mathscr{L}(H)$;

2° E_λ is an orthogonal projection on H;

3° $\lambda\leqslant\mu\Rightarrow E_\lambda E_\mu=E_\lambda$;

4° $|E_\lambda x-x|_H\to 0$ as $\lambda\to+\infty$ for $x\in H$. And $E_{-\infty}=0$;

5° $|E_\lambda x-E_\mu x|_H\to 0$ as $\lambda\to\mu+0$ for $x\in H$ and $\mu\in(-\infty,\ +\infty)$.

First we show that $E_\mu\in\mathscr{L}(H)$ for each $\mu\in(-\infty,\ +\infty)$. Indeed for $v\in V$,

$$|E_\mu v|_H^2=\int_0^{+\infty}\lambda d|E_\lambda E_\mu v|_{-a}^2$$
$$=\int_0^{\mu}\lambda d|E_\lambda v|_{-a}^2\leqslant\int_0^{+\infty}\lambda d|E_\lambda v|_{-a}^2=|v|_H^2,$$

that is

$$|E_\mu v|_H\leqslant|v|_H \text{ for } v\in V. \tag{2.3}$$

For each $x\in H$, there exists a sequence $\{v_n\}$ in V which satisfies $|v_n-x|_H\to 0$ as $n\to+\infty$ by $V\subsetneq H$ and hence $|v_n-x|_{V'_a}\to 0$ by $H\subsetneq V'_a$. So $|E_\mu v_n-E_\mu x|_{V'_a}\to 0$ as $n\to+\infty$ by $E_\mu\in\mathscr{L}(V'_a)$. By (2.3), we have

$$|E_\mu v_n-E_\mu v_m|_H\leqslant|v_n-v_m|_H\to 0 \text{ as } n,\ m\to+\infty.$$

Thus there is $y\in H$ so that $\|E_\mu v_n-y\|_H\to 0$ as $n\to+\infty$. So $y=E_\mu x$. Consequently, $|E_\mu v_n-E_\mu x|_H\to 0$ as $n\to+\infty$. Substituting $v=v_n$ into (2.3), we have

$$|E_\mu v_n|_H\leqslant|v_n|_H.$$

As $n\to+\infty$, it follows that

$$|E_\mu x|_H\leqslant|x|_H \text{ for } x\in H.$$

Tnus $E_\mu\in\mathscr{L}(H)$.

Now we show 2°. For $u,\ v\in V$

$$(E_\mu u,\ v)_H=(\mathscr{A}E_\mu u,\ v)_{-a}=\int_0^{\mu}\lambda d(E_\lambda u,\ E_\lambda v)_{-a}\quad(\text{by }(1.9))$$
$$=(u,\ \mathscr{A}E_\mu v)_{-a}=(u,\ E_\mu v)_H.$$

Hence, by the limiting process and 1°, E_μ is symmetric on H. From the definition of E_λ, we see $E_\mu^2=E_\mu$. From above, we have proved that E_μ is an orthogonal projection on H.

We omit the proof of 3°—5° here.

Lemma 2. *For each $\theta\geqslant-1$ and $f\in V'$ we have*

$$\int_0^{+\infty}\lambda^{\theta+1}d|E_\lambda f|_{-1}^2=\int_0^{+\infty}\lambda^\theta d|E_\lambda f|_H^2. \tag{2.4}$$

Proof If $f=E_Ng$, where $g\in V'$, then (2.5) can be obtained by computation. For each $f\in V'$, (2.4) may be followed by the limiting process.

From Lemma 2, it follows that

$$V_\theta=\left\{f\in V';\ \int_0^{+\infty}\lambda^\theta d|E_\lambda f|_H^2<+\infty\right\}, \tag{2.5}$$

$$(f,\ g)_\theta=\int_0^{+\infty}\lambda^\theta d(E_\lambda f,\ E_\lambda g)_H \text{ for } f,\ g\in V_\theta. \tag{2.6}$$

Lemma 3. *If $f\in V'$, then $f\in H$ iff $\lim\limits_{\lambda\to+\infty}|E_\lambda f|_H<+\infty$.*

Note that $|E_\lambda f|_H\leqslant|E_\mu f|_H$ for each pair $(\lambda,\ \mu):\lambda\leqslant\mu$.

Proof The "only if" part is obvious. The "if" part is given as follows. From $\lim\limits_{\lambda\to+\infty}|E_\lambda f|_H<+\infty$, it follows that

$$|E_\lambda f-E_\mu f|_H^2=|E_\lambda f|_H^2-|E_\mu f|_H^2\to0 \text{ as } \lambda\geqslant\mu\to+\infty$$

and hence there exists exactly only one element $x\in H$ such that

$$|E_\lambda f-x|_H\to0 \text{ as } \lambda\to+\infty.$$

Thus $|E_\lambda f-x|_{V'}\to0$ and so $f=x\in H$. ($\{E_\lambda\}$ is a resolution of identity on V', hence $|E_\lambda f-f|_{V'}\to0$ as $\lambda\to+\infty$ for $f,\ g\in V'$).

We have

$$V=V_1 \text{ and } (u,\ v)_v=(u,\ v)_1 \text{ for } u,\ v\in V; \tag{2.7}$$

(by (1.12) and (2.1))

$$V'=V_{-1} \text{ and } (f,\ g)_{V'}=(f,\ g)_{-1} \text{ for } f,\ g\in V', \tag{2.8}$$

$$H=V_0 \text{ and } (x,\ y)_H=(x,\ y)_0 \text{ for } x,\ y\in H, \tag{2.9}$$

$$D(A)=V_2. \tag{2.10}$$

The proof of (2.9). We have

$$\begin{aligned}H&=\{f\in V';\ \lim_{\lambda\to+\infty}|E_\lambda f|_H^2<+\infty\} \quad \text{(by Lemma 3)}\\&=\left\{f\in V';\ \int_0^{+\infty}d|E_\lambda f|_H^2<+\infty\right\}\\&=\left\{f\in V';\ \int_0^{+\infty}\lambda d|E_\lambda f|_{-1}^2<+\infty\right\}=V_0 \quad \text{(by Lemma 2).}\end{aligned}$$

For $x\in H$,

$$|x|_H^2=\int_0^{+\infty}d|E_\lambda x|_H^2=\int_0^{+\infty}\lambda d|E_\lambda x|_{-1}^2=|x|_0^2, \tag{2.11}$$

and hence for $x,\ y\in H$, $(x,\ y)_H=(x,\ y)_0$.

The proof of (2.10).

$$D(A)=\{f\in V'; f\in V \text{ and } f\in H\}$$
$$=\left\{f\in V';\ \int_0^{+\infty}\lambda d|E_\lambda f|_H<+\infty \text{ and } \int_0^{+\infty} d|E_\lambda \mathscr{A} f|_H^2<+\infty\right\}$$
$$=\left\{f\in V';\ \int_0^{+\infty}\lambda^2 d|E_\lambda f|_H^2<+\infty\right|=V_2$$
$$\left(\text{by } \int_0^{+\infty} d|E_\lambda \mathscr{A} f|_H^2=\int_0^{+\infty}\lambda^2 d|E_\lambda f|_H^2\right).$$
$$A=(H)\int_0^{+\infty}\lambda\, dE_\lambda. \tag{2.12}$$

Proof We have

$$(H)\int_0^{+\infty}\lambda dE_\lambda\subset(V_a')\int_0^{+\infty}\lambda\, dE_\lambda=\mathscr{A}$$

and the domain of $(H)\int_0^{+\infty}\lambda dE_\lambda$ is

$$D\left((H)\int_0^{+\infty}\lambda\, dE_\lambda\right)=\left\{f\in V';\ \int_0^{+\infty}\lambda^2 d|E_\lambda f|_H^2<+\infty\right\}=V_2=D(A).$$

Hence (2.12) is valid.

Remark. Similarly, we have

1. $E_\lambda|_{V_\theta}$ is a resolusion of identity on V_θ;
2. If $\mathscr{A}_0=\mathscr{A}|_{D(\mathscr{A}_0)}$, $D(\mathscr{A}_0)=\mathscr{A}^{-1}V_\theta$, then

$$\mathscr{A}_\theta=(V_\theta)\int_0^{+\infty}\lambda\, dE_\lambda$$

and $-\mathscr{A}_\theta$ is a generator of the analytic semigroup $e^{-\mathscr{A}_\theta t}$ on V_θ and

$$e^{-\mathscr{A}_\theta t}=(V_\theta)\int_0^{+\infty}e^{-\lambda t}\, dE_\lambda;$$

3. $e^{-\mathscr{A}_{\theta'} t}|_{V_\theta}=e^{-\mathscr{A}_\theta t}$ for each pair $(\theta,\ \theta'):\theta\geqslant\theta'\geqslant-1$.

§ 3. Some Properties of $e^{-\mathscr{A}t}$

Theorem 2. *For each $\theta\geqslant-1$, $e^{-\mathscr{A}t}$ given by* (1.14) *has the following properties:*

1° *For each t: $\mathrm{Re}t>0$, the range$^{-\mathscr{A}t}\subset V_\theta$ and $e^{-\mathscr{A}t}\in\mathscr{L}(V',\ V_\theta)$;*

2° *For each $f\in V'$ and each $\varepsilon\in(0,\ \pi/2)$*

$$\lim_{t\to0}\|t^{(1+\theta)/2}e^{-\mathscr{A}t}f\|_{V_\theta}=0;$$
$$(\mathrm{Re}t>0,\ |\arg t|<\pi/2-\varepsilon)$$

3° *For each $f\in V'$, $e^{-\mathscr{A}t}f$ is V_0-analytic function of t on* $\mathrm{Re}\, t>0$.

Proof From (1.14), 1°—3° can be obtained by computation.

§ 4. Perturbation Results

Suppose

$$b(u,\ v)=\text{continuous sesquilinear form on } V_\theta\times V \text{ for some } \theta\geqslant-1. \tag{4.1}$$

Set $\mathscr{B}$: $b(u, v)=\langle \mathscr{B}u, v\rangle$ for $u, v\in V$. By Lax-Milgram theorem, $\mathscr{B}\in\mathscr{L}(V_\theta, V')$, and we have the following perturbation results:

Theorem 3. *Under the assumptions of* (1.1, 2, 3, 4), (4.1) *and* $1>\theta\geqslant-1$, $-\mathscr{A}-\mathscr{B}$ *is a generator of a C_0-semigroup U_t on V' and it has the following properties*

1° rang $U_t\subset V_\theta$ *for* $t>0$;

2° $t^{(1+\theta)/2}U_t f\in C([0, +\infty), V_\theta)$ *for* $f\in V_\theta$;

3° $U_\theta|_{V_\theta}$ *is a C_0-Semigroup on V_θ with generator*

$$A_\theta=-(\mathscr{A}+\mathscr{B})|_{D(A_\theta)},\ \textit{where}\ D(A_\theta)=(\mathscr{A}+\mathscr{B})^{-1}V_\theta.$$

If $\theta=0$, then $U_t|_H$ is a C_0-semigroup on H with generator

$$A_0=-(\mathscr{A}+\mathscr{B})|_{D(A_0)},\ \textit{where}\ D(A_0)=(\mathscr{A}+\mathscr{B})^{-1}H.$$

Proof Let us consider the integral equation on V_θ

$$x(t)=e^{-\mathscr{A}t}f+(V_\theta)\int_0^t e^{-\mathscr{A}(t-s)}(-\mathscr{B})x(s)ds,\ \text{for}\ t>0, \tag{4.2}$$

where $f\in V'$ and the integral "$(V_\theta)\int_0^t$" is well-defined Bochner integral in V_θ. Set $\alpha=(1+\theta)/2(<1)$. Theorem 2 shows that $\|e^{-\mathscr{A}t}\|_{\mathscr{L}(V',V_\theta)}\leqslant\text{const}/t^\alpha$ for $t\in(0, t_1)$, t_1-an given arbitrary positive number and $e^{-\mathscr{A}t}f\in C((0, +\infty), V_\theta)$. Hence

$$\|e^{-\mathscr{A}(t-s)}\|_{\mathscr{L}(V_\theta)}\leqslant\text{const}(t-s)^{-\alpha}$$

and $e^{-\mathscr{A}(t-s)}\mathscr{B}$ is strongly continuous on $t>s\geqslant0$. So there is an unique solution

$$x(t)\in C((0, +\infty), V_\theta)\cap L_1(0, +\infty; V_\theta) \tag{4.3}$$

of (4.2) which may be represented by

$$x(t)=e^{-\mathscr{A}t}f+(V_\theta)\int_0^t G(t-s)e^{-\mathscr{A}s}\,\text{fds for}\ t>0 \tag{4.4}$$

and

$$G(t)=G_0(t)/t^\alpha\in\mathscr{L}(V_\theta)\ \text{for}\ t>0, \tag{4.5}$$

here $G_0(t)\in\mathscr{L}(V_\theta)$ and it is strongly V_θ-continuous on $t\in[0, +\infty)$, and $G(t)$ is the unique solution of

$$G(t)=e^{-\mathscr{A}t}(-\mathscr{B})+(V_\theta)\int_0^t e^{-\mathscr{A}(t-s)}(-\mathscr{B})G(s)ds,\ \text{for}\ t\geqslant0. \tag{4.6}$$

Set

$$x(t)=U_t f\ \text{for}\ f\in V'. \tag{4.7}$$

Hence from (4.3) we have

$$U_t=e^{-\mathscr{A}t}+(V_\theta)\int_0^t G(t-s)e^{-\mathscr{A}s}\,ds,$$

where the integral is in the strong sense.

Using Theorem 2, from (4.4, 5, 6) it follows 1°, 2° of Theorem 3.

From (4.2) and (4.7), by $V_\theta\subsetneq V'$ we have

$$x(t)=U_t f=e^{-\mathscr{A}t}f+(V')\int_0^t e^{-\mathscr{A}(t-s)}(-\mathscr{B})U_s f\,ds\ \text{for}\ t\geqslant0.$$

So the V'-continuity of $U_t f$ on the left on $t\in[0, +\infty)$ follows from the strong V'-continuity of $e^{-\mathscr{A}t}$ on $t\in[0, +\infty)$ and the fact that

$$-\mathscr{B}U_s f\in L_1(0, t_1; V')\quad(\text{by 2° of Theorem 3})$$

for $t_1>0$.

For the semigroup property of U_t we can easily show, similar to [1] p. 277,

$$(U_{t+s}-U_tU_s)f=(V_\theta)\int_0^t e^{-\mathscr{A}(t-\tau)}(-\mathscr{B})(U_{\tau+s}-U_\tau U_s)f\,d\tau$$

for $t,\ s\geqslant 0$. Hence, by the uniqueness of the solution,

$$U_{t+s}f=U_tU_sf \text{ for } t,\ s\geqslant 0 \text{ and } f\in V'.$$

Similary

$$G(t)=e^{-\mathscr{A}t}(-\mathscr{B})f+(V')\int_0^t e^{-\mathscr{A}(t-s)}(-\mathscr{B})G(s)ds$$

and $G(t)\in L_1(0,\ +\infty;\ V_\theta)$, and hence $G(t)$ is strongly V'-continuous on $t\in[0,\ +\infty)$.

Denote the generator of U_t by $\mathscr{K}$. From (4.4, 7) for $f\in V$

$$V'-\lim_{t\to+0}\frac{U_tf-f}{t}=V'-\lim_{t\to+0}\frac{e^{-\mathscr{A}t}-I}{t}f+V'-\lim_{t\to+0}\frac{1}{t}\int_0^t G(t-s)e^{-\mathscr{A}s}f\,ds$$
$$=-\mathscr{A}f+G(0)f=-\mathscr{A}f+(-\mathscr{B})f \quad (\text{by } (4.6)).$$

Hence

$$\mathscr{K}\supset-\mathscr{A}-\mathscr{B}. \tag{4.8}$$

Set

$$b_\mu(u,\ v)=\mu(u,\ v)_H+a(u,\ v)+b(u,\ v) \quad \text{for } u,\ v\in V.$$

We have

$$|u_\theta^2|=\int_0^{+\infty}\lambda^\theta d|E_\lambda u|_H^2$$
$$\leqslant\left(\int_0^{+\infty}(\lambda^\theta)^{1/\theta}d|E_\lambda u|_H^2\right)^\theta\left(\int_0^{+\infty}d|E_\lambda u|_H^2\right)^{1-\theta} \quad (\text{by Hölder inequality})$$
$$=|u|_1^{2\theta}|u|_H^{2(1-\theta)}\leqslant c_1|u|_V^{2\theta}|u|_H^{2(1-\theta)}.$$

There is a sufficiently large $\mu_0>0$ such that $\mu\geqslant\mu_0$ implies

$$cy^2/2-c_2y^{1+\theta}+\mu-\lambda=g(y)>0 \text{ for } y\geqslant 0$$

and $\mu-\mathscr{K}$ is one to one (by the property of generator). Hence for some $c_2>0$, we have

$$|b_\mu(u,\ u)|\geqslant|\mu|u|_H^2+a(u,\ u)|-|b(u,\ u)|$$
$$\geqslant\mu|u|_H^2+c\|u\|_V^2-\lambda|u|_H^2-\text{const}|u|_\theta\cdot|u|_V$$
$$\geqslant c\|u\|_V^2/2+\|u\|_V^2g(|u|_H^2/\|u\|_V^2)\geqslant c\|u\|_V^2/2 \text{ for } u\in V$$

and hence, by Lax-Milgram theorem $\mu+\mathscr{A}+\mathscr{B}$ is surjective for $\mu\geqslant\mu_0>0$. From (4.8), we have

$$\mu-\mathscr{K}\supset\mu+\mathscr{A}+\mathscr{B},$$

where $\mu+\mathscr{A}+\mathscr{B}$ is surjective and $\mu-\mathscr{K}$ is one to one, so $\mu-\mathscr{K}=\mu+\mathscr{A}+\mathscr{B}$, that is $\mathscr{K}=-\mathscr{A}-\mathscr{B}$.

Reference

[1] Cartain, R. F. & Pritchard, A. T., Lecture Notes in Control and Information Sciences, Vol. 8, Springer-Verlag, 1978.