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ITERATION OF ANALYTIC NORMAL FUNCTIONS OF MATRICES

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Abstract

In this paper, the author proves that the classical theorem of Wolff in the theory of complex functions may be extended to the class of operator-valued functions f, where f is an analytic function from the open unit disc Δ in the complex plane into a family of commutative normal operators on a certain n-dimensional complex Hilbert space, and $\|f(z)\|<1$ holds for every z in Δ.

Let H be a complex Hilbert space. Let $L(H)$ be the Banach space of all bounded linear operators on H. If a function f on the open unit disc Δ in the complex plane into $L(H)$ is of the form

$$f(z)=\sum_{n=0}^{\infty} B_n z^n \text{ for } z\in\Delta,$$

where the series is convergent in the uniform operator topology and $\{B_n\}$ is a sequence of normal operators on H, commuting pairwise, then we call f an analytic normal function of operators and denote by $N_H(\Delta)$ the set of all such functions. For two Hermitian operators A, B on H, by $A\geqslant B$, we mean $A-B$ is positive. The notation $A>B$ indicate that $A-B$ is both positive and invertible.

In [2] and [3], K. Fan extended a classical theorem of J. Wolff[7] to a result in functional calculus. The purpose of this note is to generalize in finite-dimensional Hilbert space H K. Fan's theorem in [3] to functions in $N_H(\Delta)$, i. e. analytic normal functions of matrices.

We begin with three lemmas.

Lemma 1. *Let H be a complex Hilbert space, and let $f\in N_H(\Delta)$ with $\|f(z)\|<r<1$ for all z in Δ. Then there is a unique normal operator B such that $B=f(B)$, $\|B\|<r$, and B commutes with $T\in L(H)$ if T commutes with f (i. e. $Tf(z)=f(z)T$ for all z in Δ).*

Remark. Here, $f(B)$ is defined by $\sum_{n=0}^{\infty} B_n B^n$ if $f(z)=\sum_{n=0}^{\infty} B_n z^n$ ([5], Lemma 2.2).

Proof Suppose

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$$f(z)=\sum_{n=0}^{\infty}B_n z^n \quad \text{for } z \text{ in } \Delta,$$

where $\{B_n\}$ is a sequence of normal operators commuting pairwise. Let E be the closed, normal subalgebra of $L(H)$, generated by $\{B_n\}$ and I (the identity operator). Clearly, $AT=TA$ for $A\in E$, $T\in L(H)$ if T commutes with f. Let

$$X=\{A\in E: \|A\|<1\}.$$

It is seen that E is a Banach space and X is a bounded connected open set in E. Theorem 1 in [1] says that if $g:X\to X$ is analytic and $g(X)$ lies strictly inside X, then g has a unique fixed point. To complete the proof, it is enough to show that $F:X\to X$, defined by $F(A)=f(A)$, satisfies all the hypotheses. Clearly, each element S of E is normal and commutes with f. Thus, by Theorem 3.1 in [5], we have

$$\|F(A)\|=\|f(A)\|<r<1 \quad \text{for all } A \text{ in } X.$$

Furthermore, $F(A)\in E$ for $A\in X$ and hence $F(X)$ lies stricly inside X. We will show that F is analytic in X. Since E is a commutative complex Banach algebra with a unit element, we prove that for any A_0 in X, there exists a positive number $\delta>0$ such that F is (L)-analytic (i. e. analytic in the Lorch sense [4], § 3.19) in $\{A\in E: \|A-A_0\|<\delta\}$. Let $A_0\in X$. Take $\delta=1-\|A_0\|>0$. Then

$$F(A)=f(A)=\sum_{n=0}^{\infty}B_nA^n=\sum_{n=0}^{\infty}B_n(A_0+A-A_0)^n.$$

By Cauchy's estimates ([4], p. 97), we have $\|B_n\|\leqslant 1$, $n=0, 1, 2, \cdots$. For $\|A-A_0\|<\delta$,

$$\|F(A)\|=B_0+\sum_{n=1}^{\infty}B_n\sum_{k=1}^{n}\binom{n}{k}A_0^k(A-A_0)^{n-k}\leqslant 1+\sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\|A_0\|^k\|A-A_0\|^{n-k}$$
$$=1+\sum_{n=1}^{\infty}[\|A_0\|+\|A-A_0\|]^n<\infty.$$

Thus, we may rearrange the terms of the series $F(A)$ in powers of $(A-A_0)$, and then

$$F(A)=\sum_{n=0}^{\infty}C_n(A-A_0)^n \text{ for } \|A-A_0\|<\delta,$$

where

$$C_n=\sum_{m\geqslant n}\binom{m}{n}B_mA_0^{m-n}.$$

Since $\sum C_n(A-A_0)^n$ absolutely converges for all A with $\|A-A_0\|<\delta$, F is (L)-analytic in $\{A\in E: \|A-A_0\|<\delta\}$ by Theorem 3.19.1 in [4]. Thus F is analytic in X ([4], p. 115). The proof is complete.

Remark. Lemma 1 is an extension of Rouche's theorem in sanalytic operator functions.

Lemma 2. *Suppose A, B are positive and invertible operators on a complex Hilbert space H. Then $A\geqslant B$ is equivalent to $A^{-1}\leqslant B^{-1}$.*

Proof By Lemma 5.1 in [5], $A\geqslant B$ i. e. $A^{\frac{1}{2}}A^{\frac{1}{2}}\geqslant B^{\frac{1}{2}}B^{\frac{1}{2}}$ if and only if

$$\|A^{-\frac{1}{2}}B^{\frac{1}{2}}\|\leqslant 1$$

which is equivalent to $A^{-\frac{1}{2}}A^{-\frac{1}{2}}\leqslant B^{-\frac{1}{2}}B^{-\frac{1}{2}}$, i. e. $A^{-1}\leqslant B^{-1}$. This ends the proof.

Lemma 3. *Let B be a normal operator on a complex Hilbert space H with $\|B\|\leqslant 1$. Suppose $T\in L(H)$ commutes with B and $\|T\|<1$. Then validity of the inequality*

$$\|(I-B^*T)(I-T^*T)^{-1}(I-T^*B)\|\leqslant d \tag{1}$$

implies both

$$\|T-B(dI+B^*B)^{-1}\|\leqslant d^{\frac{1}{2}}\|[B^*B+(d-1)I]^{\frac{1}{2}}(dI+B^*B)^{-1}\|, \tag{2}$$

and

$$\|(T-B)(I-B^*T)^{-1}\|\leqslant d^{-\frac{1}{2}}\|(d-1)I+B^*B\|^{\frac{1}{2}}. \tag{3}$$

Proof Clearly, the constant d in (1) must be positive. Suppose (1) holds. Then

$$(I-B^*T)(I-T^*T)^{-1}(I-T^*B)\leqslant dI,$$

i. e.

$$(I-T^*T)^{-1}\leqslant d(I-B^*T)^{-1}(I-T^*B)^{-1}.$$

Since the operators on both sides are positive and invertible, it follows from Lemma 2 that

$$(I-T^*B)(I-B^*T)\leqslant d(I-T^*T), \tag{4}$$

which is equivalent to

$$\{T^*-B^*(dI+B^*B)^{-1}\}\{T-B(dI+B^*B)^{-1}\}\leqslant d[B^*B+(d-1)I](dI+B^*B)^{-2}.$$

Thus

$$\|T-B(dI+B^*B)^{-1}\|\leqslant d^{\frac{1}{2}}\|[B^*B+(d-1)I]^{\frac{1}{2}}(dI+B^*B)^{-1}\|.$$

This proves that (2) follows from (1).

It is seen that (1) is equivalent to (4), which can be written as

$$(dI+B^*B)T^*T-T^*B-B^*T+(1-d)I\leqslant 0.$$

Then

$$(I-B^*B)\{(dI+B^*B)T^*T-T^*B-B^*T+(1-d)I\}\leqslant 0,$$

or

$$(T^*-B^*)(T-B)\leqslant d^{-1}[(d-1)I+B^*B](I-T^*B)(I-B^*T).$$

Hence

$$(I-T^*B)\{d^{-1}[(d-1)I+B^*B]-(I-T^*B)^{-1}(T^*-B^*)(T-B)\cdot(I-B^*T)^{-1}\}(I-B^*T)\geqslant 0,$$

$$(I-T^*B)^{-1}(T^*-B^*)(T-B)(I-B^*T)^{-1}\leqslant d^{-1}[(d-1)I+B^*B],$$

which implies (3). The proof is complete.

Remark. The space H in the lemmas above is not necessarily finite-dimensional, while H in the following theorem has to be of finite-dimension.

Theorem. *Let H be a complex Hilbert space of finite-dimension and let $f\in$*

$N_H(\Delta)$ with $\|f(z)\|<1$ for all z in $\Delta=\{z:|z|<1\}$. Suppose $f^{[n]}$ $(n=1, 2, \cdots)$ stands for the n-th iterate of f, i. e. $f^{[1]}(z)=f(z)$, $f^{[n]}(z)=f(f^{[n-1]}(z))$ for z in Δ and $n\geqslant 2$. Then there exists a normal operator A on H with $\|A\|\leqslant 1$ such that the following relations hold for any $T\in L(H)$, commuting with f and $\|T\|<1$:

$$(I-A^*A)\leqslant\{I-A^*f(T)\}\{I-f(T)^*f(T)\}^{-1}\{I-f(T)^*A\}$$
$$\leqslant(I-A^*T)(I-T^*T)^{-1}(I-T^*A), \tag{5}$$
$$(I-A^*A)\leqslant\{I-A^*f(T)\}\{(I+AA^*)-A^*f(T)-f(T)^*A\}^{-1}\{I-f(T)^*A\}$$
$$\leqslant\{I-A^*T\}\{(I+A^*A)-A^*T-T^*A\}^{-1}\{I-T^*A\}, \tag{6}$$
$$\|f^{[n]}(T)-A[d(A, T)I+A^*A]^{-1}\|$$
$$\leqslant\|\{d(A, T)[A^*A+(d(A, T)-1)I]\}^{\frac{1}{2}}\{d(A, T)I+A^*A\}^{-1}\|, \tag{7}$$
$$\|[f^{[n]}(T)-A][I-A^*f^{[n]}(T)]^{-1}\|\leqslant\left\{\frac{\|(d(A, T)-1)+A^*A\|}{d(A, T)}\right\}^{\frac{1}{2}}, \tag{8}$$

where $n=1, 2, 3, \cdots$ and

$$d(A, T)=\|(I-A^*T)(I-T^*T)^{-1}(I-T^*A)\|. \tag{9}$$

Besides, $f(A)=A$ if $\|A\|<1$.

Remark. Here $f\in N_H(\Delta)$ means that f is an analytic normal function of matrices.

Proof Since $\|f(z)\|<1$ for $z\in\Delta$ implies $\|f^{[2]}(z)\|=\|f(f(z))\|<1$ by Theorem 3.1 in [5], it follows from Lemma 2.5 and Theorem 3.1 in [5] that $f^{[n]}(n=2, 3, \cdots)$ are well-defined and $\|f^{[n]}(z)\|<1$ by induction.

Now choose a sequence of positive numbers $\{a_m\}$ such that $0<a_m<1$ and

$$\lim_{m\to\infty} a_m=1.$$

Put $f_m=a_mf$. Then by Lemma 1, we have a sequence of normal operators $\{A_m\}$ such that $A_m=a_mf(A_m)$, $\|A_m\|<a_m$, and A_m commutes with both T and f. Thanks to the fact that H is finite-dimensional, one may assume $\{A_m\}$ converges in the uniform operator topology (Replace the sequence by a convergent subsequence if necessary). Suppose

$$A=\lim_{m\to\infty} A_m.$$

Then $\|A\|\leqslant 1$ and A commutes with both T and f. We show that $f(A)=A$ whenever $\|A\|<1$. Let

$$f(z)=\sum_{n=0}^{\infty} B_nz^n \text{ for } z\in\Delta.$$

By Cauchy's estimates, we have $\|B_n\|\leqslant 1$ $(n=0, 1, 2, \cdots)$ ([4], p. 97). Since $\{A_m\}$ are commuting pairwise and hence each A_m commutes with A, it follows that for all such m that $\|A-A_m\|<\frac{1-\|A\|}{2}$ and any positive integer n,

$$\|A_m^n-A^n\|\leqslant\|A_m-A\|n[\|A\|+\|A-A_m\|]^{n-1}\leqslant\|A_m-A\|n\left[\frac{1+\|A\|}{2}\right]^{n-1}.$$

Then

$$\|f(A_m)-f(A)\|=\left\|\sum_{n=0}^{\infty}B_nA_m^n-\sum_{n=0}^{\infty}B_nA^n\right\|\leqslant\sum_{n=0}^{\infty}\|A_m^n-A^n\|$$
$$\leqslant\|A_m-A\|\sum_{n=1}^{\infty}n\left[\frac{1+\|A\|}{2}\right]^{n-1}\to 0,$$

as $m\to\infty$. On the other hand

$$\|A_m-f(A)\|=\|a_mf(A_m)-f(A)\|\leqslant(1-a_m)\|f(A)\|+a_m\|f(A_m)-f(A)\|.$$

Tnus $A=f(A)$ in case $\|A\|<1$.

It is easy to verify that if B is a normal operator on H with $\|B\|\leqslant 1$ and S is any operator on H with $\|S\|<1$ such that $SB=BS$ (hence $SB^*=B^*S$ by Fuglede–Putnam–Rosenplum's theorem [6]), then the following relations (10), (11), (12) hold

$$(I-S^*B)^{-1}(S^*-B^*)(S-B)(I-B^*S)^{-1}$$
$$=I-(I-B^*B)(I-S^*B)^{-1}(I-S^*S)(I-B^*S)^{-1}, \tag{10}$$
$$I-B^*B(I-S^*B)^{-1}(S^*-B^*)(S-B)(I-B^*S)^{-1}$$
$$=(I-B^*B)(I-S^*B)^{-1}\{(I+B^*B)-S^*B-B^*S\}(I-B^*S)^{-1}, \tag{11}$$
$$I+B^*B-S^*B-B^*S>0. \tag{12}$$

We check (12) only. In fact

$$I+B^*B-S^*B-B^*S=(I-S^*B)(I-B^*S)+B^*B(I-S^*S)$$
$$\geqslant(I-S^*B)(I-B^*S)>0.$$

As T commutes with f and $\|T\|<1$, Theorem 3.1 in [5] asserts $\|f(T)\|<1$. An application of the identity (10) to A and $f(T)$ gives

$$(I-A^*A)(I-f(T)^*A)^{-1}(I-f(T)^*f(T))(I-A^*f(T))^{-1}\leqslant I, \tag{13}$$

since the left side of (10) is a positive operator. Observe that for two Hermitian operators B_1, B on H, if $B_1>0$, $B_1B\geqslant 0$, then $B\geqslant 0$. Inequality (13) may be written as

$$(I-f(T)^*A)^{-1}(I-f(T)^*f(T))(I-A^*f(T))^{-1}$$
$$\cdot\{(I-A^*f(T))(I-f(T)^*f(T))^{-1}(I-f(T)^*A)-(I-A^*A)\}\geqslant 0,$$

hence
$$(I-A^*f(T))(I-f(T)^*f(T))^{-1}(I-f(T)^*A)-(I-A^*A)\geqslant 0. \tag{14}$$

This is the first inequality in (5).

Similarly, applying the identity (11) to A and $f(T)$, we obtain

$$(I-A^*A)(I-f(T)^*A)^{-1}[(I+A^*A)-f(T)^*A-A^*f(T)](I-A^*f(T))^{-1}\leqslant I,$$

and hence (notice (12) and the observation following (13))

$$(I-A^*A)\leqslant(I-A^*f(T))\{(I+A^*A)-f(T)^*A-A^*f(T)\}^{-1}(I-f(T)^*A).$$

This is the first inequality in (6).

Since $\|f_m(T)\|<1$ ([5], Theorem 3.1) and A_m is normal and commutes with f_m and T, it follows from Theorem 5.2 in [5] that

$$\{I-f_m(T)^*A_m\}^{-1}\{f_m(T)^*-A_m^*\}\{f_m(T)-A_m\}\{I-A_m^*f_m(T)\}^{-1}$$
$$=\{I-f_m(T)^*f_m(A_m)\}^{-1}\{f_m(T)^*-f_m(A_m)^*\}\{f_m(T)-f_m(A_m)\}$$
$$\times\{I-f_m(A_m)^*f_m(T)\}^{-1}$$
$$\leqslant\{I-T^*A_m\}^{-1}\{T^*-A_m^*\}\{T-A_m\}\{I-A_m^*T\}^{-1} \tag{15}$$

which may be written, by the identity (10), as

$$I-(I-A_m^*A_m)(I-f_m(T)^*A_m)^{-1}(I-f_m(T)^*f_m(T))(I-A_m^*f_m(T))^{-1}$$
$$\leqslant I-(I-A_m^*A_m)(I-T^*A_m)^{-1}(I-T^*T)(I-A_m^*T)^{-1}.$$

Thus
$$(I-T^*A_m)^{-1}(I-T^*T)(I-A_m^*T)^{-1}$$
$$\leqslant (I-f_m(T)^*A_m)^{-1}(I-f_m(T)^*f_m(T))(I-A_m^*f_m(T))^{-1}$$

(by the observation following (13) and $I-A_m^*A_m>0$) and letting $m\to\infty$,

$$(I-T^*A)^{-1}(I-T^*T)(I-A^*T)^{-1}$$
$$\leqslant (I-f(T)^*A)^{-1}(I-f(T)^*f(T))(I-A^*f(T))^{-1}. \tag{16}$$

Lemma 2 shows

$$(I-A^*f(T))(I-f(T)^*f(T))^{-1}(I-f(T)^*A)\leqslant (I-A^*T)(I-T^*T)^{-1}(I-T^*A),$$

which is the second inequality in (5).

Similarly, using (11) and following (13), we deduce from (15)

$$(I-T^*A_m)^{-1}\{(I+A_m^*A_m)-T^*A_m-A_m^*T\}(I-A_m^*T)^{-1}$$
$$\leqslant \{I-f_m(T)^*A_m\}^{-1}\{(I+A_m^*A_m)-f_m(T)^*A_m-A_m^*f_m(T)\}\{I-A_m^*f_m(T)\}^{-1}.$$

Hence a passage to the limit shows that

$$(I-T^*A)^{-1}\{(I+A^*A)-T^*A-A^*T\}(I-A^*T)^{-1}$$
$$\leqslant \{I-f(T)^*A\}^{-1}\{(I+A^*A)-f(T)^*A-A^*f(T)\}\{I-A^*f(T)\}^{-1}.$$

This implies the second inequality in (6) (by (12) and Lemma 2).

We have shown that $\|f^{[n]}(z)\|<1$ for all n and $z\in\Delta$, and hence $\|f^{[n]}(T)\|<1$ (Theorem 3.1 in [5]). Lemma 2.5 in [5] tells us that $f^{[n]}(T)=f(f^{[n-1]}(T))$. Clearly, $f^{[n]}(T)$ commutes with f for all n. Applications of (5) show that for $n\geqslant 1$,

$$\{I-A^*f^{[n]}(T)\}\{I-f^{[n]}(T)^*f^{[n]}(T)\}^{-1}\{I-f^{[n]}(T)^*A\}$$
$$\leqslant \{I-A^*f^{[n-1]}(T)\}\{I-f^{[n-1]}(T)^*f^{[n-1]}(T)\}^{-1}\{I-f^{[n-1]}(T)^*A\}\leqslant\cdots$$
$$\leqslant (I-A^*T)(I-T^*T)^{-1}(I-T^*A).$$

Hence

$$\|\{I-A^*f^{[n]}(T)\}\{I-f^{[n]}(T)^*f^{[n]}(T)\}^{-1}\{I-f^{[n]}(T)^*A\}\|\leqslant d(A,T). \tag{17}$$

Then (7), (8) are attained by Lemma 3 and (17). This completes the proof.

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