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COMPARISON OF INHOMOGENEOUS POISSON PROCESSES

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Abstract

This note illustrates the use of various partial orderings for point processes by comparing inhomogeneous Poisson processes, and studies the relationships among these orderings mainly in the case of inhomogeneous Poisson processes. Whitt has detailed several definitions of partial orderings which have been applied mainly to renewal processes (see, e. g., Whitt [14] and Miller[4]) and semi-Markov processe (see, e. g., Sonderman [10]).

§ 1. Introduction

Comparisons of random processes are useful in applied probability theory. In particular, these comparisons are often useful in providing bounds and approximations for intractable systems. So far, most work on comparing point processes. e. g. [4, 10, 14], have been devoted to comparing renewal processes and semi-Markov processes. Stoyan[11,12] simply gave definition and a few examples on comparison of point processes. Miller[4] and Sonderman[10] focused on renewal processes and semi-Markov processes respectively. In [14] Whitt introduced several partial orderings for point processes, and then studied the comparison mainly of renewal processes.

It seems desirable to study the comparison of inhomogeneous Poisson processes in detail because such a process is not in general a renewal process and because this kind of point process is important in both theory and practice. In this paper we first illustrate various partial orderings of point processes which have been applied mainly to renewal processes [4], [14] and semi-Markov processes [10], and discuss their relationship in the case of inhomogeneous Poisson processes. Then some questions concerning both limit operation and ordering are studied briefly. Finally, on the basis of results for inhomogeneous Poisson processes, we obtain two results concerning the comparison of compound Poisson processes.

We introduce the following notation.

Let $N\equiv\{N(t),\ t\geqslant 0\}$ be a counting process on $[0,\ \infty)$, where $N(t)$ denotes the

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number of points occuring in the interval $(0, t]$ for this process (assume that $\Pr\{N(0)=0\}=1$, i. e., the probability that no point occurs at $t=0$ is equal to 1). And let $T\equiv\{T(n), n\geqslant 0\}$ be the sequence of occurrence times associated with N, where

$$T(n)=\inf\{s\geqslant 0: N(s)\geqslant n\},\ n\geqslant 1 \tag{1.1}$$

with $T(0)=0$ and $T(n)=\infty$ if $N(t)<n$ for all $t\geqslant 0$. Obviously

$$0=T(0)\leqslant T(1)\leqslant T(2)\leqslant\cdots. \tag{1.2}$$

Let $L(N)$ denote the probability law of the random process N on the space of its sample paths.

In this paper, a counting process $N\equiv\{N(t), 0\leqslant t<\infty\}$ is called a (inhomogeneous) Poisson process if

(i) N has independent increments, and

(ii) for any $t>s\geqslant 0$, the increment $N(s, t]=N(t)-N(s)$ is Poisson distributed with parameter $\Lambda(t)-\Lambda(s)$, i. e.,

$$\Pr\{N(s, t]=n\}=(n!)^{-1}[\Lambda(t)-\Lambda(s)]^n\exp\{-[\Lambda(t)-\Lambda(s)]\} \text{ for } n\geqslant 0,$$

where $\Lambda(t)$ is a nonnegative nondecreasing right-continuous function of t. If $\Lambda(t)$ is continuous, the process N is said to be simple or orderly. We call $\Lambda(t)$ the accumulative mean or expectation function of the process N because

$$EN(s, t]=\Lambda(t)-\Lambda(s).$$

If $\Lambda(t)$ is differentiable and its derivative is $\lambda(t)$, then $\Lambda(t)$ can be expressed as

$$\Lambda(t)=\int_0^t \lambda(u)du,$$

and $\lambda\equiv\{\lambda(t);\ t\geqslant 0\}$ is called the intensity of the process N.

It is known, see, e. g., Snyder[8] or Miller[4], that if N is an inhomogeneous Poisson process with intensity λ, then for any $t>s\geqslant 0$ the conditional joint distrib ution of occurence times $T(N(s)+1), T(N(s)+2), \cdots, T(N(s)+k)$, given $N(s,t]=k$, is the same as the distribution of the order statistics of n independent identically distributed random variables with the common distribution function $(\Lambda(u)-\Lambda(s))/(\Lambda(t)-\Lambda(s))$ on $(s, t]$, $s<u\leqslant t$, and the density of this conditional joint distribution is

$$f_{s,t}(t_1, t_2, \cdots, t_k|k)=k!\prod_{i=1}^{k}\lambda(t_i)/(\Lambda(t)-\Lambda(s))^k,\ s<t_1\leqslant\cdots\leqslant t_k\leqslant t. \tag{1.3}$$

§ 2 Monotonicity of inhomogeneous Poisson processes

At first, following Whitt [14] we introduce five different partial orderings for counting processes as follows.

Definition 1. *For two counting processes N_1 and N_2,*

(1) $N_1\leqslant_f N_2$ *means that the conditional distribution* $\Pr\{T(N_i(t)+1)-t\leqslant x|N(s),$

$0 \leqslant s \leqslant t\}$, *which is the distribution function of the forward recurrence time at t conditional on the entire history of* N_i *up to t, have failure rate* $\gamma_i(x, t)$ *for each t and* $i=1, 2$ *(almost surely with respect to* N_i*), and for some* $\lambda(t)$ *the failure rate* $\gamma_i(x, t)$ *for* $i=1$ $(i=2)$ *is bounded above (below) by* $\lambda(t)$ *for each* $t \geqslant 0$.

(2) $N_1 \leqslant_{inc} N_2$ *means that there exist two processes* $\tilde{N}_1$ *and* $\tilde{N}_2$ *on a common probability space with associated occurrence time sequences* $\tilde{T}_1$ *and* $\tilde{T}_2$ *such that* $L(N_i) = L(\tilde{N}_i)$ *for each i and*

$$\{\tilde{T}_1(1), \tilde{T}_1(2), \cdots\} \subseteq \{\tilde{T}_2(1), \tilde{T}_2(2), \cdots\} \tag{2.1}$$

for all sample paths.

(3) $N_1 \leqslant_{int} N_2$ *means that there exist two processes* $\tilde{N}_1$ *and* $\tilde{N}_2$ *on a common probability space such that* $L(N_i) = L(\tilde{N}_i)$ *for each i and*

$$\tilde{T}_1(n) - \tilde{T}_1(n-1) \geqslant \tilde{T}_2(n) - \tilde{T}_2(n-1) \tag{2.2}$$

for all $n \geqslant 1$ *and all sample paths.*

(4) $N_1 \leqslant_n N_2$ *means that there exist two processes* $\tilde{N}_1$ *and* $\tilde{N}_2$ *on a common probability space such that* $L(N_i) = L(\tilde{N}_i)$ *for each i and*

$$\tilde{N}_1(t) \leqslant \tilde{N}_2(t) \tag{2.3}$$

for all $t \geqslant 0$ *and all sample paths.*

(5) $N_1 \leqslant_d N_2$ *means that*

$$\Pr\{N_1(t) > x\} \leqslant \Pr\{N_2(t) > x\} \tag{2.4}$$

for all x and t.

Remark 1. Recall that a nonnegative random variable has a failure rate $\gamma(t)$ if its distribution function $F(x)$ is absolutely continuous with respect to Lebesgue measure (a counting measure) and has a density (a probability mass function) $f(t)$, then $r(t) = f(t)/(1-F(t))$ for all t such that $F(t) < 1$, see, e. g., Barlow and Proschan [1].

The failure rate $\gamma(t)$ as well as the density $f(t)$ is defined uniquely, neglecting values on a null set with respect to Lebesgue measure. Therefore, e. g., in the definition of $N_1 \leqslant_f N_2$ the condition of boundedness may be restated more exactly as follows: For some $\lambda(t)$ the failure rate $\gamma_i(x, t)$ for $i=1$ $(i=2)$ is bounded above (below) by $\lambda(t)$ for each $t \geqslant 0$ and almost all x. Because two failure rates which are equal almost everywhere correspond to the same distribution function, we may identify statistically such two failure rates, and so we do not mention the words "almost all" or "almost everywhere" repeatedly in similar situations for simplicity and clarity, but we have to bear this remark in mind.

Remark 2. On talking about the orderings of random processes, an extension of a result of Strassen which shows that the stochastic order of all finite dimensional distributions of two random processes is equivalent to the possibility to construct two stochastically equivalent processes being compared on a common probability space so that each sample paths of one process lies below the corresponding sample path of the other process plays an important role, see, e. g. Kamal, Krengel and O'Brien[8] or Sonderman[9].

Remark 3. As Whitt[14] notes, we use the term 'partial ordering' loosely. The ordering $\leqslant_f$ is not reflexive. The ordering $\leqslant_d$ is not antisymmetric, and the quasi-Poisson processes of Szasz[13] afford a simple example illustrating this fact.

Remark 4. Whitt[14] added the requirement that for each $t\geqslant 0$ and all sample paths

$$\widetilde{N}_1(t)-\widetilde{N}_1(t_-)\leqslant\widetilde{N}_2(t)-\widetilde{N}_2(t_-) \tag{2.5}$$

to the definition of the ordering $N_1\leqslant_{ino}N_2$. It should be pointed out that (2.5) is not necessary for counting processes. Noting (1.1) and (1.2) we are able to see that (2.5) is automatically satisfied while (2.1) holds in this case. However, (2.5) should be added for comparing marked point processes which will be concerned with below.

Now, turn to discussion of the relationships among various orderings in Definition 1.

Theorem 1. *For general point processes N_1 and N_2,*

$$N_1\leqslant_f N_2\Rightarrow N_1\leqslant_{ino}N_2\Rightarrow N_1\leqslant_n N_2\Rightarrow N_1\leqslant_d N_2$$

and

$$N_1\leqslant_{int}N_2\Rightarrow N_1\leqslant_n N_2,$$

for renewal processes N_1 and N_2,

$$N_1\leqslant_f N_2\Rightarrow N_1\leqslant_{ino}N_2\Rightarrow N_1\leqslant_{int}N_2\Leftrightarrow N_1\leqslant_n N_2\Leftrightarrow N_1\leqslant_d N_2,$$

where "$\Rightarrow$" means "to imply" and "$\Leftrightarrow$" means "equivalent to".

This theorem without proof can be found in [14], here we just point out the following examples as a supplementary explanation.

The example given in Schmidt [7] shows that the ordering $\leqslant_d$ does not imply the ordering $\leqslant_{int}$. Furthermore, the quasi-Poisson processes of Szasz [13] is also able to provide an example which indicates the ordering $\leqslant_d$ does not imply the other orderings.

In [2] Daley gave two examples. One of them illustrates that $N_1\leqslant_f N_2$ may be false while $N_1\leqslant_{ino}N_2$, where N_1 and N_2 both are doubly stochastic Poisson processes. The another indicates that $N_1\leqslant_n N_2$ does not imply $N_1\leqslant_{int}N_2$, where N_1 is a renewal process and N_2 is an alternating renewal process.

By defining the occurrence time sequences T_1 and T_2 directly, it is easy to construct examples which show the ordering $\leqslant_{ino}$ and the ordering $\leqslant_{int}$ do not imply each other for general point processes.

The ordering $\leqslant_f$ is the strongest among five orderings. Let N_i be a renewal process associated with interarrival time X_i which has failure rate $Y_i(t)$ for $i=1, 2$. Whitt (Theorem 2 of [14]) pointed out the following results:

(1) $N_1 \leqslant_f N_2$ if and only if $\inf_{t\geqslant 0} \gamma_2(t) \geqslant \sup_{t\geqslant 0} \gamma_1(t)$;

(2) $N_1 \leqslant_{int} N_2$ if and only if $P(X_1 > x) \leqslant P(X_2 > x)$ for all x.

From this result it can be seen that the ordering $\leqslant_f$ is strongest.

However, we have a very nice result in the case of homogeneous processes.

Theorem 2. *For homogeneous Poisson processs all five orderings in Definition 1 are equivalent.*

The conclusion of Theorem 2 follows immediately from the fact that a homogeneous Poisson process is defined statistically by its constant intensity λ. But, this conclusion is false for inhomogeneous Poisson processes, moreover, in this case the results for renewal proaesses in Theorem 1 are not available because an inhomogeneous Poisson process in general is not a renewal process. Hence we have to study how the orderings are related for inhomogeneous Poisson processes.

We first introduce the following orderings $\leqslant_{sn}$ and $\leqslant_{sd}$ which seem to be more natural in the case of inhomogeneous point processes.

Definition 2. *For two counting point processes N_1 and N_2,*

(1) *$N_1 \leqslant_{sn} N_2$ means that there exist two processes $\tilde{N}_1$ and $\tilde{N}_2$ on a common probability space such that $L(N_i) = L(\tilde{N}_i)$ for each i and*

$$\tilde{N}_1(s, t] \leqslant \tilde{N}_2(s, t] \tag{2.6}$$

for all $t > s \geqslant 0$ and all sample paths.

(2) *$N_1 \leqslant_{sd} N_2$ means that for all x and all $t > s \geqslant 0$*

$$\Pr\{N_1(s, t] > x\} \leqslant \Pr\{N_2(s, t] > x\}. \tag{2.7}$$

Note that the relations (2.1) and (2.5) may be specified as follows: Every jump in $\tilde{N}_1$ is also a jump in $\tilde{N}_2$ and its size in $\tilde{N}_1$ is at most as large as in $\tilde{N}_2$ for all sample paths, in other words, the inequality

$$\tilde{N}_1(t, t+dt] \leqslant \tilde{N}_2(t, t+dt] \tag{2.1'}$$

holds for all t and all sample paths. Obviously, (2.1') is equivalent to (2.6), i. e., $N_1 \leqslant_{sn} N_2$ is identical with $N_1 \leqslant_{inc} N_2$.

Theorem 3. *Let N_1 and N_2 be two inhomogeneous Poisson processes wite intensities λ_1 and λ_2 respectively, then $N_1 \leqslant_f N_2$ if and only if*

$$\sup_{t\geqslant 0} \lambda_1(t) \leqslant \inf_{t\geqslant 0} \lambda_2(t).$$

Proof For an inhomogeneous Poisson process the failure rate of the conditonal distribution $\Pr\{T(N(t)+1) - t \leqslant x \mid N(s), 0 \leqslant s \leqslant t\}$ is equal to

$$\gamma(t, x) = \frac{\lambda(t+x)\exp\left\{-\int_t^{t+x} \lambda(u)\,du\right\}}{1-\left(1-\exp\left\{-\int_t^{t+x} \lambda(u)\,du\right\}\right)} = \lambda(t+x) \tag{2.8}$$

for all x and all t. By the definition of the ordering $\leqslant_f$ we partioularly have

$$\gamma_1(0, x) \leqslant \lambda(0) \leqslant \gamma_2(0, x)$$

for all x, where $\lambda(0)$ is some real number, so

$$\sup_{x\geq 0}\lambda_1(x)\leqslant\inf_{x\geq 0}\lambda_2(x).$$

On the other hand, if

$$\sup_{t\geq 0}\lambda_1(t)\leqslant\inf_{t\geq 0}\lambda_2(t)$$

holds, then

$$\sup_{x\geq 0}\gamma_1(t,\ x)\leqslant\inf_{x\geq 0}\gamma_2(t,\ x)$$

for all t because

$$\sup_{x\geq 0}\gamma_1(t,\ x)=\sup_{x\geq 0}\lambda_1(t+x)\leqslant\sup_{x\geq 0}\lambda_1(x)\leqslant\inf_{x\geq 0}\lambda_2(x)\leqslant\inf_{x\geq 0}\lambda_2(t+x)$$
$$\leqslant\inf_{x\geq 0}\gamma_2(t,\ x)$$

for all $t\geqslant 0$, so we are able to find some $\lambda(t)$ such that $\lambda_1(t.\ x)\leqslant\lambda(t)\leqslant\lambda_2(t,\ x)$ for all t, i. e., $N_1\leqslant_f N_2$.

Theorem 4. *Let N_1 and N_2 be two inhomogeneous Poisson processes with intensities λ_1 and λ_2 respectively, then*

$$N_1\leqslant_{inc}N_2\Leftrightarrow N_1\leqslant_{sd}N_2\Leftrightarrow\lambda_1\leqslant\lambda_2.$$

Proof We first prove that $N_1\leqslant_{sd}N_2\Rightarrow\lambda_1\leqslant\lambda_2$. In fact, from $\Pr\{N_1(s,\ t]>0\}\leqslant\Pr\{N_2(s,\ t]>0\}$ for all $t>s\geqslant 0$ follows that

$$1-\exp\left\{-\int_s^t\lambda_1(u)du\right\}\leqslant 1-\exp\left\{-\int_s^t\lambda_2(u)du\right\}\text{ for all }0\leqslant s<t,$$

then

$$\int_s^t\lambda_1(u)du\leqslant\int_s^t\lambda_2(u)du\qquad\text{for all }0\leqslant s<t.$$

It is equivalent to $\lambda_1(t)\leqslant\lambda_2(t)$ for all $t\geqslant 0$.

To complete the proof, we have to prove that $\lambda_1\leqslant\lambda_2\Rightarrow N_1\leqslant_{inc}N_2$. For this end we show that N_1 can be constructed by thinning N_2, i. e., let $T_2=\{T_2(1),\ T_2(2),\ \cdots\}$ be the occurrence time sequence associated with N_2, it is possible to construct an inhomogeneous Poisson process $\tilde{N}_1$ on the same probability space such that $L(N_1)=L(\tilde{N}_1)$ and the occurrence time sequence $\tilde{T}_1=\{\tilde{T}_1(1),\ \tilde{T}_1(2),\ \cdots\}$ associated with $\tilde{N}_1$ is a subsequence of T_2. Indeed, we may construct $\tilde{T}_1=\{\tilde{T}_1(1),\tilde{T}_1(2),\cdots\}$ by thinning (removing) the points $T_2(n)$, $n=1,\ 2,\ \cdots$, with probability $1-\lambda_1(T_2(n))/\lambda_2(T_2(n))$ in sequence. Obviously, the sequence $\tilde{T}_1$ obtained by thinning T_2 is a subsequence of T_2 and the thinning of the points $T_2(n)$, $n=1,\ 2,\ \cdots$, are independent mutually. For any fixed $t>s\geqslant 0$, we first consider the probability

$$P_{o,k}=\Pr\{\tilde{N}_1(s,\ t]=0\,|\,s<T_2(N_2(s)+1)=t_1\leqslant\cdots\leqslant T_2(N_2(s)+k)=t_k;$$
$$N_2(s,\ t]=k\}.$$

According to the rule of thinning

$$P_{o,k}=\begin{cases}\prod_{i=1}^{k}(1-\lambda_1(t_i)/\lambda_2(t_i)), & \text{for } k\geqslant 1,\\ 1, & \text{for } k=0.\end{cases}\tag{2.9}$$

Then, by (1.3) and (2.9) it follows that

$$\Pr\{\tilde{N}_1(s,\ t]=0\}=\sum_{k=0}^{\infty}\Pr\{\tilde{N}_1(s,\ t]=0,\ N_2(s,\ t]=k\}$$
$$=\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}+\sum_{k=1}^{\infty}\int_{s<t_1\leqslant\cdots\leqslant t_k\leqslant t}P_{o,k}\cdot f_{s,t}^{(2)}(t_1,\ \cdots,\ t_k|k)$$
$$\times\Pr\{N_2(s,\ t]=k\}dt_1\cdots dt_k$$
$$=\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}+\sum_{k=1}^{\infty}\int_{s<t_1\leqslant\cdots\leqslant t_k\leqslant t}\prod_{i=1}^{k}\Big(1-\frac{\lambda_1(t_i)}{\lambda_2(t_i)}\Big)k!\prod_{i=1}^{k}\frac{\lambda_2(t_i)}{\Lambda_2(t)-\Lambda_2(s)}$$
$$\times\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}[\Lambda_2(t)-\Lambda_2(s)]^k/k!\,dt_1\cdots dt_k$$
$$=\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}+\sum_{k=1}^{\infty}\int_{s<t_1,\cdots,t_k\leqslant t}\prod_{i=1}^{k}[\lambda_2(t_i)-\lambda_1(t_i)]$$
$$\times\frac{\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}}{k!}dt_1\cdots dt_k.$$

By symmetry the last sum equals

$$\sum_{k=1}^{\infty}\Big[\int_s^t(\lambda_2(u)-\lambda_1(u))du\Big]^k\frac{\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}}{k!}$$
$$=\sum_{k=1}^{\infty}\frac{\{[\Lambda_2(t)-\Lambda_2(s)]-[\Lambda_1(t)-\Lambda_1(s)]\}^k}{k!}\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}$$
$$=(\exp\{[\Lambda_2(t)-\Lambda_2(s)]-[\Lambda_1(t)-\Lambda_1(s)]\}-1)\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\},$$

hence

$$\Pr(\{\tilde{N}_1(s,\ t]=0\}=\exp\{-[\Lambda_1(t)-\Lambda_1(s)]\}.$$

Secondly, for any $m\geqslant1$

$$\Pr\{\tilde{N}_1(s,\ t]=m\}=\sum_{k=m}^{\infty}\Pr\{\tilde{N}_1(s,\ t]=m,\ N_2(s,\ t]=k\}$$
$$=\sum_{k=m}^{\infty}\int_{s<t_1\leqslant\cdots\leqslant t_k\leqslant t}\Pr\{\tilde{N}_1(s,\ t]=m|s<T_2(N_2(s)+1)=t_1\leqslant\cdots$$
$$\leqslant T_2(N_2(s)+k)=t_k\leqslant t,\ N_2(s,\ t]=k\}$$
$$\times f_{s,t}^{(2)}(t_1,\ \cdots,\ t_k|k)\Pr\{N_2(s,\ t]=k\}dt_1\cdots dt_k$$
$$=\sum_{k=m}^{\infty}\int_{s<t_1\leqslant\cdots\leqslant t_k\leqslant t}\sum_{\pi(t_{n_1},\cdots,t_{n_m})}\prod_{\substack{i=1\\ t_{n_i}\in\{t_1,\cdots,t_k\}}}^{m}\frac{\lambda_1(t_{n_i})}{\lambda_2(t_{n_i})}$$
$$\times\prod_{\substack{j=1\\ t_{n_j}\in\{t_1,\cdots,t_k\}\setminus\{t_{n_1},\cdots,t_{n_m}\}}}^{k-m}\Big(1-\frac{\lambda_1(t_{n_j})}{\lambda_2(t_{n_j})}\Big)$$
$$\times k!\prod_{l=1}^{k}\frac{\lambda_2(t_l)}{\Lambda_2(t)-\Lambda_2(s)}\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}$$
$$\times\frac{[\Lambda_2(t)-\Lambda_2(s)]^k}{k!}dt_1\cdots dt_k,$$

where $\sum_{\pi(t_{n_1},\cdots,t_{n_m})}$ denotes the summation over all combinations of $t_{n_1},\ \cdots,\ t_{n_m}$ out of $t_1,\ \cdots,\ t_k$. By using symmetry again it follows that

$$\Pr\{\tilde{N}_1(s,\ t]=m\}=\frac{k!}{m!\,(k-m)!}\Big[\int_s^t\lambda_1(u)du\Big]^m\Big[\int_s^t[\lambda_2(u)-\lambda_1(u)]du\Big]^{k-m}$$
$$\times\exp\{-\Lambda_2(t)-\Lambda_2(s)]\}/k!$$

$$=\frac{1}{m!}\left\{\sum_{k=m}^{\infty}\frac{\{[\Lambda_2(t)-\Lambda_2(s)]-[\Lambda_1(t)-\Lambda_1(s)]\}^{k-m}}{(k-m)!}\right.$$
$$\left.\times[\Lambda_1(t)-\Lambda_1(s)]^m\exp\{-[\Lambda_2(t)-\Lambda_2(s)]\}\right\}$$
$$=\frac{[\Lambda_1(t)-\Lambda_1(s)]^m}{m!}\exp\{-[\Lambda_1(t)-\Lambda_1(s)]\}.$$

Finally, from the construction and the fact that N_2 has independent increments follows that $\tilde{N}_1$ also has independent increments, so $\tilde{N}_1$ is an inhomogeneous Poisson process with intensity λ_1.

Theorem 5. *Let N_1 and N_2 be two inhomogeneous Poisson processes with cumulative mean functions $\Lambda_1(t)$ and $\Lambda_2(t)$ respectively. If*

$$\Lambda_1^{-1}(u)-\Lambda_1^{-1}(v)\geqslant\Lambda_2^{-1}(u)-\Lambda_2^{-1}(v)\qquad \textit{for any } u>v\geqslant 0, \tag{2.10}$$

at which the inverse functions are well-defined, then

$$N_1\leqslant_{int}N_2,$$

where $\Lambda_i^{-1}(u)$ is the inverse function of $\Lambda_i(t)$, which is defined by

$$\Lambda_i^{-1}(u)=\inf\{t:\Lambda_i(t)\geqslant u\}.$$

Proof The proof depends on the fact that an inhomogeneous Poisson process $N(t)$ with cumulative mean function $\Lambda(t)$ may be represented as $N(t)=M(\Lambda(t))$, where $M(t)$ is a homogeneous Poisson process with unit intensity, see, e. g., Snyder([8]p.62). In other words, an inhomogeneous Poisson process can be constructed by rescaling the time coordinate from a homogeneous Poisson process $M(t)$ with unit intensity. Therefore, we may construct the processes N_i, $i=1, 2$, in this manner. Let $M(t)$ be a homogeneous Poisson process with unit intensity and let the occurrence time sequence be $S_1, S_2, \cdots$. For $i=1, 2$, translate the points S_n associated with $M(t)$ to $T_i(n)=\Lambda_i^{-1}(S_n)$ for $n=1, 2, \cdots$. It is easy to see that the process N_i with associated occurrence time sequence $T_i(1), T_i(2), \cdots$ is an inhomogeneous Poisson process with cumulative mean function $\Lambda_i(t)$ for $i=1, 2$, and

$$T_1(n)-T_1(n-1)\geqslant T_2(n)-T_2(n-1)$$

for all $n\geqslant 1$ from (2.10) and the construction of N_1 and N_2.

Finally, it should be noted that the fact that $\Lambda_i^{-1}(S_n)$ is not well-defined means the point $T_i(n)$ actually does not exist in the process N_i.

Theorem 6. *Let N_i be an inhomogeneous Poisson process with intensity λ_i for $i=1, 2$. If $\lambda_1\leqslant\lambda_2$ and either λ_1 or λ_2 is nonincreasing, then $N_1\leqslant_{int}N_2$.*

Proof It suffices to prove that (2.10) is satisfied. At first, suppose that both $\lambda_1(t)$ and $\lambda_2(t)$ are not equal to zero. From the differential rule of inverse function and definition of intensity follows that for any $\varepsilon>0$ and $\omega\geqslant 0$, there exists $\delta(\omega, \varepsilon)>0$ such that

$$\frac{\Lambda_1^{-1}(\omega')-\Lambda_1^{-1}(\omega)}{\omega'-\omega}\geqslant\frac{d^+\Lambda_1^{-1}(\omega)}{d\omega}-\frac{\varepsilon}{2}=\frac{1}{\lambda_1(\Lambda_1^{-1}(\omega)+0)}-\frac{\varepsilon}{2}\quad\text{for } \omega'\in(\omega,\ \omega+\delta),$$

$$\frac{\Lambda_2^{-1}(\omega')-\Lambda_2^{-1}(\omega)}{\omega'-\omega}\leqslant\frac{d^+\Lambda_2^{-1}(\omega)}{d\omega}+\frac{\varepsilon}{2}=\frac{1}{\lambda_2(\Lambda_2^{-1}(\omega)+0)}+\frac{\varepsilon}{2}\quad\text{for } \omega'\in(\omega,\ \omega+\delta),\tag{2.11}$$

$$\frac{\Lambda_1^{-1}(\omega')-\Lambda_1^{-1}(\omega)}{\omega'-\omega}\geqslant\frac{d^-\Lambda_1^{-1}(\omega)}{d\omega}-\frac{\varepsilon}{2}=\frac{1}{\lambda_1(\Lambda_1^{-1}(\omega)-0)}-\frac{\varepsilon}{2}\quad\text{for } \omega'\in(\omega-\delta,\ \omega),$$

$$\frac{\Lambda_2^{-1}(\omega')-\Lambda_2^{-1}(\omega)}{\omega'-\omega}\leqslant\frac{d^-\Lambda_2^{-1}(\omega)}{d\omega}+\frac{\varepsilon}{2}=\frac{1}{\lambda_2(\Lambda_2^{-1}(\omega)-0)}+\frac{\varepsilon}{2}\quad\text{for } \omega'\in(\omega-\delta,\ \omega).$$

From the condition $\lambda_1\leqslant\lambda_2$ follows that

$$\Lambda_1(t)\leqslant\Lambda_2(t)\qquad\text{for all } t\geqslant 0,$$

then

$$\Lambda_1^{-1}(\omega)\geqslant\Lambda_2^{-1}(\omega)\qquad\text{for all } \omega\geqslant 0.$$

Under the assumption that either λ_1 or λ_2 is non-increasing, we have

$$1/\lambda_1(\Lambda_1^{-1}(\omega))\geqslant 1/\lambda_2(\Lambda_2^{-1}(\omega))\qquad\text{for all } \omega\geqslant 0.\tag{2.12}$$

Let $I(\omega)=I_-(\omega)\cup\{\omega\}\cup I_+(\omega)=(\omega-\sigma,\ \omega+\sigma)$, where $I_-(\omega)=(\omega-\sigma,\ \omega)$ and $I_+(\omega)=(\omega,\ \omega+\sigma)$, from (2.11) and (2.12) it follows that

$$\frac{\Lambda_1^{-1}(\omega')-\Lambda_1^{-1}(\omega)}{\omega'-\omega}\geqslant\frac{\Lambda_2^{-1}(\omega')-\Lambda_2^{-1}(\omega)}{\omega'-\omega}-\varepsilon\qquad\text{for } \omega'\in I(\omega)-\{\omega\},\tag{2.13}$$

i. e.,

$$\Lambda_1^{-1}(\omega')-\Lambda_1^{-1}(\omega)\geqslant\Lambda_2^{-1}(\omega')-\Lambda_2^{-1}(\omega)-\varepsilon(\omega'-\omega)\qquad\text{for } \omega'\in I_+(\omega);$$

$$\Lambda_1^{-1}(\omega)-\Lambda_1^{-1}(\omega')\geqslant\Lambda_2^{-1}(\omega)-\Lambda_2^{-1}(\omega')-\varepsilon(\omega-\omega')\qquad\text{for } \omega'\in I_-(\omega).\tag{2.14}$$

For any fixed $u>v>0$, at which the inverse functions Λ_1^{-1} and Λ_2^{-1} are well defined, the family of open intervals $\{I(\omega);\ \omega\in[v,\ u]\}$ constitutes an open cover for the closed interval $[v,\ u]$. By a well-known theorem there exists a finite subcover $\{I(\omega_0),\ I(\omega_1),\ \cdots,\ I(\omega_n)\}$ of $\{I(\omega);\ \omega\in[u,\ v]\}$ for $[v,\ u]$, without loss of generality assume that $v=\omega_0<\omega_1<\cdots<\omega_n=u$ and there is at least one common point $\omega^{(i)}$ in $I_+(\omega_i)\cap I_-(\omega_{i+1})$ for $i=0,\ 1,\ \cdots,\ n-1$. By writing

$$\Lambda_i^{-1}(u)-\Lambda_i^{-1}(v)=[\Lambda_i^{-1}(u)-\Lambda_i^{-1}(\omega^{(n-1)})]+[\Lambda_i^{-1}(\omega^{(n-1)})-\Lambda_i^{-1}(\omega_{n-1})]+\cdots$$
$$+[\Lambda_i^{-1}(\omega_1)-\Lambda_i^{-1}(\omega^{(0)})]+[\Lambda_i^{-1}(\omega^{(0)})-\Lambda_i^{-1}(v)]$$

for $i=1$, 2 and using (2.14), it is easy to see that

$$\Lambda_1^{-1}(u)-\Lambda_1^{-1}(v)\geqslant\Lambda_2^{-1}(u)-\Lambda_2^{-1}(v)+\varepsilon(u-v).$$

Because of the arbitrarity of ε it follows that

$$\Lambda_1^{-1}(u)-\Lambda_1^{-1}(v)\geqslant\Lambda_2^{-1}(u)-\Lambda_2^{-1}(v).$$

Finally, note that the inequality (2.13) still holds only under the assumption that $\lambda_2(t)>0$ for all $t\geqslant 0$ ($\lambda_1(t)$ may be equal to zero). Turn to consideing the case that $\lambda_2(t_0)=0$ for some t_0. If $\lambda_1(t)$ is non-increasing, then it follows from

$$\lambda_2(t_0)\geqslant\lambda_1(t_0)$$

that $\lambda_1(t_0)=0$. Let $t_1=\inf\{t;\ \lambda_1(t)=0\}$, by hypothesis about $\lambda_1(t)$ it is clear that

$\lambda_1(t)=0$ for $t\in[t_1, \infty)$, and $\lambda_1(t)>0$ then $\lambda_2(t)>0$ as well for $t\in[0, t_1)$. So we may compare N_1 and N_2 only in $[0, t_1)$ because the situation is trivial in $[t_1, \infty)$. Similarly, in the case that $\lambda_2(t)$ is non-increasing we have $\lambda_2(t)>0$ for $t\in[0, t_2)$, where $t_2=\inf\{t; \lambda_2(t)=0\}$. At this time we may compare N_1 and N_2 only in $[0, t_2)$ because both $\lambda_1(t)$ and $\lambda_2(t)$ equal zero in $[t_2, \infty)$.

Examples. Let N_i be an inhomogeneous Poisson process with intensity $\lambda_i(t)$ for $i=1, 2$,

(1) Suppose that

$$\lambda_1(t)=\begin{cases} a_1, & 0\leqslant t<t_0; \\ t^{b_1}, & t\geqslant t_0, \end{cases}$$

and

$$\lambda_2(t)=\begin{cases} a_2, & t\leqslant t<t_0; \\ t^{b_2}, & t\geqslant t_0, \end{cases}$$

here $a_1(\geqslant 0)$, $a_2(\geqslant 0)$, b_1, b_2 and t_0 are constants.

If (i) $a_1\geqslant t^{b_1}$, $b_1<0$, $a_1\leqslant a_2$ and $b_1\leqslant b_2$,

or

(ii) $a_2\geqslant t_0^{b_2}$, $b_2<0$, $a_1\leqslant a_2$ and $b_1\leqslant b_2$,

then $N_1\leqslant_{int}N_2$.

(2) Suppose that

$$\lambda_1(t)=b_1e^{a_1t}$$

and

$$\lambda_2(t)=b_2e^{a_2t},$$

where a_1, a_2, $b_1(\geqslant 0)$ and $b_2(\geqslant 0)$ are constants.

If $a_1\leqslant 0$, $a_1\leqslant a_2$ and $b_1\leqslant b_2$, then $N_1\leqslant_{int}N_2$.

Obviously, Theorem 6 gives just some kind of the suffioiet condition for $N_1\leqslant N_2$. It is not difficult to construct examples demonstrating the fact that the relation $N_1\leqslant_{int}N_2$, in general, does not imply the inequality $\lambda_1\leqslant\lambda_2$.

Summarizing Theorem 3, 4 and 6, we have the following conclusin.

Theorem 7. *Let N_i be an inhomogeneous Poisson process with intensity λ_i for $i=1, 2$, then*

$$N_1\leqslant_f N_2\Rightarrow N_1\leqslant_{inc}N_2\Leftrightarrow N_1\leqslant_{sd}N_2,$$

and

$$N_1\leqslant_f N_2\Rightarrow N_1\leqslant_{int}N_2.$$

The proof of the theorem is immediate, here we want to point out that if the condition of Theorem 3 is satisfied, then the inequality (2.13) still holds, and from this follows that $N_1\leqslant_{int}N_2$.

Now, we deal with some questions concerning both limit operation and ordering briefly.

Theorem 8. (1) *Let* N^m, $m=1, 2, \cdots$ *and* N *be counting point processes. If* $N^m(t) \to N(t)$ *as* $m\to\infty$ *for each* t *almost surely, then* $T^m(n) \to T(n)$ *for all* n *almost sure, where* $T^m(n)$ *and* $T(n)$ *are* n*-th occurrence times associated with* N^m *and* N *respectively.*

(2) *Let* N_i^m *and* N_i *be counting point processes for* $i=1, 2$ *and* $m=1, 2, \cdots$. *If* $N_i^m(t) \to N_i(t)$ *for each* i *and every* $t\geqslant 0$ *almost surely and* $\{T_1^m(1), T_1^m(2), \cdots\}\subseteq\{T_2^m(1), T_2^m(2), \cdots\}$ $(T_1^m(n)-T_1^m(n-1)\geqslant T_2^m(n)-T_2^m(n-1)$ *for all* $n\geqslant 1)$ *for each* m, *then* $\{T_1(1), T_1(2), \cdots\}\subseteq\{T_2(1), T_2(2), \cdots\}$ $(T_1(n)-T_1(n-1)\geqslant T_2(n)-T_2(n-1)$ *for all* $n\geqslant 1)$ *almost surely, i. e.* $N_1\leqslant_{inc}N_2$ $(N_1\leqslant_{int}N_2)$.

Proof

(1) Because the sample paths of N^m and N belong to the function space $D[0, \infty)$, it follows from the right continuity of sample paths that $N^m(t)\to N(t)$ implies the following fact: there exists an integer K, which may be different for different t and different sample path ω, such that $N^m(t)=N(t)$ for $m>K$. To prove $T^m(n)\to T(n)$ for all n almost surely, it suffices to show that $T^m(n)\to T(n)$ for each n almost surely. Were this convergence not true, i. e., if there would exist a $\delta>0$ and an ω-set B such that $\Pr(B)>0$ and for $\omega\in B$ and every m, one could find an integer $M_k>m$ such that $|T(n, \omega)-T^{M_k}(n, \omega)|>\delta$. First assume that $T(n, \omega)-T^{M_k}(n, \omega)>\delta$, i. e., $T^{M_k}<T(n, \omega)-\delta$, writing $T(n, \omega)=t_n$, we have $N(t_n-\delta)\leqslant n-1$. On the other hand, it follows from $T^{M_k}(n, \omega)<t_n-\delta$ that $N^{M_k}(t_n-\delta, \omega)\geqslant n$, then

$$N^{M_k}(t_n-\delta, \omega)>N(t_n-\delta, \omega)$$

for $\omega\in B$. This is contradictory to that $N^m(t)=N(t)$ for all t and enough large m almost surely. In the case that $T^{M_k}(n, \omega)-T(n, \omega)>\delta$ a similar contradiction can be obtained. The proof of (1) is complete.

(2) From $T_i^m(n)\to T_i(n)$ as $m\to\infty$ for $i=1, 2$ and $\{T_1^m(n)\}\subseteq\{T_2^m(n)\}$ follows that $T_2(n)\leqslant T_1(n)$ for every n. Obviously, it is enough to discuss the case of $T_1(n)<\infty$. Denote $T_1(n)$ by t_n. In the finite interval $[0, t_n]$, there are only finitely many, n' say, points of N_2 by the hypothesis of finiteness of counting processes. On the other hand, $T_1^m(n)$ is a point in the sequence $\{T_2^m(1), T_2^m(2), \cdots\}$ for every m, i. e., $T_1^m(n)=T_2^m(n_m)$ for some $n_m\geqslant n$. It is easy to see that $n_m\leqslant n'$ for large m. We now consider the sequence $\{T_2^1(n_1), T_2^2(n_2), \cdots, T_2^k(n_k), \cdots\}$. There are infinitely many n_m which take a common value, n'' say, because $n_m\leqslant n'$ for large m. That is, there exists a subsequence $\{T_2^{m_1}(n_{m_1}), T_2^{m_2}(n_{m_2}), \cdots\}$ of the sequence $\{T_2^1(n_1), T_2^2(n_2), \cdots\}$ such that $T_2^{m_k}(n_{m_k})=T_2^{m_k}(n'')$ for all m_k, then $T_2^{m_k}(n'')\to T_2(n'')$ as $m_k\to\infty$ follows from $T_2^m(n'')\to T_2(n'')$ as $m\to\infty$. However, the sequence $\{T_2^1(n_1), T_2^2(n_2), \cdots\}$ is the sequence $\{T_1^1(n), T_1^2(n), \cdots\}$ originally. So $T_2^{m_k}(n'')=T_1^{m_k}(n)\to T_1(n)$ as $m_k\to\infty$. Consequently. it follows from the uniqueness of limit that $T_1(n)=T_2(n'')$, i. e.,

$$\{T_1(1), T_1(2), \cdots\}\subseteq\{T_2(1), T_2(2), \cdots\}.$$

It is obvious that $T_1(n)-T_1(n-1)\geqslant T_2(n)-T_2(n-1)$ for all $n\geqslant 1$ if

$$T_1^m(n)-T_1^m(n-1)\geqslant T_2^m(n)-T_2^m(n-1)$$

for all $n\geqslant 1$ and all m.

Remark. If N_1^m is constructed by thinning N_2^m on a common probability space for each m and $N_i^m\to N_i$ a. s. for $i=1, 2$, then the condition of Theorem 8 is satisfied, so $N_1\leqslant N_{ino}N_2$.

§ 3. Comparing compound Poisson processes

Finally, we discuss the problem on comparison of compound Poisson processes. Let $N=\{N(t), t\geqslant 0\}$ be an inhomogeneous Poisson process with intensity $\lambda(t)$. Label an auxiliary variable u_j, which is called a mark, to j-th point t_j of N for $j=1, 2, \cdots$. Suppose that $\{u_j\}$ is a sequence of mutually independent identically distributed random variables which is also independent of N, and take their value in a mark space U which is a closed partially ordered Polish space (see, e. g., Nachbin [5]) and the addition on U is well-defined. The process

$$A\equiv\left\{A(t)=\sum_{j=1}^{N(t)}u_j,\ t\geqslant 0\right\}$$

with its value in U is called compound Poisson process (assume that $u_j\neq 0$ for all n). Obviously, the class of compound Poisson is a kind of simple marked point processes.

Theorem 9. *Let $A_i\equiv\{A_i(t), t\geqslant 0\}$ be a compound Poisson process with associated inhomogeneous Poisson process N_i which has intensity λ_i for $i=1, 2$, and let the mark variable u_{ij} have the distribution $P_i(\cdot)$ for $i=1, 2$. If $\lambda_1(t)\leqslant\lambda_2(t)$ for all t and $P_1(\cdot)\leqslant_d P_2(\cdot)$ (for the definition of the ordering $\leqslant_d$ for distributions of random variables, see, e. g., [3]), then $A_1\leqslant_{ino}A_2$.*

Proof We use the constructive argument to prove the desired conclusion as follows. $N_1\leqslant_{ino}N_2$ results from $\lambda_1\leqslant\lambda_2$ and Theorem 4. Hence we are able to construct two inhomogeneous Poissnn processes $\widetilde{N}_1$ and $\widetilde{N}_2$ with intensities λ_1 and λ_2, respectively, on a common probability space (Ω^0, F^0, P^0) such that the occurrence time sequence $\widetilde{T}_1=\{\widetilde{T}_1(1), \widetilde{T}_1(2), \cdots\}$ associated with $\widetilde{N}_1$ is a subsequence of the occurrence time sequence $\widetilde{T}_2=\{\widetilde{T}_2(1), \widetilde{T}_2(2),\cdots\}$ associated with N_2, i. e.,

$$\widetilde{T}_1(1)=\widetilde{T}_2(n_1),\ \widetilde{T}_1(2)=\widetilde{T}_2(n_2)\cdots.$$

On the other hand, for each natural number j it is possible to construct two random variables V_{1j} and V_{2j} (independently in j) on a common probability space (Ω^i, F^i, P^i) such that V_{1j} and V_{2j} have the distribution $P_1(\cdot)$ and $P_2(\cdot)$ respectively, and $V_{1j}\leqslant V_{2j}$. Then, by defining $\widetilde{U}_{2j}=V_{2j}$ and $\widetilde{U}_{1j}=V_{1n_j}$ we obtain two processes

$$\widetilde{A}_2=\sum_{j=1}^{\widetilde{N}_2(t)}\widetilde{u}_{2j}\ \text{and}\ \widetilde{A}_1(t)=\sum_{j=1}^{\widetilde{N}_1(t)}\widetilde{u}_{1j}$$

on the product space

$$(\Omega^0\times\Omega'\times\Omega'\times\cdots,\ F^0\times F'\times F'\times\cdots;\ P^0\times P'\times P'\times\cdots).$$

According to the construction of $\widetilde{A}_i$ and the definition of compound Poisson process, it is clear that $L(\widetilde{A}_i)=L(\widetilde{A}_i)$ for $i=1, 2$, then $A_1\leqslant_{inc}A_2$.

When the mark space U is discrete, i. e. $U=\{U_1, U_2, \cdots\}$, we have the following theorem.

Theorem 10. *If the mark variable u_{ij} of the j-th point in A_i takes values U_k with probability P_{ik} for $i=1, 2$, $j=1, 2, \cdots$ and $k=1, 2, \cdots$, and*

$$P_{1k}\lambda_1(t)\leqslant P_{2k}\lambda_2(t) \tag{3.1}$$

for each k and each t, then $A_1\leqslant_{inc}A_2$.

Proof Note that A_i, $i=1, 2$, may be represented, see e. g., Snyder [8], p. 140, as

$$A_i(t)=\sum_{k=1}^{\infty}U_kN_{ik}(t), \tag{3.2}$$

where $N_{ik}(t)$, $k=1, 2, \cdots$, are mutually independent (inhomogeneous) Poisson processes with $P_{ik}\lambda_i(t)$ for $i=1, 2$ and U_k is mark for $k=1, 2, \cdots$. From (3.1) and Theorem 4 follows that

$$N_{1k}\leqslant_{inc}N_{2k} \qquad \text{for each } k,$$

where N_{1k} can be constructed by thinning N_{2k}. Then

$$\sum_{k=1}^{m}U_kN_{1k}\leqslant_{inc}\sum_{k=1}^{m}U_kN_{2k} \qquad \text{for all } m.$$

Consequently, by Theorem 8 (it is clear that the conclusion of Theorem 8 still holds for the processes concerned with here) follows that

$$A_1\leqslant_{inc}A_2.$$

Remark. (1) Theorems 9 and 10 do not imply each other. However, (3.1) implies that $\lambda_1\leqslant\lambda_2$.

(2) In Theorems 9 and 10, in addition to the corresponding stated conditions, we suppose that either $\lambda_1(t)$ or $\lambda_2(t)$ is non-incrersing, then $A_1\leqslant_{int}A_2$ follows immediately from the definition of compound Poisson processes and Theorem 6.

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References

[1] Barlow, R. E. and Proschan, F., *Statistical Theory of Reliability and Life Testing*, Holt, Rinehavt and Winston,. New York (1975).

[2] Daley, D. J., Distances of random variables and point processes. Institute of Statistics Mimeo Series #1331, Dept. of Statist. University of North Carolina, Chapel Hill, (1981).

[3] Kamae, T., Krengel, U. and O'Brien, G. L., Stochastic inequalities on partially ordered spaces. *Ann. Prob.*, 5 (1977), 899—912.

[4] Miller, D. R., Almost sure comparisons of renewal processes and Poisson processes with applications to reliability theory. *Math. Operat. Res.*, **4** (1979), 406—413.

[5] Nachbin, L. *Topology and Order*. Van Nostrand, New York, (1965).

[6] Parthasarathy, K. R. *Probability Measures on Metric Space*, Academic Press, New York, (1967).

[7] Schmidt, V., On the non-equivalence of two criteria of comparability of stationary point processes, *Zast. Math.*, 15 (1976), 33—37.

[8] Snyder, D. L., *Random Point Processes*, Wiley, New York, (1975).

[9] Sonderman, D., Comparing multi-server queues with finite waiting rooms, I: same number of servers. *Adv. Appl. Prob.*, **11** (1979), 439—447.

[10] Sonderman, D., Comparing semi-Markov processes. *Math. Operat. Res.*, **5** (1980), 110—119.

[11] Stoyan, D., Bounds and approximations in queueing through monotonicity and continuity, *Operat. Res.*, **25** (1977), 851—863.

[12] Stoyan, D., *Comparison Methods for Queues and other Stochastic Models*. Wiley, New York, (1983).

[13] Szasz, D., Once more on the Poisson processes, *Studia Sci. Math. Hungar.*, 5 (1970), 441—444.

[14] Whitt, W., Comparing counting processes and queues. *Adv. Appl. Prob.*, **13** (1981), 207—220.