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LINEAR GROUP OVER A CLASS OF RING R

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Abstract

Let R be a commutative ring[1] with identical element 1 and maximal ideal M_i where $i \in N$ and N is an ordered indicatrix set. Let the mapping

$$f: R \to \prod_{i \in N} R/M_i$$

be a ring homomorphism from R onto $\prod_{i \in N} R/M_i$, where $\prod_{i \in N} R/M_i$ is the direct product of residual fields R/M_i. In this paper, it is proved that if $A \in GL_n(R)$, then $A = BH_1 \cdots\cdots H_{k-1}$, where res $B=1$ and $H_1, \cdots, H_{k-1}$ are the symmetries. Furthermore, the bound of the positive integer number K is investigated. In particular, the author gives the smallest number $l(A)$ of symmetric factors in the products which expresses the elements of $G_n = \{A \in GL_n(R) \mid \det A = \pm 1\}$. Consequently, the $l(A)$ problems discussed in [2、3、4] are special cases of this paper.

§ 1. Introduction

In 1975, H. Radjavi[2] showed that $l(A) \leqslant 2n-1$, if $R=F$ is a field and $A \in G_n$. He also conjectured that $l(A) \geqslant 2n-2$. In 1978, F. S. Cater[3] showed that this conjecture is false and proved that $l(A) \leqslant n+2$. He conjectured furthermore that $l(A) \geqslant n+1$. However in 1979, D. Ž. Djoković and J. Malzan[4], negatived the above conjecture again and solved the problem on the field F thoroughly.

In this paper, we extend the results in [4] to a class of ring R, give the symmetry generation theorem for $GL_n(R)$ and the method of constructing symmetric factors. The method of this paper also can be applied to the discussion for the length of generation of $GL_n(R)$ by transvections. In particular, the Theorem 1 and Theorem 2 in our paper still hold if symmetries are replaced by transvections. Since local ring is included in ring R in our paper, as a matter of fact, this paper has given the lower bound of the length of generation by transvections. This question has never been solved over local ring before now.

§ 2. Main results

Let $A \in GL_n(R)$ and F_A be the set of all submodules in R^n that contain the

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columns of $I-A$. S_A is the submodule of F_A, containing the least number of generators. $m(S_A)$ denotes the least number of the vectors of S_A which generate S_A. $m_T(A)$ denotes the least number of the column vectors of $I\text{-}TAT^{-1}$ which generate $I\text{-}TAT^{-1}$ (where $T\in GL_n(R)$). We define the residual number of A as follows

$$\operatorname{res} A=\min\{m_T(A)\mid T\in GL_n(R)\}.$$

If R is a field, we have

$$m(S_A)=\operatorname{res} A=\operatorname{rank}(I-A).$$

We know that the homomorphism of group f_i: $GL_n(R)\to GL_n(R/M_i)$ will be induced by homomorphism f_i: $R\to R/M_i$.

An element $A\in GL_n(R)$ is called a symmetry if $\operatorname{res} A=1$ with $\det A=-1$. Assume that $A\in GL_n(R)$ can be expressed as $A=BH_1\cdots H_{k-1}$, where H_i's are symmetries and $\operatorname{res} B=1$. Then we write the least number K of the factors in the preceding expression as $K=\delta(A)$. $E_{ij}(b)$ denotes the $n\times n$ matrix with b in the (i, j) position and o's everywhere else.

If $A\in GL_n(R)$ and $\operatorname{res}A=r$, there exists $T\in GL_n(R)$ such that

$$TAT^{-1}=\left(\begin{array}{cc|c} D^{(r)} & & \\ \hline * & * & I^{(n-r)} \\ * & * & \end{array}\right). \tag{1}$$

(1) is called the normalization of A.

Obviously, if A is similar to matrix B i. e. $A\backsim B$, then we have $\delta(A)=\delta(B)$. Therefore, we will use the same notation for both similarity and equality.

Lemma 1. *For any $m\in N$, there exists an element $K\in\bigcap_{j<m} M_j$ but $K\notin\bigcup_{j>m} M_j$.*

Proof It is sufficient that take

$$K=f^{-1}(0,\ 0,\ \cdots,\ 0,\ 1+M_m,\ \cdots).$$

Lemma 2. *Let $D^{(r)}\in GL_r(R)$, and let $f_iD^{(r)}\neq aI$, $\forall a\in R/M_i$ and $\forall i\in N$. Then, there exists an element $T\in GL_r(R)$ such that the element at the position $(r', r-1)$ of $TD^{(r)}T^{-1}$ is* 1.

Proof Let $D^{(r)}=(a_{ij})$, $i, j=1, 2, \cdots, r$.

1) If $a_{r,r-1}$ is a unit, we take $T=I+E_{r-1,r-1}(a_{r,r-1}-1)$.

2) Now let us assume that $a_{r,r-1}$ is not a unit. Then, for any $t\in N$, from $f_tD^{(r)}\neq aI$ $\forall a\in R/M_t$, there exists $T_t\in GL_r(R/M_t)$ over field R/M_t, such that the element of $T_t(f_tD^{(r)})T_t^{-1}$ at the position $(r, r-1)$ is 1. Since the mapping f is a surjection from R onto $\prod_{i\in N} R/M_i$, there exists $T\in GL_r(R)$ such that $f_tT=T_t$, $\forall t\in N$. Therefore, the element of $TD^{(r)}T^{-1}$ at the position $(r, r-1)$ is a unit. This amount to the case of 1).

Lemma 3. *Let $A\in GL_n(R)$, and let $A=B_1\cdots B_k$, where* $\operatorname{res} B_i=1$. $i=1, 2, \cdots, k$. *Then $m(S_A)\leqslant k$.*

Proof 1) Let us first prove that $m(S_A)=m(S_{TAT^{-1}})$ for any $A,\ T\in GL_n(R)$. Let $d_1,\ d_2,\ \cdots,\ d_t$ be the least group of generators of S_A. Then

$$I-A=(d_1,\ d_2,\ \cdots,\ d_t)\begin{pmatrix} r_{11}\cdots r_{1n} \\ \cdots\cdots\cdots \\ r_{t1}\cdots r_{tn}\end{pmatrix},$$

where

$$d_i=\begin{pmatrix} d_{1i} \\ d_{2i} \\ \vdots \\ d_{ni}\end{pmatrix},\ i=1,\ 2,\ \cdots,\ t.$$

$$I\text{-}TAT^{-1}=T(I-A)T^{-1}=T(d_1,\ d_2,\ \cdots,\ d_t)\begin{pmatrix} r_{11}\cdots r_{1n} \\ \cdots\cdots\cdots \\ r_{t1}\cdots r_{tn}\end{pmatrix}T^{-1}$$

$$=(r_1,\ r_2,\ \cdots,\ r_t)\begin{pmatrix} r'_{11}\cdots r'_{1n} \\ \cdots\cdots\cdots \\ r'_{t1}\cdots r'_{tn}\end{pmatrix}. \tag{$*$}$$

Formula $(*)$ implies that $r_1,\ r_2,\ \cdots,\ r_t$ can generate the column vectors of $I\text{-}TAT^{-1}$. Hereby $m(S_{TAT^{-1}})\leqslant t=m(S_A)$.

Similarly, we can prove $m(S_A)\leqslant m(S_{TAT^{-1}})$. So $m(S_A)=m(S_{TAT^{-1}})$.

2) We prove Lemma by induction on k. The result is obvious for $k=1$.

Suppose the result hold for $k-1$. Clearly

$$B_k\sim B'_k=\begin{pmatrix} * & & & \\ * & 1 & & \\ \vdots & & \ddots & \\ * & & & 1\end{pmatrix},$$

i. e. there exists $T\in GL_n(R)$ such that $TB_kT^{-1}=B'_k$. So

$$TAT^{-1}=TB_1T^{-1}TB_2T^{-1}\cdots TB_kT^{-1}=B'_1B'_2\cdots B'_k,$$

where res $B'_i=1,\ i=1,\ 2,\ \cdots,\ k$. Set

$$B=B'_1B'_2\cdots B'_{k-1}=\begin{pmatrix} b_{11}\cdots b_{1n} \\ \cdots\cdots\cdots \\ b_{n1}\cdots b_{nn}\end{pmatrix}.$$

Then

$$TAT^{-1}=BB'_k=\begin{pmatrix} *\ b_{12}\cdots b_{1n} \\ \cdots\cdots\cdots\cdots \\ *\ b_{n2}\cdots b_{nn}\end{pmatrix}. \tag{$**$}$$

By the assumption of induction, it follows that $m(S_B)\leqslant k-1$. By formula $(**)$, it is easy to have $m(S_{TAT^{-1}})\leqslant k$. From 1), we obtain $m(S_A)\leqslant k$.

Lemma 4. *Let A be an element of $GL_n(R)$ with* res$A=r$. *If A satisfies any one of the following conditions:*

1) *In the normalization of A, $f_jD^{(r)}=aI$ for some $j\in N$, where $a\neq\pm1$.*

2) *Assume that in the normalization of A, $f_jD^{(r)}=I$,* res$(f_jA)=r$ *and $2\notin M_j$ for some $j\in N$. Then $\delta(A)\neq r$.*

Proof There is no loss of generality in assuming

$$A=\left(\begin{array}{c|c} D^{(r)} & \\ \hline * & I^{(n-r)} \end{array}\right).$$

Suppose $A=BH_1\cdots H_{r-1}$, where res $B=1$ and $H_i(i=1, 2, \cdots, r-1)$ is a symmetry. Then in field R/M_j,

$$f_jA=(f_jB)(f_jH_1), \cdots, (f_jH_{r-1})=\overline{B}\overline{H}_1\cdots\overline{H}_{r-1},$$

where $\overline{H}_i=f_jH_i$ is a symmetry and res $\overline{B}=$res $(f_jB)\leqslant 1$. From assumption we have $m(S_{f_jA})=r$. According to Lemma 3 we have res $\overline{B}=1$.

In field R/M_j, we consider $I-\overline{H}_{r-1}$. Since res $\overline{H}_{r-1}=1$, we can set

$$I-\overline{H}_{r-1}=\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}(b_1, \cdots, b_{r+1}, \cdots, b_n) \tag{$*$}$$

where c_i, $b_j\in R/M_j$.

(1) If the elements of the last $n-r$ columns of $I-\overline{H}_{r-1}$ are not all equal to zero. There is no loss of generality in assuming that the elements of the $r+1$ column are not all equal to zero. So $b_{r+1}\neq 0$. Set

$$T^{-1}=\left(\begin{array}{ccc|ccccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \hline \frac{-b_1}{b_{r+1}} & \cdots & \frac{-b_r}{b_{r+1}} & 1 & \frac{-b_{r+2}}{b_{r+1}} & \cdots & \frac{-b_n}{b_{r+1}} & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \end{array}\right).$$

Then

$$T(I-\overline{H}_{r-1})T^{-1}=I-T\overline{H}_{r-1}T^{-1}=\left(\begin{array}{ccc|cccc} 0 & \cdots & 0 & * & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & 0 & \cdots & 0 \end{array}\right).$$

Therefore, we have

$$T\overline{H}_{r-1}T^{-1}=\left(\begin{array}{c|cccc} I^{(r)} & * & & & \\ & \vdots & & & \\ \hline & * & & & \\ & \vdots & 1 & & \\ & \vdots & & \ddots & \\ & * & & & 1 \end{array}\right)=\overline{H}'_{r-1}.$$

Since

$$f_jA=\left(\begin{array}{ccc|c} a & & & \\ & \ddots & & \\ & & a & \\ \hline & * & & I^{(n-r)} \end{array}\right),$$

We have

$$T(f;A)T^{-1}=\left(\begin{array}{ccc|c} a & & & \\ & \ddots & & \\ & & a & \\ \hline & C & & I^{(n-r)} \end{array}\right)=A_1.$$

But,

$$A_1=T\overline{B}T^{-1}T\overline{H}_1T^{-1}\cdots T\overline{H}_{r-1}T^{-1}=\overline{B}'\overline{H}'_1\cdots\overline{H}'_{r-1},$$

$$A_1\overline{H}'_{r-1}=\overline{B}\overline{H}'_1\cdots\overline{H}'_{r-2}.$$

From Lemma 3 we know $m(S_{A_1\overline{H}'_{r-1}})\leqslant r-1$. But,

$$A_1\overline{H}'_{r-1}=\left(\begin{array}{ccc|cccc} a & & & * & & & \\ & \ddots & & \vdots & & & \\ & & a & * & & & \\ \hline & & & * & & & \\ & C & & \vdots & 1 & & \\ & & & \vdots & & \ddots & \\ & & & * & & & 1 \end{array}\right).$$

Then, if $a\neq 1$, it is obvious that $m(S_{A_1\overline{H}'_{r-1}})\geqslant r$; if $a=1$ from the hypothesis that $\operatorname{res}(f;A)=r$, we have $\operatorname{res}(A_1)=r$. Furthermore, the r column vectors of C are linear independent. Hereby $m(S_{A_1\overline{H}'_{r-1}})\geqslant r$. This contradicts $m(S_{A_1\overline{H}'_{r-1}})\leqslant r-1$.

(2) If all elements of the last $n-r$ columns of $I-\overline{H}_{r-1}$ are equal to zero. There is no loss of generality in assuming that the elements of the first column of $I-\overline{H}_{r-1}$ are not all equal to zero. Then we have $b_1\neq 0$, $b_{r+1}=\cdots=b_n=0$ in the formula (*). Put

$$T^{-1}=\left(\begin{array}{cccc|c} 1 & \frac{-b_2}{b_1} & \cdots & \frac{-b_r}{b_1} & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \hline & & & & I^{(n-r)} \end{array}\right).$$

Then

$$T(I-\overline{H}_{r-1})T^{-1}=I-T\overline{H}_{r-1}T^{-1}=\begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix}.$$

So we can let

$$T\overline{H}_{r-1}T^{-1}=\left(\begin{array}{cccc|c} -1 & & & & \\ * & 1 & & & \\ \vdots & & \ddots & & \\ * & & & 1 & \\ \hline * & & & & \\ \vdots & & & & I^{(n-r)} \\ * & & & & \end{array}\right)=\overline{H}'_{r-1}.$$

Then

$$T(f;A)T^{-1}=\left(\begin{array}{ccc|c} a & & & \\ & \ddots & & \\ & & a & \\ \hline & C' & & I^{(n-r)} \end{array}\right)=A_1,$$

$$A_1 = TBT^{-1}T\bar{H}_1T^{-1}\cdots T\bar{H}_{r-1}T^{-1} = \bar{B}'\bar{H}'_1\cdots\bar{H}'_{r-1},$$
$$A_1\bar{H}'_{r-1} = \bar{B}'\bar{H}'_1\cdots\bar{H}'_{r-2}.$$

According to Lemma 3 we have $m(S_{A_1\bar{H}'_{r-1}}) \leqslant r-1$. But

$$A_1\bar{H}'_{r-1} = \left(\begin{array}{cccc|c} -a & & & & \\ * & a & & & \\ \vdots & & \ddots & & \\ \vdots & & & a & \\ \hline * & & & & \\ \vdots & & C_1 & & I^{(n-r)} \\ * & & & & \end{array}\right).$$

i) If $a \neq \pm 1$, $m(S_{A_1\bar{H}'_{r-1}}) = r$. This contradicts $m(S_{A_1\bar{H}'_{r-1}}) \leqslant r-1$.

ii) If $a=1$ and $2 \notin M_j$, then

$$I - A_1\bar{H}'_{r-1} = \left(\begin{array}{cc|c} 2 & & \\ -* & & \\ \vdots & & \\ -* & & \\ \hline -* & & \\ \vdots & -C_1 & \\ -* & & \end{array}\right). \qquad (**)$$

From the hypothesis of Lemma i. e. $m(S_{f_jA}) = r$ and from 1) of Lemma 3, we obtain $m(S_{f_jA}) = m(S_{A_1}) = r$. Then we know that r column vectors in C' are linear independent. Since $C' = \begin{pmatrix} *' & \\ \vdots & C_1 \\ *' & \end{pmatrix}$, $r-1$ column vectors in C_1 are linear independent. Hence $m(S_{A_1\bar{H}'_{r-1}}) = r$ by formula $(**)$. This contradicts $m(S_{A_1\bar{H}'_{r-1}}) \leqslant r-1$. So $\delta(A) \neq r$.

Theorem 1. *Let A be an element of $GL_n(R)$ with $\operatorname{res}A = r$ and $m(S_A) = t$. In the normalization of A, if $f_jD^{(r)} \neq aI$ $\forall a \in R/M_j$ and $\forall j \in N$, then*

$$A = BH_1\cdots H_{q-1},$$

where $q \leqslant r$, $\operatorname{res}B = 1$, $H_i(i=1, \cdots, q-1)$ is a symmetry and $t \leqslant \delta(A) \leqslant r$. In particular, we have $\delta(A) = t$, if there exists some $i \in N$ such that $\operatorname{res}(f_iA) = r$.

Proof We use induction on $r = \operatorname{res} A$. If $r=1$, there is nothing to prove. So we assume that the results hold for all A with $\operatorname{res} A \leqslant r-1$.

Since $\operatorname{res} A = r$, there is no loss of generality in assuming that

$$A = \left(\begin{array}{c|c} D^{(r)} & \\ \hline * & I^{(n-r)} \end{array}\right).$$

By Lemma 2, there exists $T^{(r)} \in GL_r(R)$ such that 1 lies at the position $(r, r-1)$ of $T^{(r)}D^{(r)}T^{(r)-1} = D_1^{(r)}$. Set

$$T = \left(\begin{array}{c|c} T^{(r)} & \\ \hline & I^{(n-r)} \end{array}\right).$$

Then

$$TAT^{-1}=\left(\begin{array}{c|c} D_1^{(r)} & \\ \hline * & I^{(n-r)} \end{array}\right)=\left(\begin{array}{ccc|c} * \cdots\cdots & a'_{1,r-1} & * & \\ \vdots & \vdots & \vdots & \\ * \cdots\cdots & a'_{r-1,r-1} & * & \\ * \cdots\cdots & 1 & * & \\ \hline * \cdots\cdots & a'_{r+1,r-1} & * & \\ \vdots & \vdots & \vdots & I^{(n-r)} \\ * \cdots\cdots & a'_{n,r-1} & * & \end{array}\right)=A_1.$$

Set

$$T_1=\left(\begin{array}{ccc|c} 1 & & -a'_{1,r-1} & \\ & \ddots & \vdots & \\ & 1 & -a'_{r-1,r-1} & \\ & & 1 & \\ \hline & & -a'_{r+1,r-1} & \\ & & \vdots & I^{(n-r)} \\ & & -a'_{n,r-1} & \end{array}\right).$$

Then

$$T_1A_1T_1^{-1}=\left(\begin{array}{cccc|c} a_{11}\cdots a_{1,r-2} & 0 & a_{1r} & \\ \vdots \quad\quad \vdots & \vdots & \vdots & \\ a_{r1}\cdots a_{r,r-2} & 1 & a_{rr} & \\ \hline * \cdots\cdots * & 0 & * & \\ \vdots \quad\quad \vdots & \vdots & \vdots & I^{(n-r)} \\ * \cdots\cdots * & 0 & * & \end{array}\right)=A_2.$$

Set

$$H'_{q-1}=\left(\begin{array}{cccc|c} 1 & & & & \\ & \ddots & & & \\ & 1 & 0 & 0 & \\ & 0 & 0 & 1 & \\ & 0 & 1 & 0 & \\ \hline & & & & I^{(n-r)} \end{array}\right).$$

We have

$$A_2H'_{q-1}=\left(\begin{array}{cc|c} * \cdots * & & \\ \vdots \quad \vdots & & \\ * \cdots * & 1 & \\ \hline * \cdots * & & \\ \vdots \quad \vdots & & I^{(n-r)} \\ * \cdots * & & \end{array}\right)=\left(\begin{array}{c|c} D_1^{(r-1)} & \\ \hline * \cdots * & \\ \vdots \quad \vdots & I^{(n-r+1)} \\ * \cdots * & \end{array}\right),$$

where

$$D_1^{(r-1)}=\begin{pmatrix} a_{11} & \cdots & a_{1,r-2} & a_{1r} \\ \vdots & & \vdots & \vdots \\ a_{r-1,1} & \cdots & a_{r-1,r-2} & a_{r-1,r} \end{pmatrix}.$$

i) If $f_jD_1^{(r-1)}\neq aI$, $\forall j\in N$ and $\forall a\in R/M_j$ by the assumption of induction, it follows that

$$A_2H'_{q-1}=B'H'_1\cdots H'_{q-2},$$

where

$$q-1\leqslant \operatorname{res}(A_2H'_{q-1})\leqslant r-1.$$

So

$$A_2=B'H'_1\cdots H'_{q-2}H'_{q-1}.$$

Since $A_2=T_1TAT^{-1}T_1^{-1}$, putting $T_2^{-1}=T_1T$, we have

$$A=T_2B'T_2^{-1}T_2H_1'T_2^{-1}\cdots T_2H_{q-1}'T_2^{-1}=BH_1\cdots H_{q-1},$$

where $\mathrm{res}B=1$, H_i is a symmetry, $q\leqslant r$. So $t\leqslant\delta(A)\leqslant r$ by Lemma 3.

ii) Assume that $f_jD_1^{(r-1)}=b_jI$ $b_j\in R/M_j$, $\forall j\in N_1$ and $f_iD_1^{(r-1)}\neq b_iI$ $\forall b_i\in R/M_i$, $\forall i\in N_2$, where $N_1\cup N_2=N$, $N_1\cap N_2=\emptyset$. Then, we replace H_{q-1}' by

$$H_{q-1}''=\left(\begin{array}{cccc|c} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -k & 0\ \ 1 & \\ & & k & 1\ \ 0 & \\ \hline & & & & I^{(n-r)} \end{array}\right),$$

where $k\in\bigcap_{i\in N_2}M_i$, $k\notin\bigcup_{i\in N_1}M_i$. It follows that

$$A_2H_{q-1}''=\left(\begin{array}{c|c} D_2^{(r-1)} & \\ \hline \begin{matrix} *&\cdots&* \\ \vdots & & \vdots \\ *&\cdots&* \end{matrix} & I^{(n-r+1)} \end{array}\right).$$

If $i\in N_2$, we have

$$f_i(H_{q-1}'')=f_i(H_{q-1}')=H_{q-1}'.$$

Hereby, we may deduce

$$f_i(A_2H_{q-1}'')=f_i(A_2H_{q-1}').$$

Therefore

$$f_iD_2^{(r-1)}=f_iD_1^{(r-1)}\neq b_iI,\ \forall b_i\in R/M_i.$$

Assume that $i\in N_1$. Since $a_{r-1,r-2}+ka_{r-1,r}$ lies at the position $(r-1,\ r-2)$ of $D_2^{(r-1)}$, and $a_{r-1,r}$ lies at the diagonal of $D_1^{(r-1)}$, we have

$$f_i(a_{r-1,r-2}+ka_{r-1,r})=f_i(ka_{r-1,r})\neq 0.$$

Hence

$$f_iD_2^{(r-1)}\neq b_iI,\ \ \forall i\in N,\ \forall b_i\in R/M_i.$$

This amount to the case of i).

If $\mathrm{res}(f_iA)=r$, obviously, we have $t=r$. Hence $\delta(A)=t$.

Theorem 2. *Assumethat $A\in GL_n(R)$, $m(S_A)=t$ and $\mathrm{res}A=r$. We have $A=BH_1\cdots H_q$ in the normalization of A, if $f_jD^{(r)}=b_jI$ for some $j\in N$, where $\mathrm{res}B=1$, H_i is a symmetry, $q\leqslant r$ and $t\leqslant\delta(A)\leqslant r+1$. In particular, if $b_j\neq\pm1$ or $b_j=1$ but $\mathrm{res}(f_jA)=r$ and $2\notin M_j$, we have $\delta(A)=r+1$.*

Proof There is no loss of generality in assuming that

$$A=\left(\begin{array}{c|c} D^{(r)} & \\ \hline * & I^{(n-r)} \end{array}\right),$$

where

$$D^{(r)}=\begin{pmatrix} a_{11}&\cdots&a_{1r} \\ \vdots & & \vdots \\ a_{r1}&\cdots&a_{rr}\end{pmatrix}.$$

Set

$$H_q=\left(\begin{array}{ccccc|c} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ \hline & & & & & I^{(n-r)} \end{array}\right).$$

Then

$$AH_q=\left(\begin{array}{c|c} D_1^{(r)} & \\ \hline * & I^{(n-r)} \end{array}\right).$$

(1) If $f_jD_1^{(r)}\neq b_jI$, $\forall j\in N$, $\forall b_j\in R/M_j$, then $AH_q=BH_1\cdots H_{q-1}$ by Theorem 1, where res $B=1$, H_i is a symmetry, $q\leqslant \text{res}(AH_q)\leqslant r$. So $A=BH_1\cdots H_q$ and $t\leqslant\delta(A)\leqslant r+1$ by Lemma 3.

(2) Assume that $f_jD_1^{(r)}=b_jI$, $\forall j\in N_1$, where $b_j\in R/M_j$ and $f_jD_1^{(r)}\neq b_jI$, $\forall b_j\in R/M_j$ and $\forall j\in N_2$, where $N_1\cup N_2=N$, $N_1\cap N_2=\phi$. We replace H_q by

$$H_q'=\left(\begin{array}{ccccc|c} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -k & 0 & 1 & \\ & & k & 1 & 0 & \\ \hline & & & & & I^{(n-r)} \end{array}\right),$$

where $k\in\bigcap_{i\in N_2}M_i$, $k\notin\bigcup_{i\in N_1}M_i$. It follows that

$$AH_q'=\left(\begin{array}{c|c} D_2^{(r)} & \\ \hline * & I^{(n-r)} \end{array}\right)$$

and $a_{r,r-2}-ka_{r,r-1}+ka_{rr}$ lies at the position $(r, r-2)$ of $D_2^{(r)}$. From

$$f_j(a_{r,r-2}-ka_{r,r-1}+ka_{rr})=f_j(-ka_{r,r-1})\neq 0,\ \forall j\in N_1$$

and

$$f_jD_2^{(r)}=f_jD_1^{(r)}\neq b_jI,\ \forall j\in N_2 \text{ and } \forall b_j\in R/M_j,$$

we deduce that

$$f_iD_2\neq b_iI,\ \forall i\in N \text{ and } \forall b_i\in R/M_i.$$

This amount to the case of (1).

If $b_j\neq\pm1$ or $b_j=1$, $\text{res}(f_jA)=r$ and $2\notin M_j$, it may be seen that $r=t$ and $\delta(A)=r+1$ by Lemma 4.

Now, assume that G_n denotes the subgroup generated by all the symmetries of $GL_n(R)$, i. e.,

$$G_n=\{A\in GL_n(R)\,|\det A=\pm1\}.$$

If $1\neq-1$ in R, we define $g(A)$ for $A\in G_n$ by

$$g(A)=\begin{cases}0 & \text{when } \det A=(-1)^{\text{res}A},\\ 1 & \text{otherwise}.\end{cases}$$

Theorem 3. *Let A be an element of G_n with* res $A=r$ *and* $m(S_A)=t$.

(1) *If* $1\neq -1$, *we have* $t\leqslant l(A)\leqslant r+2$; *if* $1=-1$, *we have* $t\leqslant l(A)\leqslant r+1$.

(2) *Let* $r=t$. *In the normalization of* A, *if* $f_iD\neq b_iI$, $\forall i\in N$ *and* $\forall b_i\in R/M_i$, *we have*

$$l(A)=\begin{cases} r+g(A) & \text{if } 1\neq -1,\\ r & \text{if } 1=-1. \end{cases}$$

Proof (1) If $1\neq -1$, we first point out that if $\det B=1$ and $\operatorname{res} B=1$, then B can be expressed as the product of two symmetries. Since there exists $T\in GL_n(R)$ such that

$$TBT^{-1}=\left(\begin{array}{c|c} 1 & \\ \hline * & \\ \vdots & I^{(n-1)} \\ * & \end{array}\right)=\left(\begin{array}{c|c} -1 & \\ \hline -* & \\ \vdots & I^{(n-1)} \\ -* & \end{array}\right)\left(\begin{array}{c|c} -1 & \\ \hline & \\ & I^{(n-1)} \end{array}\right)=H_1'H_2',$$

we have $B=T^{-1}H_1'TT^{-1}H_2'T=H_1H_2$. Let $1=-1$. If $\operatorname{res} B=1$ and $\det B=1$, then B is a symmetry. Therefore the result of (1) can be proved by Theorems 1, 2, and Lemma 3 immediately.

(2) From Theorem 1 we have

$$A=BH_1\cdots H_{r-1}, \tag{*}$$

where $H_i(i=1, \cdots, r-1)$ is a symmetry, $\operatorname{res} B=1$.

If $1=-1$, B is a symmetry. Then, from Lemma 3 we deduce $l(A)=r$.

Now, let us assume $1\neq -1$. We proceed in two steps.

1) $\det A=1$.

If r is even, then $\det A=(-1)^r$, $g(A)=0$. According to formula (*), we have $\det B=-1$. So B is a symmetry. By Lemma 3, we have $l(A)=r+g(A)$.

If r is odd, then $\det A\neq(-1)^r$, $g(A)=1$. According to formula (*), we obtain $\det B=1$. But then B can be represented as a product of two symmetries. Therefore, from Lemma 3 and $\det A=1$ we deduce $l(A)=r+g(A)$.

2) $\det A=-1$.

If r is even, then $\det A\neq(-1)^r$ and $g(A)=1$. From formula (*) we obtain $\det B=1$. Then B can be expressed as a product of two symmetries. Therefore, according to Lemma 3 and $\det A=-1$, we deduce that $l(A)=r+g(A)$.

If r is ond, then, from formula (*) we have $\det B=-1$. B is a symmetry. By Lemma 3, $l(A)=r$. Hence, from $\det A=(-1)^r$ and $g(A)=0$, we have $l(A)=r+g(A)$.

Theorem 4. *Let* A *be an element of* G_n *with* $\operatorname{res} A=r$ *and* $m(S_A)=t$. *Then*

(1) *Let* $1\neq -1$. *We have* $l(A)=r+2-g(A)$ *in the normalization of* A, *if* A *satisfies any one of the following conditions*:

(a) *There exists a* $j\in N$ *such that*

$$f_jD^{(r)}=aI \text{ where } a\neq \pm 1;$$

(b) *There exists a* $j\in N$ *such that*

$$f_jD^{(r)}=I \text{ and } \operatorname{res}(f_jA)=r \text{ where } 2\notin M_j.$$

(2) *Let* $1\neq -1$. *We have* $l(A)=r$, *if* $D^{(r)}=-I$.

(3) *Let* $1=-1$. *We have* $l(A)=r+1$, *if there exists a* $j\in N$ *such that* $f_jD^{(r)}=b_jI$ *where* $b_j\neq 1$.

(4) *Let* $1=-1$. *We have* $l(A)=r$, *if* $D^{(r)}=I$ *and* $t=r$.

Proof It is easy to verify that $t=r$.

(1) By Theorem 2, we have

$$A=BH_1\cdots H_r \ (**),$$

where $\mathrm{res}B=1$ and $H_i(i=1,\cdots,r)$ is a symmetry.

There are two cases:

i) $\det A=1$.

If r is odd, then $\det A\neq(-1)^r$ and $g(A)=1$. By formula $(**)$, we have $\det B=-1$, i. e., B is a symmetry. Therefore, from Lemma 4, we deduce $l(A)=r+2-g(A)$.

If r is even, then $\det A=(-1)^r$ and $g(A)=0$. Thus, $\det B=1$ and B can be represented as a product of two symmetries. According to Lemma 4 and $\det A=1$ we have $l(A)=r+2-g(A)$.

ii) $\det A=-1$.

If r is odd, then $\det A=(-1)^r$ and $g(A)=0$. By formula $(**)$ we have $\det B=1$. So B can be represented as a product of two symmetries. Then according to Lemma 4 we have $l(A)=r+2-g(A)$.

If r is even, then $\det A\neq(-1)^r$, $g(A)=1$ and $\det B=-1$ (i. e., B is a symmetry). By Lemma 4, we obtain $l(A)=r+2-g(A)$.

(2) If $D^{(r)}=-I$, then, we have

$$\left(\begin{array}{ccc|c} -1 & & & \\ & -1 & & \\ & & \ddots\ -1 & \\ \hline * & \cdots & * & \\ \vdots & \ddots & \vdots & I^{(n-r)} \\ * & \cdots & * & \end{array}\right)=\left(\begin{array}{ccc|c} -1 & & & \\ & 1 & & \\ & & \ddots\ 1 & \\ \hline * & & & \\ \vdots & & & I^{(n-r)} \\ * & & & \end{array}\right)\left(\begin{array}{ccc|c} 1 & & & \\ & -1 & & \\ & & \ddots\ 1 & \\ \hline & * & & \\ & \vdots & & I^{(n-r)} \\ & * & & \end{array}\right)\cdots\left(\begin{array}{ccc|c} 1 & & & \\ & 1 & & \\ & & \ddots\ -1 & \\ \hline & & * & \\ & & \vdots & I^{(n-r)} \\ & & * & \end{array}\right)$$

$$=H_1H_2\cdots H_r,$$

i. e., $l(A)=r$.

(3) According to Theorem 2 we have $A=B\,H_1\cdots H_r$. Because of $1=-1$, B is a symmetry. Then $l(A)=r+1$ by Lemma 4.

(4) The proof of (4) is similar to the proof of (2).

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