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THE TRANSFORMATION OPERATOR I OF NONLINEAR EVOLUTION EQUATIONS

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Abstract

In this paper, the equivalence of two classes of nonlinear evolution equations is proved by introducing the transformation operator S and its inverse operator S^{-1}. By the transformation operator S, some properties of one class of these equations, such as the infinite number of conserved quantities, Bäcklund transformations, etc, are deduced from the corresponding known properties of its equivalent class.

§ 1. Introduction

Associated with the eigenvalue problem

$$\varphi'_x = M'\varphi', \quad M' = \begin{pmatrix} -i\xi & q' \\ r' & i\xi \end{pmatrix}, \ \varphi' = \begin{pmatrix} \varphi'_1 \\ \varphi'_2 \end{pmatrix}, \tag{1.1}$$

one can get a class of nonlinear evolution equations[1]

$$\begin{pmatrix} -q'_t \\ r'_t \end{pmatrix} = 2i\left\{ \sum_{j=0}^{n} \alpha_j(t) L'^{n-j} \begin{pmatrix} q' \\ r' \end{pmatrix} + \sum_{j=0}^{n-1} k_j(t) L'^{n-j} \begin{pmatrix} xq' \\ xr' \end{pmatrix} \right\}, \tag{1.2}$$

where

$$L' = \frac{1}{2i}\left(D\sigma - 2\begin{pmatrix} q' \\ r' \end{pmatrix} I(r', \ -q') \right) = \frac{1}{2i}\begin{pmatrix} -D - 2q'Ir' & 2q'Iq' \\ -2r'Ir' & D + 2r'Iq' \end{pmatrix}, \tag{1.3}$$

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.4}$$

D is differential operator and I is integral operator, i. e.

$$I = \int_x^\infty \cdots dx, \text{ and } DI = ID = -1. \tag{1.5}$$

Associated with the eigenvalue problem

$$\varphi_x = M\varphi, \quad M = \begin{pmatrix} -i\xi & \xi q \\ r & i\xi \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1.6}$$

one can get a class of nonlinear evolution equations[1]

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$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = D\sum_{j=0}^{n-1}\alpha_j(t)L^{n-j-1}\begin{pmatrix} q \\ r \end{pmatrix} + 2i\alpha_n(t)\begin{pmatrix} -q \\ r \end{pmatrix} + D\sum_{j=0}^{n-1}k_j(t)L^{n-j-1}\begin{pmatrix} xq+Iq \\ xr \end{pmatrix}, \tag{1.7}$$

where

$$L = \frac{1}{2i}\left(D\sigma - i\begin{pmatrix} q \\ r \end{pmatrix} \quad I(r, q)D\right) = \frac{1}{2i}\begin{pmatrix} -D-iqIrD & -iqIqD \\ -irIrD & D-irIqD \end{pmatrix}. \tag{1.8}$$

In [2], it is proved that the transformation

$$\varphi' = T\varphi, \quad T = \begin{pmatrix} \lambda & \frac{i}{2}q\lambda \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda = e^{\frac{i}{2}Iqr} \tag{1.9}$$

maps the eigenvalue problem (1.6) into the eigenvalue problem (1.1), and the potentials satisfy the following relations

$$q' = \lambda^2\left(\frac{i}{2}q_x + \frac{i}{4}q^2 r\right), \quad r' = \lambda^{-2}r. \tag{1.10}$$

It is natural to ask whether the solutions of two classes of nonlinear evolution equation (1.2), (1.7) associated with the eigenvalue problems (1.1) and (1.6) satisfy the transformation (1.10), and can some properties of the equation (1.7), such as the infinite number of conserved quantities, Bäcklund transformation etc be derived from the corresponding properties of the equation (1.2)? The first problem has been considered in [1]. In this paper we give an exact answer by introducing the transformation operator S of the operators L' and L.

§ 2. The Transformation Operator

Unless otherwise specified, it is always assumed in the following that functions (q', r') and (q, r) have continuous derivatives of any possible order which occur in equation (1.2) and equation (1.7) respectively and they have all asymptotic behavior $O(|x|^{-2-\varepsilon})$ $(\varepsilon>0)$ when $x\to\infty$. For convenience we simply say that (q', r') and (q, r) satisfy the fundamental condition $\mathscr{E}$.

Now we define the transformation operator of the operators L and L', rewrite transformation (1.10) in matrix form

$$\begin{pmatrix} q' \\ r' \end{pmatrix} = \begin{pmatrix} \frac{i}{2}\lambda^2 D & \frac{1}{4}\lambda^2 q^2 \\ 0 & \lambda^{-2} \end{pmatrix}\begin{pmatrix} q \\ r \end{pmatrix}. \tag{2.1}$$

Set

$$S = \begin{pmatrix} \frac{i}{2}\lambda^2 D & \frac{1}{4}\lambda^2 q^2 \\ 0 & \lambda^{-2} \end{pmatrix}, \tag{2.2}$$

then the expression (2.1) can be written Simply

$$\begin{pmatrix} q' \\ r' \end{pmatrix} = S \begin{pmatrix} q \\ r \end{pmatrix}. \tag{2.3}$$

One can easily prove that if (q, r) satisfies the fundamental condition $\mathscr{E}$, then (q', r') defined by transformation (2.3) will satisfy the fundamental condition $\mathscr{E}$ too, and the following equalities will hold

$$S\begin{pmatrix} q \\ r \end{pmatrix} I = \begin{pmatrix} q' \\ r' \end{pmatrix} I - \frac{i\lambda^2}{2}\begin{pmatrix} q \\ 0 \end{pmatrix}, \tag{2.4}$$

$$S\begin{pmatrix} xq+Iq \\ xr \end{pmatrix} = \begin{pmatrix} xq' \\ xr' \end{pmatrix}. \tag{2.5}$$

Moreover, we have also the equality

$$2iSLI\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \begin{pmatrix} q'_t \\ -r'_t \end{pmatrix}. \tag{2.6}$$

In fact

$$2iSLI\begin{pmatrix} q_t \\ r_t \end{pmatrix} = S\left(-\sigma + i\begin{pmatrix} q \\ r \end{pmatrix} \quad I(r,\ q)\right)\begin{pmatrix} q_t \\ r_t \end{pmatrix},$$

using formula (2.4), we have

$$2iSLI\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \begin{pmatrix} \frac{i}{2}\lambda^2 D & -\frac{1}{4}\lambda^2 q^2 \\ 0 & -\lambda^{-2} \end{pmatrix}\begin{pmatrix} q_t \\ r_t \end{pmatrix} + i\begin{pmatrix} q' \\ r' \end{pmatrix} I(qr)_t + \frac{\lambda^2}{2}\begin{pmatrix} q \\ 0 \end{pmatrix}(qr)_t.$$

Paying attention to the transformation (1.10), one gets the formula (2.6) immediatly

$$2iSLI\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \begin{pmatrix} \lambda^2(\lambda^{-2}q')_t + iq'I(qr)_t \\ -r'_t \end{pmatrix} = \begin{pmatrix} q'_t \\ -r'_t \end{pmatrix}.$$

At last, we prove the fundamental relation between the operators L and L'.

Theorem 1. *Let the operators L and L' be expressed by (1.8) and (1.3) respectively, and let the potentials (q', r') and (q, r) satisfy the transformation (1.10), then*

$$L'S = SL. \tag{2.7}$$

Proof

$$L'S = \frac{1}{2i}\left\{D\sigma - 2\begin{pmatrix} q' \\ r' \end{pmatrix} I(r',\ -q')\right\}\begin{pmatrix} \frac{i}{2}\lambda^2 D & \frac{1}{4}\lambda^2 q^2 \\ 0 & \lambda^{-2} \end{pmatrix}$$

$$= \frac{1}{2i}\left\{\begin{pmatrix} -\frac{i}{2}D\lambda^2 D & -\frac{1}{4}D\lambda^2 q^2 \\ 0 & D\lambda^{-2} \end{pmatrix} - i\begin{pmatrix} q' \\ r' \end{pmatrix} I(rD,\ -q_x)\right\},$$

using the relations

$$D\lambda^2 D = \lambda^2 D^2 - i\lambda^2 qrD, \quad -\frac{1}{4}D\lambda^2 q^2 = -\frac{1}{4}\lambda^2 q^2 D + iqq',$$

$$D\lambda^{-2} = \lambda^{-2}D + iqr', \quad q_x = Dq - qD,$$

we have

$$L'S=\frac{1}{2i}\left\{SD\sigma+\begin{pmatrix}-\frac{1}{2}\lambda^2qrD & iqq'-\frac{1}{2}\lambda^2q^2D\\ 0 & iqr'\end{pmatrix}\right.$$

$$\left.-i\begin{pmatrix}q'\\ r'\end{pmatrix}I(r,\ q)D-i\begin{pmatrix}q'\\ r'\end{pmatrix}(0,\ q)\right\}$$

$$=\frac{1}{2i}\left\{SD\sigma-\frac{1}{2}\lambda^2\begin{pmatrix}q\\ 0\end{pmatrix}(r,\ q)D-i\begin{pmatrix}q'\\ r'\end{pmatrix}I(r,\ q)D\right\}.$$

Paying attention to formula (2.4), we get the relation (2.7)

$$L'S=\frac{1}{2i}\left\{SD\sigma-iS\begin{pmatrix}q\\ r\end{pmatrix}I(r,\ q)D\right\}=SL.$$

Hereafter, the operator S is called the transformation operator from L' to L. From the formula (2.7), it is easily seen that for any positive integer n and polynomial $P(z)$ there are

$$L'^nS=SL^n, \tag{2.8}$$

$$P(L')S=SP(L). \tag{2.9}$$

§ 3. Inverse Transformation Operator

It is easy to verify that

$$S^{-1}=\begin{pmatrix}2iI\lambda^{-2} & -\frac{i}{2}I\lambda^2q^2\\ 0 & \lambda^2\end{pmatrix} \tag{3.1}$$

is a right inverse operator of S, i. e.

$$S^{-1}S=e, \tag{3.2}$$

where e is the 2×2 unit matrix. Moreover if $ID=-1$, then S^{-1} is a left inverse operator of S, i. e.

$$SS^{-1}=e. \tag{3.3}$$

Now, we prove that this inverse operator S^{-1} is defined uniquely by the potentials $(q',\ r')$, if given functions $(q',\ r')$ satisfy the conditions

$$\int_x^\infty|(1+y)q'(y,\ t)|dy<\infty,$$

$$\int_x^\infty|(1+y)r'(y,\ t)|dy<\infty. \tag{3.4}$$

By the method of successive approximations one can prove that there is only one solution $(\varphi_1,\ \varphi_2)$ of the equation (1.1) with $\xi=0$ and the boundary condition

$$(\varphi_1,\ \varphi_2)\to(0,\ 1),\ (x\to\infty).$$

Set

$$q=-2i\varphi_1\varphi_2,\quad r=r'\varphi_2^{-2}, \tag{3.5}$$

we can verify easily that the given functions $(q',\ r')$ and the functions $(q,\ r)$ defined by (3.5) satisfy the equation (1.10) and $(q,\ r)$ tend to $(0,\ 0)$ as $x\to\infty$.

If there is another set of functions $(\tilde{q},\ \tilde{r})$ which satisfy the equation (1.10) and tend to $(0,\ 0)$ when $x\to\infty$, then we set

$$\tilde{\varphi}_1=\frac{i}{2}\exp\left(\frac{i}{2}I\tilde{q}\tilde{r}\right)\tilde{q},\quad \tilde{\varphi}_2=\exp\left(-\frac{i}{2}I\tilde{q}\tilde{r}\right), \tag{3.6}$$

thus get easily that $(\tilde{\varphi}_1,\ \tilde{\varphi}_2)$ satisfied the eigenvalue problem (1.1) with $\xi=0$ and tend to $(0,\ 1)$ when $x\to\infty$. According to the uniqueness, we have $\tilde{\varphi}_1=\varphi_1,\ \tilde{\varphi}_2=\varphi_2$, thus $\tilde{q}=q,\ \tilde{r}=r$. So we have proved following theorem.

Theorem 2. *If potentials $(q',\ r')$ which satisfy the conditions (3.4) are given and $(\varphi_1,\ \varphi_2)$ is a set of solutions of the eigenvalue problem (1.1) with $\xi=0$ under boundary condition $(\varphi_1,\ \varphi_2)\to(0,\ 1)$ as $x\to\infty$., then unique solution $(q,\ r)$ of the equation (1.10) can be expressed by (3.5) and the inverse operator S^{-1} of the transformation operator S can be expressed by implicit form of the functions (q',r').*

By inverse operator formulae (2.3), (2.5) and (2.6) can be written as follows

$$\begin{pmatrix} q \\ r \end{pmatrix}=S^{-1}\begin{pmatrix} q' \\ r' \end{pmatrix}, \tag{3.7}$$

$$\begin{pmatrix} xq+Iq \\ xr \end{pmatrix}=S^{-1}\begin{pmatrix} xq' \\ xr' \end{pmatrix}, \tag{3.8}$$

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix}=\frac{1}{2i}DL^{-1}S^{-1}\begin{pmatrix} -q'_t \\ r'_t \end{pmatrix}, \tag{3.9}$$

where L^{-1} is the inverse operator which reads

$$L^{-1}=2i\left(-\sigma I+iI\begin{pmatrix} q \\ -r \end{pmatrix}\quad I(-r,\ q)\right)=2i\begin{pmatrix} I-iIqIr & iIqIq \\ iIrIr & -I-iIrIq \end{pmatrix}, \tag{3.10}$$

i. e.

$$LL^{-1}=L^{-1}L=e. \tag{3.11}$$

It is easily proved that S^{-1} is the transformation operator from L to L', i. e.

$$LS^{-1}=S^{-1}L'. \tag{3.12}$$

In fact, multiplying the operator S^{-1} from the left and from the right, respectively, in the transformation relation (2.9), we get this formula immediately.

Thus we also have

$$L^nS^{-1}=S^{-1}L'^n, \tag{3.13}$$

$$P(L)S^{-1}=S^{-1}P(L'), \tag{3.14}$$

where n is arbitrary positive integer. $P(z)$ is a polynomial of z.

§ 4. The Equivalence of the Equation (1.2) and the Equation (1.7)

After the properties of the trantformation operator and its inverse transformation operator have been explained we begin to discuss the relations of the solution of nonlinear evolution equations (1.2) and (1.7).

Theorem 3. *The nonlinear evolution equations* (1.2) *and* (1.7) *are equivalence under the transformation* (1.10), *i. e. if* (q, r) *is a set of solutions of the equation* (1.7), *then a set of functions* (q', r') *defined by* (1.10) *is solution of the equation* (1.2).

Conversely, if (q', r') *is a solution of the equation* (1.2), *then a set of functions* (q, r) *solved from* (1.10) *is a solution of the equation* (1.7).

Proof Suppose (q, r) belongs to $\mathscr{E}$ and is a solution of the equation (1.7). Then (q', r') defined by (2.3) belong to $\mathscr{E}$ too. Multiplying the operator $-2iSLI$ in both sides of (1.7), using the formula (2.8) the right hand side of (1.7) turn into

$$\text{right side}=2i\left\{\sum_{j=0}^{n-1}\alpha_j(t)L'^{n-j}S\begin{pmatrix}q\\r\end{pmatrix}+\alpha_n(t)S\begin{pmatrix}q\\r\end{pmatrix}+\sum_{j=0}^{n-1}k_j(t)L'^{n-j}S\begin{pmatrix}xq+Iq\\xr\end{pmatrix}\right\}.$$

Paying attention to the formulae (2.3), (2.5) and (2.6), it is just the equation (1.2). So we have proved that the set of functions (q', r') defined by (1.10) is a solution of equation (1.2).

Conversely, if (q', r') belongs to $\mathscr{E}$ and is a solution of the equation (1.2), then the function expressed by (3.5) belongs to $\mathscr{E}$ too[1]. From Theorem 2 we know that inverse operators S^{-1} and L^{-1} exist. By multiplying the operator $\frac{1}{2i}DL^{-1}S^{-1}$ in both sides of the equation (1.2) and using the formula (3.13), the right side turns into

$$\text{right side}=D\sum_{j=0}^{n-1}\alpha_j(t)L^{n-j-1}S^{-1}\begin{pmatrix}q'\\r'\end{pmatrix}$$
$$+\alpha_n(t)DL^{-1}S^{-1}\begin{pmatrix}q'\\r'\end{pmatrix}+D\sum_{j=0}^{n-1}k_j(t)L^{n-j-1}S^{-1}\begin{pmatrix}xq'\\xr'\end{pmatrix}.$$

Paying attention to the formulae (3.7), (3.8), and (3.9), it is just the equation (1.7). So we have proved that the functions (q, r) which defined by (3.5) is the set of equations (1.7).

§ 5. Conserved Quantities

When $k_j(t)=0$, $j\geqslant 0$, the equations (1.2) and (1.7) reduce to that

$$\begin{pmatrix}-q'_t\\r'_t\end{pmatrix}=2i\sum_{j=0}^{n}\alpha_j(t)L'^{n-j}\begin{pmatrix}q'\\r'\end{pmatrix},\tag{5.1}$$

$$\begin{pmatrix}q_t\\r_t\end{pmatrix}=D\sum_{j=0}^{n-1}\alpha_j(t)L^{n-j}\begin{pmatrix}q\\r\end{pmatrix}+2i\alpha_n(t)\begin{pmatrix}-q\\r\end{pmatrix}.\tag{5.2}$$

It is known that the (5.1) has an infinite number of conserved quantities

$$C'_m=\int_{-\infty}^{\infty}\left(\begin{pmatrix}q'\\r'\end{pmatrix},\ i\sigma_2 L'^m\begin{pmatrix}xq'\\xr'\end{pmatrix}\right)dx \quad (m=1,\ 2,\ \cdots), \tag{5.3}$$

where

$$\sigma_2=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix},\ \left(\begin{pmatrix}u_1\\v_1\end{pmatrix},\ \begin{pmatrix}u_2\\v_2\end{pmatrix}\right)=u_1u_2+v_1v_2. \tag{5.4}$$

In order to derive an infinite number of conserved quantities of evolution equation (5.2) by transformation operator S from formulae (5.3) we mention the following definition.

If functions $(q',\ r')$ and $(q,\ r)$ have continuous derivatives of any possible order which occur in equation (5.1) and equation (5.2) respectively, and if they have all asymptotic behavior $O(|x|^{-1-\varepsilon})$ when $|x|\to\infty$, then $(q',\ r')$ and $(q,\ r)$ are considered satisfying the fundamental condition $\mathscr{F}$.

Since (5.1) and (5.2) are equivalent equations under the transformation (1.10), we see that when $(q',\ r')$ is a set of the solutions of equation (5.1) and satisfy condition $\mathscr{F}$, the functions $(q,\ r)$ defined by transformation (2.3) must be the solution of equation (5.2) and satisfy condition $\mathscr{F}$, and vice versa.

It is easily seen from (2.3), (2.5) and (2.8) that the quantities

$$\begin{aligned}C_m&=\int_{-\infty}^{\infty}\left(S\begin{pmatrix}q\\r\end{pmatrix},\ i\sigma_2 S L^m\begin{pmatrix}xq+Iq\\xr\end{pmatrix}\right)dx\\&=\int_{-\infty}^{\infty}\left(\begin{pmatrix}q'\\r'\end{pmatrix},\ i\sigma_2 L'^m\begin{pmatrix}xq'\\xr'\end{pmatrix}\right)dx \quad (m=1,\ 2,\ \cdots)\end{aligned} \tag{5.5}$$

are independent of t. It means that $C_m(m=1,\ 2,\ \cdots)$ are the conserved quantities of (5.2).

By considering the formula

$$\int_{-\infty}^{\infty}\left(S\begin{pmatrix}u_1\\v_1\end{pmatrix},\ \sigma_2 S\begin{pmatrix}u_2\\v_2\end{pmatrix}\right)dx=\frac{1}{2}\int_{-\infty}^{\infty}\left(\begin{pmatrix}D & 0\\0 & 1\end{pmatrix}\begin{pmatrix}u_1\\v_1\end{pmatrix},\ \begin{pmatrix}0 & 1\\-D & 0\end{pmatrix}\begin{pmatrix}u_2\\v_2\end{pmatrix}\right)dx \tag{5.6}$$

and the condition $\lim\limits_{|x|\to\infty}(q,\ r)=0$, (5.5) can be rewritten as follows

$$C_m=\frac{1}{2i}\int_{-\infty}^{\infty}\left(\begin{pmatrix}r\\q\end{pmatrix},\ DL^m\begin{pmatrix}xq+Iq\\xr\end{pmatrix}\right)dx \quad (m=1,\ 2,\ \cdots). \tag{5.7}$$

By the way, (5.2) has still a conserved quantity besides (5.7), that is

$$C_0=\int_{-\infty}^{\infty}r(x)q(x)dx.$$

§ 6. Bäcklund Transformations

The Bäcklund transformations[3] which connect two setes of potentials $(q',\ r')$ and $(q,\ r)$ that satisfy the same equation (5.1) are

$$H_+(\Lambda')\begin{pmatrix} q' \\ r' \end{pmatrix} + H_-(\Lambda')\begin{pmatrix} \tilde{q}' \\ \tilde{r}' \end{pmatrix} = 0, \tag{6.1}$$

with

$$H_\pm(z) = g(z) \mp f(z)\sigma.$$

$g(z)$ and $f(z)$ are two arbitrary entire functions and

$$\Lambda' = \frac{1}{2i}\begin{pmatrix} -D-\tilde{q}'I\tilde{r}'-q'Ir' & \tilde{q}'Iq'+q'I\tilde{q}' \\ -\tilde{r}'Ir'-r'I\tilde{r}' & D+\tilde{r}'I\tilde{q}'+r'Iq' \end{pmatrix}$$
$$= \frac{1}{2}L' + \frac{1}{2}\tilde{L}' + \frac{1}{2i}\begin{pmatrix} 0 & -(q'-\tilde{q}')I(q'-\tilde{q}') \\ (r'-\tilde{r}')I(r'-\tilde{r}') & 0 \end{pmatrix}$$
$$= \frac{1}{2}L' + \frac{1}{2}\tilde{L}' + \frac{1}{2i}\eta\begin{pmatrix} q'-\tilde{q}' \\ r'-\tilde{r}' \end{pmatrix}.$$

The operator η defined as follows

$$\eta\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -uIu \\ vIv & 0 \end{pmatrix}.$$

Substituting the following equations

$$\begin{pmatrix} q' \\ r' \end{pmatrix} = S\begin{pmatrix} q \\ r \end{pmatrix}, \quad \begin{pmatrix} \tilde{q}' \\ \tilde{r}' \end{pmatrix} = \tilde{S}\begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix} \tag{6.3}$$

into (6.1), we get the Bäcklund transformations of (5.2). By the transformation operator, they can be written in explicit form. Actually, Λ' may be expressed in terms of (q, r) and $(\tilde{q}, \tilde{r})$, by means of (6.3) and (2.7), as follows

$$\Lambda = \frac{1}{2}SLS^{-1} + \frac{1}{2}\tilde{S}\tilde{L}\tilde{S}^{-1} + \frac{1}{2i}\eta\left(S\begin{pmatrix} q \\ r \end{pmatrix} - \tilde{S}\begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix}\right). \tag{6.4}$$

Here we use the notation Λ instead Λ'. It is then clear that the equations

$$H_+(\Lambda)S\begin{pmatrix} q \\ r \end{pmatrix} + H_-(\Lambda)\tilde{S}\begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix} = 0 \tag{6.5}$$

are the Bäcklund transformations of (5.2). In fact, if two pairs of potentials (q, r) and $(\tilde{q}, \tilde{r})$ are related by (6.5), and (q, r) satisfies the (5.2), then (q', r') and $(\tilde{q}', \tilde{r}')$ obtained from (6.3) satisfy the eq uation (6.1), and (q', r') is a solution of (5.1). This implies that $(\tilde{q}', \tilde{r}')$ is also a solution of (5.1). We may therefore conclude that $(\tilde{q}, \tilde{r})$ satisfies the equation. (5.2).

§ 7. An Important Property of L

When (q', r') satisfies the fundamental condition $\mathscr{F}$, the operator L' satisfies[3]

$$\int_{-\infty}^{\infty}\left(\begin{pmatrix} q' \\ r' \end{pmatrix}, \; i\sigma_2 L'^m\begin{pmatrix} q' \\ r' \end{pmatrix}\right)dx = 0 \quad (m=0, 1, \cdots). \tag{7.1}$$

From (7.1), (2.3) and (2.8), it yields

$$\int_{-\infty}^{\infty}\left(S\begin{pmatrix} q \\ r \end{pmatrix},\ i\sigma_2 SL^m\begin{pmatrix} q \\ r \end{pmatrix}\right)dx=0 \quad (m=0,\ 1,\ \cdots). \tag{7.2}$$

Using (5.6), (7.2) and $\lim_{|x|\to\infty}(q,\ r)=0$, we have

$$\int_{-\infty}^{\infty}\left(\begin{pmatrix} r \\ q \end{pmatrix},\ DL^m\begin{pmatrix} q \\ r \end{pmatrix}\right)dx=0 \quad (m=0,\ 1,\ \cdots). \tag{7.3}$$

§ 8. Equation (5.2) is NLPDE

We put

$$\begin{pmatrix} Q_1 \\ R_1 \end{pmatrix}=L'^n\begin{pmatrix} q' \\ r' \end{pmatrix}.$$

It is known that[3] the Q_1, R_1 do not contain any integral expression of r' and q', instead, being expressed only in terms of products r and q and of their derivatives. By virtue of the definition of L' and (1.10), one concludes (by recursion) that

$$L'^n\begin{pmatrix} q' \\ r' \end{pmatrix}=\begin{pmatrix} Q_1 \\ R_1 \end{pmatrix}=\begin{pmatrix} \lambda^2 Q_2 \\ \lambda^{-2}R_2 \end{pmatrix}, \tag{8.1}$$

where Q_2 and R_2 are polynomials of r, q and their successive derivatives. Using (2.3), (2.8) and (8.1), we have

$$SL^n\begin{pmatrix} q \\ r \end{pmatrix}=\begin{pmatrix} \lambda^2 Q_2 \\ \lambda^{-2}R_2 \end{pmatrix}. \tag{8.2}$$

From (8.2) and (3.1), it yields

$$\begin{pmatrix} Q \\ R \end{pmatrix}=DL^n\begin{pmatrix} q \\ r \end{pmatrix}=DS^{-1}\begin{pmatrix} \lambda^2 Q_2 \\ \lambda^{-2}R_2 \end{pmatrix}=\begin{pmatrix} -2iQ_2+\frac{i}{2}q^2R_2 \\ DR_2 \end{pmatrix}.$$

This shows that Q and R are polynomials of q, r and their successive derivatives i. e., the eq. (5.2) is a NLPDE.

§ 9. Hamiltonian Structure

For every real ξ, we define the following matrix solutions Φ', Ψ' and Φ, Ψ for (1.1) and (1.6), respectively, with the boundary conditions

$$\Phi'=\begin{pmatrix} \varphi_1' & \bar{\varphi}_1' \\ \varphi_2' & \bar{\varphi}_2' \end{pmatrix}\xrightarrow[x\to-\infty]{}\begin{pmatrix} e^{-i\xi x} & 0 \\ 0 & -e^{i\xi x} \end{pmatrix}, \tag{9.1}$$

$$\Psi'=\begin{pmatrix} \psi_1' & \bar{\psi}_1' \\ \psi_2' & \bar{\psi}_2' \end{pmatrix}\xrightarrow[x\to\infty]{}\begin{pmatrix} 0 & e^{i\xi x} \\ e^{-i\xi x} & 0 \end{pmatrix},$$

$$\Phi=\begin{pmatrix}\varphi_1 & \bar\varphi_1\\ \varphi_2 & \bar\varphi_2\end{pmatrix}\xrightarrow[x\to-\infty]{}\begin{pmatrix}e^{-i\xi x} & 0\\ 0 & -e^{i\xi x}\end{pmatrix},\tag{9.2}$$

$$\Psi=\begin{pmatrix}\psi_1 & \bar\psi_1\\ \psi_2 & \bar\psi_2\end{pmatrix}\xrightarrow[x\to\infty]{}\begin{pmatrix}0 & e^{-i\xi x}\\ e^{i\xi x} & 0\end{pmatrix}.$$

From (1.9), we have

$$\begin{pmatrix}\varphi_1' & \bar\varphi_1'\\ \varphi_2' & \bar\varphi_2'\end{pmatrix}=T\begin{pmatrix}\varphi_1 & \bar\varphi_1\\ \varphi_2 & \bar\varphi_2\end{pmatrix}\begin{pmatrix}\frac{1}{\lambda_0} & 0\\ 0 & \lambda_0\end{pmatrix},$$

$$\lambda_0=e^{\frac{i}{2}\int_{-\infty}^{\infty}q(x)r(x)dx},\tag{9.3}$$

$$\begin{pmatrix}\psi_1' & \bar\psi_1'\\ \psi_2' & \bar\psi_2'\end{pmatrix}=T\begin{pmatrix}\psi_1 & \bar\psi_1\\ \psi_2 & \bar\psi_2\end{pmatrix}.\tag{9.4}$$

This yields easily[1]

$$a'(\xi, t)=\frac{1}{\lambda_0}a(\xi, t),$$

$$b'(\xi, t)=\frac{1}{\lambda_0}b(\xi, t).\tag{9.5}$$

From (9.3), (9.4), (9.5), (1.6) and (2.2), we get

$$\xi\begin{pmatrix}-\frac{\varphi_1'\psi_1'}{a'}\\ \frac{\varphi_2'\psi_2'}{a'}\end{pmatrix}=S\begin{pmatrix}-\frac{\varphi_1\psi_1}{a}\\ \xi\frac{\varphi_2\psi_2}{a}\end{pmatrix}.\tag{9.6}$$

Using (9.6), (3.2), (3.7) and the following expansion[4]

$$\begin{pmatrix}-\frac{\varphi_1'\psi_1'}{a'}\\ \frac{\varphi_2'\psi_2^1}{a'}\end{pmatrix}\sim-\frac{1}{2i}\sum_{m=0}^{\infty}\frac{1}{\xi^{m+1}}L'^m\begin{pmatrix}q'\\ r'\end{pmatrix},\tag{9.7}$$

we obtain

$$\begin{pmatrix}-\frac{\varphi_1\psi_1}{a}\\ \xi\frac{\varphi_2\psi_2}{a}\end{pmatrix}\sim-\frac{1}{2i}\sum_{m=0}^{\infty}\frac{1}{\xi^m}L^m\begin{pmatrix}q\\ r\end{pmatrix}.\tag{9.8}$$

Moreover, we have[4]

$$\frac{\delta\ln a'}{\delta q'}=\frac{\varphi_2'\psi_2'}{a'},\quad \frac{\delta\ln a'}{\delta r'}=-\frac{\varphi_1'\psi_1'}{a'}.$$

By virtue of the difference between (1.1) and (1.6), it is easily seen that

$$\frac{\delta\ln a}{\delta q}=\xi\frac{\varphi_2\psi_2}{a},\quad \frac{\delta\ln a}{\delta r}=\frac{\varphi_1\psi_1}{a}.\tag{9.9}$$

But[4]

$$\ln a'\sim-\sum_{m=0}^{\infty}\frac{1}{\xi^{m+1}}C'_{m+1},\tag{9.10}$$

from (9.5) and (9.10), we have

$$\ln a \sim -\sum_{m=0}^{\infty} \frac{1}{\xi^m} C_m, \tag{9.11}$$

where the sequence $\{C_m\}_{m=0}^{\infty}$ are conserved quantities of (5.2).

Using (9.8), (9.9) and (9.11), we get

$$L^m \begin{pmatrix} q \\ r \end{pmatrix} = 2i\,\text{grad}_{r,q} C_m = 2i \begin{pmatrix} \dfrac{\delta C_m}{\delta r} \\ \dfrac{\delta C_m}{\delta q} \end{pmatrix},$$

Therefore, if

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = D \sum_{j=0}^{n-1} \alpha_j(t) L^{n-j-1} \begin{pmatrix} q \\ r \end{pmatrix}.$$

then

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = 2iD \sum_{j=0}^{n-1} \alpha_j(t)\, \text{rad}_{r,q}|C_{n-j-1},$$

and the Hamiltonian is

$$H = 2i \sum_{j=0}^{n-1} \alpha_j(t) C_{n-j-1}.$$

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References

[1] Li Yishen and Zhuang Dawei, *Scientia Sinica*, **A2** (1983), 107—118.

[2] Zhang Etang, Chen Dengyuan, Zeng Yungbo, Zhu Gu-cheng and Li Yishen. *Journal of China University of Science and Technology*, (1983) 54—62, Special Issue, (Math.).

[3] Calogero, F. and Degasperis, A. IL Nuovo Cimonto, 32B (1976), 201—242; Lett. Nuovo Cimento, **22** (1978), 263—269.

[4] Newell, A. C. in: Solitons. Topics in Current Physics, Vol 17, eds. R. Bullough and P. Caudrey. (Springer. Berlin. 1980), p. 177.