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OSCILLATORY PROPERTY OF *N*-TH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

The authors study oscillatory property of nonlinear functional differential equation

$$L_n x(t) + p(t) f(x(t), x(g(t))) = r(t), \tag{1}$$

where $L_n x(t)$ is an n–th order linear differential operator defined by

$$L_0 x(t) = x(t),$$

$$L_k x(t) = \frac{d}{dt}(a_{k-1}(t) L_{k-1} x(t)), \quad k=1, 2, \cdots, n.$$

Sufficient conditions are obtained which guarantee that all continuable solutions of (1) are oscillatory or tend to zero as $t \to \infty$.

§ 1. Introduction

In this paper, we establish criteria for oscillation or nonoscillation of solutions of n–th order functional differential equations of the type

$$L_n x(t) + p(t) f(x(t), x(g(t))) = r(t), \tag{1}$$

where L_k is a differential operator defined recursively by

$$L_0 x(t) = x(t),$$

and for $k=1, \cdots, n$,

$$L_k x(t) = \frac{d}{dt}\{a_{k-1}(t) L_{k-1} x(t)\}.$$

The functions $p(t) \geqslant 0$, $a_{k-1}(t) > 0$ and $r(t)$ are assumed to be real valued and continuous on $[0, \infty)$. The function $f(u, v)$ is also real valued and continuous on $\mathbf{R}^2$, and $a_0(t)$ is both bounded and bounded away from 0.

The results we obtain extend and improve work of J. Bradley[1], K. L. Chiou[2], G. C. T. Kung[4], W. E. Mahfoud[7], S. B. Norkin[9] and P. Waltman[10] in nonforced second order differential equations and of G. Ladas[5], D. L. Lovelady[6] and W. E. Mahfoud[8] in nonforced higher order differential equations and provide criteria for determining oscillatory properties of solutions. Thus we will only be concerned with the continuable solutions of equation (1).

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A real valued continuous function $y(t)$ is said to be oscillatory if it has arbitrarily large zeros. A solution $x(t)$ of (1) is said to be oscillatory if for sufficiently large T, $x(t)\not\equiv 0$ on $[T, \infty)$ and it has arbitrarily large zeros, otherwise, it is said to be nonoscillatory. Equation (1) is called oscillatory if every solution is oscillatory.

We assume throughout the paper the following properties of the coefficient functions and $f(u, v)$.

(i) $\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{a_k(s)}=\infty$, $k=1, \cdots, n-1$;

(ii) $\lim_{t\to\infty} g(t)=\infty$;

(iii) If $uv>0$, then $uf(u, v)>0$;

(iv) $\liminf_{u,v\to\infty} f(u, v)>0$, $\limsup_{u,v\to-\infty} f(u, v)<0$;

(v) $\lim_{t\to\infty}\int_{t_0}^{t} p(s)ds=\infty$.

Moreover, we define recursively the functions $w_k(t)$, $\phi_k(t)$, and $\psi_k(t)$, if they exist, by

$$w_0(t)=\int_t^\infty r(s)ds,\quad w_{k+1}(t)=\int_t^\infty \frac{w_k(s)}{a_{n-k-1}(s)}ds;$$

$$\phi_0(t)=\int_t^\infty p(s)ds,\quad \phi_{k+1}(t)=\int_t^\infty \frac{\phi_k(s)}{a_{n-k-1}(s)}ds;$$

$$\psi_0(t)=1,\quad \psi_{k+1}(t)=\int_{t_0}^t \frac{\psi_k(s)}{a_{k+1}(s)}ds.\quad k=0, 1, \cdots, n-2,$$

and we assume that

(vi) $w_{n-1}(t)$ is defined on $[0, \infty)$.

§ 2. Main Result

Lemma 1. *If* (i)—(vi) *hold and $x(t)$ is a nonoscillatory solution of equation* (1), *then for $k=0, 1, \cdots, n-1$,*

$$\lim_{t\to\infty} a_k(t)L_kx(t)=0.$$

Proof Let $x(t)$ be a nonoscillatory solution of equation (1). Suppose that $x(t)$ is eventually positive. A similar argument establishes the result that $x(t)$ is eventually negative. By (ii), there is sufficiently large T such that both $x(t)>0$ and $x(g(t))>0$ for $t\geq T$. Integrating both sides of equation (1), we have

$$a_{n-1}(t)L_{n-1}x(t)-a_{n-1}(T)L_{n-1}x(T)+\int_T^t p(s)f(x(s), x(g(s)))ds=\int_T^t r(s)ds. \tag{2}$$

Then by (iii) and (vi), there exists β_{n-1} such that $\lim_{t\to\infty} a_{n-1}(t)L_{n-1}x(t)=\beta_{n-1}$, where β_{n-1} is finite or $-\infty$. Now, we claim $\beta_{n-1}=0$.

In fact, suppose that $\beta_{n-1}<c<0$, where c is a constant, then there is $T_1\geq T$ such that $a_{n-1}(t)L_{n-1}x(t)\leq c$ for $t\geq T_1$.

If $n=1$, obviously, $a_0(t)x(t)\leqslant c<0$ for $t\geqslant T_1$. This is a contradiction to positiveness of $x(t)$.

If $n>1$, we have

$$a_{n-2}(t)L_{n-2}x(t)-a_{n-2}(T_1)L_{n-2}x(T_1)\leqslant c\int_{T_1}^{t}\frac{ds}{a_{n-1}(s)}.$$

Letting $t\to\infty$, we find that $\lim\limits_{t\to\infty}a_{n-2}(t)L_{n-2}x(t)=-\infty$. Repeating the same argument, we arrive at $\lim\limits_{t\to\infty}a_0(t)x(t)=-\infty$. This contradicts the fact that $x(t)$ is eventually positive.

Suppose that $\beta_{n-1}>0$, then, there is $T_2\geqslant T$ such that

$$a_{n-1}(t)L_{n-1}x(t)\geqslant\frac{\beta_{n-1}}{2},\quad t\geqslant T_2. \tag{3}$$

Assume that $n=1$, namely, $a_0(t)x(t)\geqslant\frac{\beta_0}{2}>0$, $t\geqslant T_2$. Since $a_0(t)\geqslant\tilde{c}_0>0$ for sufficiently large t, where $\tilde{c}_0$ is a constant, there exists a constant c_0 such that $x(t)>c_0$ and $x(g(t))>c_0$ for all sufficiently large t, say, $t\geqslant T_3\geqslant T_2$. Hence from (iii), if $x(t)$ is bounded, or from (iii) and (iv), if $\lim\limits_{t\to\infty}x(t)=\lim\limits_{t\to\infty}x(g(t))=\infty$, we have $f(x(t), x(g(t)))>c>0$ for sufficiently large $t\geqslant T_4\geqslant T_3$, where c is some constant. Hence

$$a_0(t)x(t)-a_0(T_4)x(T_4)+c\int_{T_4}^{t}p(s)f(x(s),\ x(g(s)))ds\leqslant\int_{T_4}^{t}r(s)ds.$$

Letting $t\to\infty$ and using (v), we will get a contradiction to the condition (vi).

Assume that $n>1$, we have from (2)

$$a_{n-2}(t)L_{n-2}(t)-a_{n-2}(T_2)L_{n-2}(T_2)\geqslant\frac{\beta_{n-1}}{2}\int_{T_2}^{t}\frac{ds}{a_{n-1}(s)}.$$

Therefore, $\lim\limits_{t\to\infty}a_{n-2}(t)L_{n-2}(t)=\infty$. Applying this argument repeatedly we finally show that $\lim\limits_{t\to\infty}a_0(t)x(t)=\infty$. Since $a_0(\mathrm{t})$ is bounded, $\lim\limits_{t\to\infty}x(t)=\lim\limits_{t\to\infty}x(g(t))=\infty$. As the proof above for $n=1$, using (2) and (iii)—(vi), we arrire at a contradiction. Thus the only remaining alternative is $\beta_{n-1}=0$, as desired.

Letting $t\to\infty$ in (2), we get

$$-a_{n-1}(t)L_{n-1}x(t)+\int_{t}^{\infty}p(s)f(x(s),\ x(g(s)))ds=w_0(t),\ t\geqslant T_2. \tag{4}$$

From (4), it follows that

$$\begin{aligned}&-a_{n-2}(t)L_{n-2}x(t)+a_{n-2}(T_2)L_{n-2}x(T_2)\\&+\int_{T_2}^{t}\frac{1}{a_{n-1}(s_2)}\int_{s_2}^{\infty}p(s_1)f(x(s_1),\ x(g(s_1)))ds_1ds_2=\int_{T_2}^{t}\frac{w_0(s)}{a_{n-1}(s)}ds.\end{aligned}$$

Hence, from (iii) and (vi) we have

$$\lim_{t\to\infty}a_{n-2}(t)L_{n-2}x(t)=\beta_{n-2},$$

where β_{n-2} is finite or ∞. Repeated application of the similar argument above shows $\beta_{n-2}=0$. Continuing in this way, we have $\lim\limits_{t\to\infty}a_k(t)L_kx(t)=\beta_k=0$ for $k=n-3,\ n-4,$

$\cdots$, 1. And similarly, we obtain $\lim_{t\to\infty} a_0(t)L_0x(t)=\beta_0$, where β_0 is finite or infinite. We can prove $\beta_0=0$. Otherwise, if $\beta_0<0$, it is impossible; if $\beta_0>0$, $\lim_{t\to\infty} x(t)>0$. As proof for $n=1$, by (iii)—(vi) and (2), we would be lead to a contradiction. This completes the proof of Lemma 1.

Remark 1. We observe from the proof of Lemma 1 that if $x(t)$ is a bounded solution of equation (1), then Lemma 1 follows readily without the condition (iv) and (v) excepting $k=0$.

Theorem 1. *Assume that* (i)—(vi) *hold. If n is even and $w_{n-1}(t)$ is oscillatory on* $[0, \infty)$, *then equation.* (1) *is oscillatory.*

If n is odd, then either every solution of equation (1) *is oscillatory or*

$$(-1)^{k+1}a_{n-k}(t)L_{n-k}x(t)+w_{k-1}(t)$$

converges to zero monotonically as $t\to\infty$, $k=1, 2, \cdots, n$. In particular, $\lim_{t\to\infty} x(t)=0$.

Proof Assume that $x(t)$ is a nonoscillatory solution of equation (1) and that $x(t)>0$ and $x(g(t))>0$ for $t\geqslant T\geqslant 0$. A similar proof will hold for the case $x(t)<0$ and $x(g(t))<0$. By Lemma 1

$$\lim_{t\to\infty} a_k(t)L_kx(t)=0 \tag{5}$$

for $k=0, 1, \cdots, n-1$. Integrating (1) on (t, ∞), $t\geqslant T$, we obtain

$$-a_{n-1}(t)L_{n-1}x(t)+\int_t^\infty p(s)f(x(s),\ x(g(s)))ds=\int_t^\infty r(s)ds=w_0(t).$$

Repeated integration, together with (5), yields

$$(-1)^k a_{n-k}(t)L_{n-k}x(t)+\int_t^\infty \frac{1}{a_{n-k+1}(s_k)}\int_{s_k}^\infty\cdots\int_{s_2}^\infty p(s_1)f(x(s_1),\ x(g(s_1)))ds_1ds_2\cdots ds_k \tag{6}$$
$$=w_{k-1}(t),\quad k=1,\ 2,\ \cdots,\ n-1.$$

Finally, we can obtain

$$(-1)^{n-1}[a_0(t)x(t)-a_0(T)x(T)]+F(t)=\int_T^t \frac{w_{n-2}(s)}{a_1(s)}ds, \tag{7}$$

where

$$F(t)=\int_T^t\frac{1}{a_1(s_n)}\int_{s_{n-1}}^\infty \frac{1}{a_2(s_{n-1})}\cdots\int_{s_2}^\infty p(s_1)f(x(s_1),\ x(g(s_1))ds_1\,ds_2\cdots ds_n.$$

By Lemma 1, $a_0(t)x(t)\to 0$, as $t\to\infty$, and $F(t)\geqslant 0$. This implies from (7) that

$$(-1)^n a_0(t)x(t)\leqslant w_{n-1}(t).$$

If n is even, we have $0<a_0(t)x(t)<w_{n-1}(t)$ for all $t\geqslant T$. This shows $w_{n-1}(t)$ is nonoscillatory, which is a contradiction. This establishes the first statement of the theorem.

The result for n to be odd is an immediate consequence of (6) and (7). This completes the proof of Theorem 1.

If $r(t)\equiv 0$ in equation (1), then the conditions on $w_{n-1}(t)$ are trivially satisfied. Thus we have from Theorem 1 the following result.

Corollary 1. *Assume that $r(t)\equiv 0$ and* (i)—(v) *hold. If n is even, then equation*

(1) is oscillatory. If n is odd, then either every solution of equation (1) is oscillatory or $a_k(t)L_kx(t)$, in particular, $x(t)$ tends to zero, $k=0, 1, \cdots, n-1$.

Remark 2. Corollary 1 extends and improves Theorem 1 and Theorem 2 of Bradley[1], as well as results of Norkin [9, pp 149—150], Kung[4], Mahfoud[8], and P. Waltman[10].

As an example, consider

$$x''+\left[\frac{1}{4}(1+t^2)\right]x=(t^{-\alpha}\sin t)'', \tag{8}$$

where α is a constant with $0<\alpha<1$. This example is treated by A. G. Kartsatos and M. N. Manougian[3] who show that every solution of equation (8) is either unbounded or bounded and oscillatory. By our Theorem 1, in fact, equation (8) is oscillatory.

Theorem 2. *Assume that* (i)—(iii), (vi) *hold and that for some j, $1\leqslant j\leqslant n-1$, $\phi_{j-1}(t)$ exists, but $\phi_j(t)$ does not exist. If n is even and $w_{n-1}(t)$ is oscillatory on $[0, \infty)$, then every bounded solution of equation (1) is oscillatory. If n is odd, then either every bounded solution of equation (1) is oscillatory or $(-1)^{k+1}a_{n-k}(t)L_{n-k}x(t)+w_{k-1}(t)$ converges to zero monotonically as $t\to\infty$, $k=1, 2, \cdots, n$. In particular, $\lim\limits_{t\to\infty}x(t)=0$.*

Proof Assume that $x(t)$ is a bounded nonoscillatory solution of equation (1), then there is T such that $x(t)>0$, $x(g(t))>0$ for $t\geqslant T$. A similar proof will hold for $x(t)<0$, $x(g(t))<0$ for $t\geqslant T$. By Lemma 1 and Remark 1, as the proof of Theorem 1, we easily obtain

$$(-1)^k a_{n-k}(t)L_{n-k}x(t)+\int_t^\infty\frac{1}{a_{n-k+1}(s_k)}\int^\infty\cdots\int_{s_2}^\infty p(s_1)f(x(s_1), x(g(s_1)))ds_1\cdots ds_k$$
$$=w_{k-1}(t),\quad k=1, 2, \cdots, n-1. \tag{6}$$

Because the first term in (6) tends to zero as $t\to\infty$, from (vi) $w_{k-1}(t)$ is continuous, therefore the integral of the left side is finite.

In (6), Let $k=j+1$, $j=0, 1, \cdots, n-2$. Since $\phi_j(t)$ does not exist, it follows from (6) that $\lim\limits_{t\to\infty}\inf f(x(t), x(g(t)))=0$.

Hence, from (iii), either $\lim\limits_{t\to\infty}\inf x(t)=0$ or $\lim\limits_{t\to\infty}\inf x(g(t))=0$. Any one of the two cases implies that

$$\lim_{t\to\infty}\inf x(t)=0. \tag{9}$$

Letting $k=n-1$ in (6) and then integrating from T to t, we obtain

$$(-1)^{n-1}[a_0(t)L_0x(t)-a_0(T)L_0x(T)]+\int_T^t\frac{ds_n}{a_1(s_n)}\int_{s_n}^\infty\frac{ds_{n-1}}{a_2(s_{n-1})}\int_{s_{n-1}}^\infty\cdots$$
$$\cdots\int_{s_2}^\infty p(s_1)f(x(s_1), x(g(s_1)))ds_1=\int_T^t\frac{w_{n-2}(s)}{a_1(s)}ds. \tag{10}$$

From (10) and boundedness of $x(t)$, $\lim\limits_{t\to\infty}a_0(t)L_0x(t)=\lim\limits_{t\to\infty}a_0(t)x(t)=\beta_0$ exists and is finite. In view of (9) and assumed condition about $a_0(t)$,

$$\lim_{t\to\infty}a_0(t)L_0x(t)=\lim_{t\to\infty}x(t)=0.$$

In (10), letting $t\to\infty$, we obtain $(-1)^n a_0(t)x(t)\leqslant w_{n-1}(t)$, $t\geqslant T$. Therefore, if n is even $0<a_0(t)x(t)\leqslant w_{n-1}(t)$, $t\geqslant T$, which contradicts the fact that $w_{n-1}(t)$ is oscillatory.

If n is odd, it is easy from (6) to get the remains of this theorem. Proof of Theorem 2 is completed.

Consider the 6-th order linear equation

$$\frac{d^3\left(e^{-t}\dfrac{d^3x}{dt^3}\right)}{dt^3}+18e^{-\left(t+\frac{\pi}{2}\right)}x\left(t-\frac{\pi}{2}\right)=-26e^{-2t}\sin t. \tag{11}$$

Here $f(u, v)=v$, (iii) is satisfied. It is easy to see that $w_5(t)$ is oscillatory, and $\phi_i(t)=18e^{-\left(t+\frac{\pi}{2}\right)}$, $i=1, 2, 3$; $\phi_4(t)$ does not exist. So equation. (11) is oscillatory, while the relative results of [3, 6, 8] fail to apply. One such solution of (11) is $x(t)=e^{-t}\sin t$.

We notice that the conditions on $w_{n-1}(t)$ hold trivially in the case that $r(t)\equiv 0$.

Corollary 2. *Assume that $r(t)\equiv 0$ and that the other conditions of Theorem 2 hold. If n is even, then every bounded solution of equation (1) is oscillatory. If n is odd, then either every bounded solution of equation (1) is oscillatory or $(-1)^{k+1}a_{n-k}(t)L_{n-k}x(t)$ tends to zero monotonically, $k=1, 2, \cdots, n$.*

Corollary 3. *Assume that $r(t)\equiv 0$, (i)—(iii), (vi) hold and $\lim\limits_{t\to\infty}\int_{t_0}^t p(s)ds<\infty$, $\lim\limits_{t\to\infty}\int_{t_0}^t \psi_{n-1}(s)p(s)ds=\infty$. If n is even, then every bounded solution of equation (1) is oscillatory. If n is odd, then either every bounded solution of equation (1) is oscillatory or $(-1)^{k+1}a_{n-k}(t)L_{n-k}x(t)$ tends to zero monotonically as $t\to\infty$, $k=1, 2, \cdots, n$.*

Proof Since $p(t)\geqslant 0$ and $a_j(t)>0$ for each j, we may apply the Fubini Theorem to interchange the order of integration and obtain

$$\int_{t_0}^t \psi_j(s)p(s)ds=\int_{t_0}^t \frac{1}{a_0(t_1)}\int_{t_1}^t \frac{1}{a_1(t_2)}\cdots\int_{t_{-1j}}^t \frac{1}{a_{j-1}(t_j)}\int_{t_j}^t p(s)ds\,dt_j\cdots dt_1,$$

and in particular, for $j=n-1$, we have

$$\int_{t_0}^t \psi_{n-1}(s)p(s)ds\leqslant \phi_{n-1}(t_0). \tag{12}$$

Letting $t\to\infty$ in (12), it follows that $\phi_{n-1}(t_0)=\infty$ and Corollary 2 applies.

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