Chin. Ann. of Math.
6B (1) 1985

ON ESTIMATE OF COMPLETE TRIGONOMETRIC SUMS

QI MINGGAO (戚鸣臬)* DING PING (丁 平)**

Abstract

In this paper, the following theorem is proved:

Let $f(x)=a_kx^k+\cdots+a_1x+a_0$ be a polynomial with integral coefficients such that $(a_1, \cdots, a_k, q)=1$, where q is a positive integer. Then, for $k\geqslant 3$,

$$\left|\sum_{x=1}^{q} e^{2\pi i f(x)/q}\right|\leqslant e^{2k}q^{1-1/k}.$$

§ 1. Introducion

Let q be an integer >1 and $f(x)=a_kx^k+\cdots+a_1x+a_0$ be a polynomial of degree k with integral coefficients such that $(a_1, \cdots, a_k, q)=1$. By a complete trigonometric sum we mean a sum of the form

$$S(q, f(x))=\sum_{x=1}^{q} e^{2\pi i f(x)/q}. \tag{1}$$

Many problems in analytic number theory desire to have precise estimates of $S(q, f(x))$ for large q. Si nce $S(q, f(x))=0$ for $k=1$ and the case $k=2$ can be settled by the theory of Gaussian sums, we suppose $k\geqslant 3$.

In 1940, Professor Hua[1] first proved that

$$S(q, f(x))=O(q^{1-\frac{1}{k}+\varepsilon}), \tag{2}$$

where the constant implied by "O" depends only on k. The main order $1-1/k$ is the best possible. Afterwards, some mathematicians are interested in the improvements of the constant implied by "O". In 1977, Chen Jingrun[2] and Стечкин, С. Б.[3] proved respectively that

$$|S(q, f(x))|\leqslant e^{6.1k}q^{1-1/k} \quad (k\geqslant 3) \tag{3}$$

and

$$|S(q, f(x))|\leqslant B(k)q^{1-1/k}, \tag{4}$$

where

$$B(k)\leqslant \exp\left\{k+O\left(\frac{k}{\log k}\right)\right\} \quad (k\to\infty). \tag{5}$$

Recently Lu Minggao[4] has proved that

Manuscript received January 8, 1983.

* Department of Mathematics, Qinghua University, Beijing, China.

** Institute of Mathematics, Academia Sinica, Beijing, China.

$$|S(q, f(x))| \leqslant e^{3k} q^{1-1/k} \quad (k \geqslant 3). \tag{6}$$

Our purpose in this paper is to prove the following

Theorem. *Let $f(x)=a_k x^k+\cdots+a_1 x+a_0$ be a polynomial with integral coefficients such that $(a_1, \cdots, a_k, q)=1$, where q is a positive integer. Then for $k\geqslant 3$ we have*

$$|S(q, f(x))| \leqslant e^{2k} q^{1-1/k}.$$

§ 2. Several Lemmas

Lemma 1.[2,4] *Let k be integer $\geqslant 3$ and $f(x)=a_k x^k+\cdots+a_1 x+a_0$ be a polynomial with ingetral coefficients such that $(a, \cdots, a_k, p)=1$, where p is a prime $>k$. Then for $l\geqslant 1$*

$$|S(p^l, f(x))| \leqslant \begin{cases} \operatorname{Max}\{1, \operatorname{Min}(p^{\frac{1}{k}}, (k-1)p^{-\frac{1}{2}+\frac{1}{k}})\} p^{l(1-\frac{1}{k})}, & p>(k-1)^{\frac{k}{k-2}} \\ (k-1)p^{\frac{3}{k}-1} p^{l(1-\frac{1}{k})}, & k<p\leqslant (k-1)^{\frac{k}{k-2}}. \end{cases} \tag{7}$$

It can also be written in the form

$$|S(p^l, f(x))| p^{-(l-\frac{1}{k})} \leqslant \begin{cases} 1, & p>(k-1)^{\frac{2k}{k-2}}; \\ (k-1)p^{-\frac{1}{2}+\frac{1}{k}}, & (k-1)^2<p\leqslant (k-1)^{\frac{2k}{k-2}}; \\ p^{\frac{1}{k}}, & (k-1)^{\frac{k}{k-2}}<p\leqslant (k-1)^2; \\ (k-1)p^{\frac{3}{k}-1}, & k<p\leqslant (k-1)^{\frac{k}{k-2}}. \end{cases} \tag{8}$$

Lemma 2.[5,6] *Define*

$$\pi(x)=\sum_{p\leqslant x} 1, \quad \vartheta(x)=\sum_{p\leqslant x} \log p.$$

Then

$$\vartheta(x)<1.001102\,x, \text{ if } x>0; \tag{9}$$

$$\pi(x)<1.2551\,\frac{x}{\log x}, \text{ if } x>1. \tag{10}$$

Lemma 3. *Let $k\geqslant 3$. Then for $p\leqslant k$ and $1\leqslant y\leqslant k-1$ we have*

$$p^{\frac{y}{k}} \leqslant y p^{\frac{1}{k}}. \tag{11}$$

Proof It is similar to the proof of Lemma 5 in [2].

Lemma 4. *Let k be an integer $\geqslant 3$ and p is a prime$\leqslant k$. Let $f(x)=a_k x^k+\cdots+a_1 x+a_0$ be a polynomial with integral coefficients such that $(a_1, \cdots, a_k, p)=1$. Then for $l\geqslant 1$, we have*

$$|S(p^l, f(x))| \leqslant (k-1) k^{\frac{2}{k}} p^{\frac{2}{k}-1} p^{l(1-\frac{1}{k})}. \tag{12}$$

Proof We define t satisfying $p^t \| (ka_k, \cdots, 2a_2, a_1)$. From $(a_1, \cdots, a_k, p)=1$, we obtain $p^t \leqslant k$. Let $\mu_1, \cdots, \mu_r$ be the different zeros modulo p of the congruence

$$p^{-t} f'(x) \equiv o \pmod p, \quad o\leqslant x<p \tag{13}$$

and let $m_1, \cdots, m_r$ be their multiplicities. Putting $m_1+\cdots+m_r=m$, it is obvious that $r\leqslant M\leqslant k-1$.

1) $l\leqslant 2t$: From $p^t\leqslant k$ we have

$$|S(p^l, f(x))|\leqslant p^l=p^{\frac{l}{k}}p^{l(1-\frac{1}{k})}\leqslant k^{\frac{2}{k}}p^{l(1-\frac{1}{k})}\leqslant(k-1)k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}. \tag{14}$$

2) $l=2t+1$: If $t=0$, then $l=1$. We get immediately

$$|S(p^l, f(x))|\leqslant p^l=p^{\frac{1}{k}}p^{l(1-\frac{1}{k})}\leqslant(k-1)k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}. \tag{15}$$

So we suppose $t\geqslant 1$. We transform the sum $S(p^l, f(x))$ by substituting $x=y+p^{l-t-1}z$, where y and z run independently through the values

$$y=1, \cdots, p^{l-t-1},\ z=0, \cdots, p^{t+1}-1.$$

Also we denote $e_q(x)=e^{2\pi ix/q}$. Thus

$$\begin{aligned}S(p^l, f(x)) &= \sum_{x=1}^{p^l} e_{p^l}(f(x))\\ &= \sum_{y=1}^{p^{l-t-1}} e_{p^l}(f(y)) \sum_{z=0}^{p^{t+1}-1} e_{p^l}\Big(p^{l-t-1}zf'(y)+\frac{1}{2}p^{2(l-t-1)}z^2f''(y)\Big)\\ &= \sum_{y=1}^{p^{l-t-1}} e_{p^l}(f(y)) \sum_{z=0}^{p^{t+1}-1} e_p\Big(z\frac{f'(y)}{p^t}+\frac{1}{2}z^2f''(y)\Big).\end{aligned} \tag{16}$$

If either p is an odd prime and $t\geqslant 1$ or $p=2$ and $t\geqslant 2$, since $p^t|f''(y)$, then $2p|f''(y)$ for all $1\leqslant y\leqslant p^{l-t-1}$. Thus from (16) we have

$$S(p^l, f(x))=\sum_{y=1}^{p^{l-t-1}} e_{p^l}(f(y)) \sum_{z=0}^{p^{t+1}-1} e_{p^{t+1}}(zf'(y)).$$

Now if $y\not\equiv\mu_j \pmod p$, $j=1, \cdots, r$, then

$$\sum_{z=0}^{p^{t+1}-1} e_{p^{t+1}}(zf'(y))=0. \tag{17}$$

Therefore we obtain

$$\begin{aligned}|S(p^l, f(x))| &\leqslant \sum_{j=1}^{r}\Bigg|\sum_{\substack{y=1\\ y\equiv\mu_j \pmod p}}^{p^l} e_{p^l}(f(y))\Bigg| = \sum_{j=1}^{r}\Bigg|\sum_{y=1}^{p^{l-1}} e_{p^l}(f(\mu_j+py))\Bigg|\leqslant rp^{l-1}\\ &= rp^{\frac{l}{k}-1}p^{l(1-\frac{1}{k})}=rp^{\frac{2t}{k}}p^{\frac{1}{k}-1}p^{l(1-\frac{1}{k})}\leqslant(k-1)k^{\frac{2}{k}}p^{\frac{1}{k}-1}p^{l(1-\frac{1}{k})}.\end{aligned} \tag{18}$$

It remains to consider $p=2$ and $t=1$. In this case $l=3$. Hence

$$|S(p^l, f(x))|\leqslant p^l=p^{\frac{3}{k}}p^{l(1-\frac{1}{k})}\leqslant(k-1)p^{\frac{3}{k}-1}p^{l(1-\frac{1}{k})}\leqslant(k-1)k^{\frac{2}{k}}p^{\frac{1}{k}-1}p^{l(1-\frac{1}{k})}. \tag{19}$$

3) $l\geqslant 2(t+1)$: By the same argument as before, we substitute $x=y+p^{l-t-1}z$, where y and z run independently through the values

$$y=1, \cdots, p^{l-t-1},\ z=0, \cdots, p^{t+1}-1.$$

Thus for $l\geqslant 2(t+1)$

$$S(p^l, f(x))=\sum_{x=1}^{p^l} e_{p^l}(f(x)) = \sum_{y=1}^{p^{l-t-1}} e_{p^l}(f(y)) \sum_{z=0}^{p^{t+1}-1} e_{p^{t+1}}(zf'(y)).$$

Hence from (17) we have

$$|S(p^l, f(x))|\leqslant\sum_{j=1}^{r}\Bigg|\sum_{\substack{y=1\\ y\equiv\mu_j \pmod p}}^{p^l} e_{p^l}(f(y))\Bigg|=\sum_{j=1}^{r}|S_{\mu_j,p^l}|,\ \text{say}. \tag{20}$$

Let σ_j Satisfy $p^{\sigma_j}\|(f(py+\mu_j)-f(\mu_j))$ and put

$$g_{\mu_j}(y)=p^{-\sigma_j}(f(py+\mu_j)-f(\mu_j)). \tag{21}$$

In view of

$$f(py+\mu_j)-f(\mu_j)=pyf'(\mu_j)+\frac{(py)^2f''(\mu_j)}{2!}+\cdots+\frac{(py)^{m_j}f^{(m_j)}(\mu_j)}{m_j!}+\cdots, \tag{22}$$

we obtain

$$2\leqslant\sigma_j\leqslant m_j+t+1. \tag{23}$$

Also from [7], we have

$$\sigma_j\leqslant k. \tag{24}$$

Let t_j satisfy $p^{t_j}\|g'_{\mu_j}(y)$. It is obvious that.

$$p^{t_j}\leqslant k. \tag{25}$$

Since $p^t\|f'(x)$, $p^t\left|\frac{f^{(h+1)}(\mu_j)}{h!}\right.$ holds for any non-negative integer h. So

$$p^{m_j+t+1}\Big\|\frac{1}{m_j!}\,p^{m_j+1}f^{(m_j+1)}(\mu_j).$$

From (21) and (22), we see that $p^{-\sigma_j}\cdot\frac{p^{m_j+1}f^{(m_j+1)}(\mu_j)}{m_j!}$ is one of the coefficients of the polynomial $g'_{\mu_j}(y)$. Hence we have

$$m_j+t+1-\sigma_j\geqslant t_j \tag{26}$$

and

$$g'_{\mu_j}(y)\equiv p^{-\sigma_j}\left(pf'(\mu_j)+\cdots+\frac{1}{m_j!}\,p^{m_j+1}y^{m_j}f^{(m_j+1)}(\mu_j)\right)(\bmod p^{t_j+1}).$$

By this expression, we conclude that the congruence $g'_{\mu_j}(y)\equiv o(\bmod p^{t_j+1})$ $(o\leqslant y\leqslant p)$ has atmost m_j solutions.

Using mathematical induction, we now prove that for $l\geqslant 2(t+1)$, the following estimate

$$|S(p^l,\ f(x))|\leqslant mk^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})} \tag{27}$$

holds for any polynomial $f(x)$ satisfying the lemma's conditions.

For $l=2(t+1)$, from (20) we have

$$|S(p^l,f(x))|\leqslant rp^{l-1}=rp^{\frac{2t}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}\leqslant mk^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}.$$

Now assume that (27) is true for each integer in interval $[2(t+1),\ l-1]$. Then for $l>2(t+1)$, let

$$\mathscr{A}_1=\{j:\ l-\sigma_j\geqslant 2t_j+2\},$$
$$\mathscr{A}_2=\{j:\ l-\sigma_j=2t_j+1\},$$
$$\mathscr{A}_3=\{j:\ l-\sigma_j\leqslant 2t_j\}$$

and write $M_1=\sum_{j\in\mathscr{A}_1}m_j$, $M_2=\sum_{j\in\mathscr{A}_2}m_j$, $M_3=\sum_{j\in\mathscr{A}_3}m_j$. We have $M_1+M_2+M_3=m$.

Since

$$|S_{\mu_j,p^l}|=\left|\sum_{y=0}^{p^{l-1}-1} e_{p^l}(f(\mu_j+py))\right|=\left|e_{p^l}(f(\mu_j))\sum_{y=0}^{p^{l-1}-1} e_{p^{l-\sigma_j}}(g_{\mu_j}(y))\right|$$
$$=p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|,$$

by the inductive hypothesis and (24), (28), we have

$$\sum_{j\in\mathscr{A}_1}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_1}p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|$$
$$\leqslant\sum_{j\in\mathscr{A}_1}p^{\sigma_j-1}m_jk^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{(l-\sigma_j)(1-\frac{1}{k})}\leqslant M_1k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}. \tag{29}$$

We consider $j\in\mathscr{A}_2$. If $t_j=0$, it follows from (28), (23), and Lemma 3 that

$$\sum_{j\in\mathscr{A}_2}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_2}p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|\leqslant\sum_{j\in\mathscr{A}_2}p^{l-1}=\sum_{j\in\mathscr{A}_2}p^{\frac{\sigma_j+1}{k}-1}p^{l(1-\frac{1}{k})}$$
$$\leqslant\sum_{j\in\mathscr{A}_2}p^{\frac{m_j+t+2}{k}-1}p^{l(1-\frac{1}{k})}\leqslant\sum_{t\in\mathscr{A}_2}m_jk^{\frac{1}{k}}p^{\frac{3}{k}-1}p^{l(1-\frac{1}{k})}$$
$$\leqslant M_2k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}. \tag{30}$$

Now we turn to $t_j\geqslant1$. At first, suppose either p is an odd prime and $t_j\geqslant1$ or $p=2$ and $t_j\geqslant2$. We proceed as in the proof of (18) but use t_j, $l-\sigma_j$ and $g_{\mu_j}(y)$ instead of t, l and $f(x)$ respectively. We have

$$|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|\leqslant sp^{l-\sigma_j-1}, \tag{31}$$

where s is the number of the different zeros modulo p of the congruence

$$g'_{\mu_j}(y)\equiv o(\text{mod } p^{t_j+1})\quad(o\leqslant y<p).$$

Hence $s\leqslant m_j$. From (28), (31) and (24) we have

$$\sum_{j\in\mathscr{A}_2}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_2}p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|\leqslant\sum_{j\in\mathscr{A}_2}m_jp^{l-2}$$
$$=\sum_{j\in\mathscr{A}_2}m_jp^{\frac{\sigma_j+2t_j+1}{k}-2}p^{l(1-\frac{1}{k})}\leqslant M_2k^{\frac{2}{k}}p^{\frac{1}{k}-1}p^{l(1-\frac{1}{k})}.$$

It remains to investigate $p=2$ and $t_j=1$. In this case, by (26)

$$\sigma_j\leqslant m_j+t+1-t_j=m_j+t,\ l=\sigma_j+2t_j+1\leqslant m_j+t+3.$$

If $k\geqslant4$, then

$$p^{\frac{2}{k}}=2^{\frac{2}{k}}=4^{\frac{1}{k}}\leqslant k^{\frac{1}{k}}.$$

Thus from (28) and Lemma 3 we have

$$\sum_{j\in\mathscr{A}_2}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_2}p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|\leqslant\sum_{j\in\mathscr{A}_2}p^{l-1}\leqslant\sum_{j\in\mathscr{A}_2}p^{\frac{m_j+t+3}{k}-1}p^{l(1-\frac{1}{k})}$$
$$\leqslant\sum_{j\in\mathscr{A}_2}m_jk^{\frac{1}{k}}p^{\frac{4}{k}-1}p^{l(1-\frac{1}{k})}\leqslant M_2k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}.$$

If $k=3$. Since $p^t\leqslant k$ and $p=2$, we have $t\leqslant1$. Also we have

$$p^{\frac{3}{k}}=2<3^{\frac{2}{3}}=k^{\frac{2}{k}}.$$

Therefore, it follows from (28) and Lemma 3 that

$$\sum_{j\in\mathscr{A}_2}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_2}p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|\leqslant\sum_{j\in\mathscr{A}_2}p^{l-1}\leqslant\sum_{j\in\mathscr{A}_2}p^{\frac{m_j+t+3}{k}-1}p^{l(1-\frac{1}{k})}$$
$$\leqslant\sum_{j\in\mathscr{A}_2}m_jp^{\frac{5}{k}-1}p^{l(1-\frac{1}{k})}\leqslant M_2k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}.$$

Hence we have in any case

$$\sum_{j\in\mathscr{A}_2}|S_{\mu_j,p^l}|\leqslant M_2k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}. \tag{32}$$

Finally it follows from (25), (26) and Lemma 3 that

$$\sum_{j\in\mathscr{A}_3}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_3}\left|\sum_{y=0}^{p^{l-1}-1}e_{p^l}(f(\mu_j+py))\right|\leqslant\sum_{j\in\mathscr{A}_3}p^{l-1}\leqslant\sum_{j\in\mathscr{A}_3}p^{\frac{\sigma_j+2t_j}{k}-1}p^{l(1-\frac{1}{k})}$$
$$\leqslant\sum_{j\in\mathscr{A}_3}p^{\frac{m_j+t+1+t_j}{k}-1}p^{l(1-\frac{1}{k})}\leqslant M_3k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}. \tag{33}$$

In view of (20), (29), (32), (33), we have for $l\geqslant 2(t+1)$

$$|S(p^l, f(x))|\leqslant\Big(\sum_{j\in\mathscr{A}_1}+\sum_{j\in\mathscr{A}_2}+\sum_{j\in\mathscr{A}_3}\Big)|S_{\mu_j,p^l}|\leqslant(M_1+M_2+M_3)k^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}$$
$$=mk^{\frac{2}{k}}p^{\frac{2}{k}-1}p^{l(1-\frac{1}{k})}.$$

Thus it follows that (27) is true for all $l\geqslant 2(t+1)$. We complete the proof of the Lemma.

§ 3. Proof of the Theorem

In view of Lemma 1, Lemma 4 and Theorem 1 0.2 in [7], we have

$$|S(q, f(x))|\leqslant\prod_{p\leqslant k}(k-1)k^{\frac{2}{k}}p^{\frac{2}{k}-1}\prod_{k<p\leqslant(k-1)^{\frac{k}{k-2}}}(k-1)p^{\frac{3}{k}-1}\times\prod_{(k-1)^{\frac{k}{k-2}}<p\leqslant(k-1)^2}p^{\frac{1}{k}}$$
$$\times\prod_{(k-1)^2\leqslant p<(k-1)^{\frac{2k}{k-2}}}(k-1)p^{-\frac{1}{2}+\frac{1}{k}}q^{1-\frac{1}{k}}=e^{F(k)}q^{1-\frac{1}{k}},\ \text{say}. \tag{44}$$

Let $x_k=(k-1)^{\frac{k}{k-2}}$. Then

$$F(k)=\Big(1-\frac{2}{k}\Big)(\pi(x_k)\log x_k-\vartheta(x_k))+\frac{2}{k}\pi(k)\log k$$
$$+\frac{1}{k}(\vartheta(x_k^2)-\vartheta(k))+\log(k-1)(\pi(x_k^2)-\pi((k-1)^2))$$
$$-\frac{1}{2}(\vartheta(x_k^2)-\vartheta((k-1)^2)). \tag{35}$$

Since

$$\pi(x)\log x-\vartheta(x)=\log x\int_2^x\frac{\vartheta(t)}{t\log^2 t}dt, \tag{36}$$

we have

$$F(k)=\log(k-1)\int_2^{x_k}\frac{\vartheta(t)}{t\log^2 t}dt+\log(k-1)\int_{(k-1)^2}^{x_k^2}\frac{\vartheta(t)}{t\log^2 t}dt$$
$$+\frac{2}{k}\pi(k)\log k-\frac{1}{k}\vartheta(k),$$

and

$$\frac{F(k)}{k}=\frac{\log(k-1)}{k}\int_2^{x_k}\frac{\vartheta(t)}{t\log^2 t}dt+\frac{\log(k-1)}{k}\int_{(k-1)^2}^{x_k^2}\frac{\vartheta(t)}{t\log^2 t}dt$$
$$+\frac{1}{k^2}(2\pi(k)\log k-\vartheta(k))=I_1(k)+I_2(k)+I_3(k),\ \text{say}. \tag{37}$$

By Lemma 2, it follows that

$$I_2(k)\leqslant 1.001102\frac{\log(k-1)}{k}\int_{(k-1)^2}^{x_k^2}\frac{dt}{\log^2 t}$$

$$=1.001102\,\frac{\log(k-1)}{k}\sum_{i=0}^{15}\int_{(k-1)^{2+\frac{0.25i}{k-2}}}^{(k-1)^{2+\frac{0.25(i+1)}{k-2}}}\frac{dt}{\log^2 t}$$

$$\leqslant 1.001102\,\frac{\log(k-1)}{k}\sum_{i=0}^{15}\frac{(k-1)^{2+\frac{0.25i}{k-1}}}{\left(2+\frac{0.25i}{k-2}\right)^2\log^2(k-1)}\left[(k-1)^{\frac{0.25}{k-2}}-1\right].$$

A Simple calculation shows that

$$e^x-1\leqslant x\left(1+\frac{x}{2}\left(1+\frac{x}{3}\left(1+\frac{x}{4}e^x\right)\right)\right),\quad (x>0), \tag{38}$$

and that $\dfrac{(k-1)^{2+\frac{0.25i}{k-2}}}{k(k-2)\left(2+\frac{0.25i}{k-2}\right)^2}$ and $(k-1)^{\frac{0.25}{k-2}}-1$ are decreasing for $k\geqslant 9$ and $0\leqslant i\leqslant 15$. Hence, for $k\geqslant 12$, we have

$$I_2(k)\leqslant 1.001102\frac{0.25}{k(k-2)}\left[1+\frac{0.25\log(k-1)}{2(k-2)}\left(1+\frac{0.25\log(k-1)}{3(k-2)}\right.\right.$$

$$\left.\left.\times\left(1+\frac{0.25\log(k-1)}{4(k-2)}(k-1)^{\frac{0.25}{k-2}}\right)\right)\right]\sum_{i=0}^{15}\frac{(k-1)^{\frac{0.25i}{k-2}+2}}{\left(2+\frac{0.25i}{k-2}\right)^2}$$

$$\leqslant 1.001102\,\frac{0.25}{12\times 10}\left[1+\frac{0.25\log 11}{20}\left(1+\frac{0.25\log 11}{30}\left(1+\frac{0.25\log 11}{40}\cdot 11^{\frac{1}{40}}\right)\right)\right]$$

$$\times\sum_{i=0}^{15}\frac{11^{\frac{i}{40}+2}}{\left(2+\frac{i}{40}\right)^2}<1.3872. \tag{39}$$

Since

$$I_1(k)+I_3(k)=\frac{k-2}{k^2}(\pi(x_k)\log x_k-\vartheta(x_k))+\frac{1}{k^2}(2\pi(k)\log k-\vartheta(k)),$$

we have

$$[x_{12}]=17,$$

$$I_1(12)+I_3(12)=\frac{10}{12^2}\left(\pi(17)\cdot\frac{12}{10}\log 11-\vartheta(17)\right)+\frac{1}{12^2}(2\pi(12)\log 12-\vartheta(12))$$

$$\leqslant\frac{10}{12^2}\left(7\times\frac{12}{10}\log 11-13.1431\right)+\frac{1}{12^2}(2\times 5\log 12-7.7450)$$

$$\leqslant 0.4861+0.1188=0.6049.$$

From this and (37), (39) it follows that

$$\frac{F(12)}{12}\leqslant 1.3872+0.6049=1.9921. \tag{40}$$

Similarly

$$I_1(13)+I_3(13)\leqslant 0.4826+0.1212=0.6038;$$

$$\frac{F(13)}{13}\leqslant 1.3872+0.6038=1.9910; \tag{41}$$

$$I_1(14)+I_3(14)\leqslant 0.4808+0.1090=0.5898;$$

$$\frac{F(14)}{14}\leqslant 1.3872+0.5898=1.9770. \tag{42}$$

The expression (36) implies that $\pi(x)\log x\leqslant\vartheta(x)$ is increasing for $x>2$. Note that $\frac{k-2}{k}$ is decreasing for $k\geqslant 4$. Hence for $15\leqslant k\leqslant 16$, we have $[x_{16}]=22$,

$$\begin{aligned}I_1(k)+I_3(k)&=\frac{k-2}{k^2}(\pi(x_k)\log x_k-\vartheta(x_k))+\frac{1}{k^2}(\pi(k)\log k+(\pi(k)\log k-\vartheta(k)))\\&\leqslant\frac{13}{15^2}\left(\pi(22)\cdot\frac{16}{14}\log 15-\vartheta(22)\right)\\&\quad+\frac{1}{15^2}(\pi(16)\log 16+\pi(16)\log 16-\vartheta(16))\\&\leqslant\frac{13}{15^2}\left(8\times\frac{16}{14}\log 15-16.0876\right)+\frac{1}{15^2}(2\times 6\log 16-10.3099)\\&\leqslant 0.5011+0.1021=0.6032,\end{aligned}$$

$$\frac{F(k)}{k}\leqslant 1.3872+0.6032=1.9904,\quad (15\leqslant k\leqslant 16). \tag{43}$$

Similarly, for $17\leqslant k\leqslant 18$, we have

$$I_1(k)+I_3(k)\leqslant 0.4912+0.0946=0.5858,$$

$$\frac{F(k)}{k}\leqslant 1.3872+0.5858=1.9730\quad (17\leqslant k\leqslant 18). \tag{44}$$

For $19\leqslant k\leqslant 22$, we have

$$I_1(k)+I_3(k)\leqslant 0.5142+0.0925=0.6067,$$

$$\frac{F(k)}{k}\leqslant 1.3872+0.6067=1.9939\quad (19\leqslant k\leqslant 22). \tag{45}$$

For $23\leqslant k\leqslant 28$, we have

$$I_1(k)+I_3(k)\leqslant 0.5169+0.0771=0.5940.$$

$$\frac{F(k)}{k}\leqslant 1.3872+0.5940=1.9812\quad (23\leqslant k\leqslant 28). \tag{46}$$

For $29\leqslant k\leqslant 40$, we have

$$I(k)+I_3(k)\leqslant 0.5422+0.0701=0.6123,$$

$$\frac{F(k)}{k}\leqslant 1.3872+0.6123=1.9995\quad (29\leqslant k\leqslant 40). \tag{47}$$

Now suppose $k\geqslant 41$. As before we have

$$\begin{aligned}I_2(k)&\leqslant 1.001102\,\frac{\log(k-1)}{k}\sum_{i=0}^{3}\int_{(k-1)^{2+\frac{i}{k-2}}}^{(k-1)^{2+\frac{i+1}{k-2}}}\frac{dt}{\log^2 t}\\&\leqslant 1.001102\,\frac{\log(k-1)}{k}\sum_{i=0}^{3}\frac{(k-1)^{2+\frac{i}{k-2}}}{\left(2+\frac{i}{k-2}\right)^2\log^2(k-1)}\left[(k-1)^{\frac{1}{k-2}}-1\right].\end{aligned}$$

Since $\dfrac{(k-1)^{2+\frac{i}{k-2}}}{k(k-2)\left(2+\frac{i}{k-2}\right)^2}$ and $(k-1)^{\frac{1}{k-2}}$ are decreasing for $k\geqslant 9$ and $0\leqslant i\leqslant 3$, it follows from (38) that for $k \geqslant 41$

$$I_2(k) \leqslant 1.001102 \frac{(k-1)^2}{k(k-2)} \left[1+\frac{\log(k-1)}{2(k-1)}\left(1+\frac{\log(k-1)}{3(k-2)}\right.\right.$$

$$\left.\left.\times\left(1+\frac{\log(k-1)}{4(k-2)}(k-1)^{\frac{1}{k-2}}\right)\right)\right] \sum_{i=0}^{3} \frac{(k-1)^{\frac{i}{k-2}}}{\left(2+\frac{i}{k-2}\right)^2}$$

$$\leqslant 1.001102 \cdot \frac{40^2}{41 \times 39}\left[1+\frac{\log 40}{2\times 39}\left(1+\frac{\log 40}{3\times 39}\left(1+\frac{\log 40}{4\times 39}\cdot 40^{\frac{1}{39}}\right)\right)\right]$$

$$\cdot \sum_{i=0}^{3} \frac{40^{\frac{i}{39}}}{\left(2+\frac{i}{39}\right)^2} \leqslant 1.16929.$$

We turn next to $I_1(k)$ for $k \geqslant 41$. In view of Lemma 2, we obtain

$$I_1(k) \leqslant 1.001102 \frac{\log(k-1)}{k} \int_2^{x_k} \frac{dt}{\log^2 t}$$

$$= 1.001102 \frac{\log(k-1)}{k}\left(\int_{k-1}^{(k-1)^{1+\frac{1}{k-2}}} \frac{dt}{\log^2 t} + \int_{(k-1)^{1+\frac{1}{k-2}}}^{(k-1)^{\frac{k}{k-2}}} \frac{dt}{\log^2 t}\right.$$

$$\left. + \int_{(k-1)^{\frac{11}{12}}}^{k-1} \frac{dt}{\log^2 t} + \sum_{i=0}^{4} \int_{(k-1)^{\frac{1}{2}+\frac{i}{12}}}^{(k-1)^{\frac{1}{2}+\frac{i+1}{12}}} \frac{dt}{\log^2 t} + \int_2^{(k-1)^{\frac{1}{2}}} \frac{dt}{\log^2 t}\right). \tag{49}$$

We shall have an upper estimate on the right hand side of (49). Since $(k-1)^{\frac{1}{k-2}}-1$, $\frac{k+1}{k\log(k-1)}$ and $\frac{(k-1)^{1+\frac{1}{k-2}}}{k\left(1+\frac{1}{k-2}\right)^2\log(k-1)}$ are decreasing for $k \geqslant 41$, we have for $k \geqslant 41$

$$1.001102 \frac{\log(k-1)}{k}\left(\int_{k-1}^{(k-1)^{1+\frac{1}{k-2}}} \frac{dt}{\log^2 t} + \int_{(k-1)^{1+\frac{1}{k-2}}}^{(k-1)^{\frac{k}{k-2}}} \frac{dt}{\log^2 t}\right)$$

$$\leqslant 1.001102 \frac{\log(k-1)}{k}\left[\frac{k-1}{\log^2(k-1)}\left((k-1)^{\frac{1}{k-2}}-1\right)\right.$$

$$\left. + \frac{(k-1)^{1+\frac{1}{k-2}}}{\left(1+\frac{1}{k-2}\right)^2 \log^2(k-1)}\left((k-1)^{\frac{1}{k-2}}-1\right)\right]$$

$$\leqslant 1.001102 \cdot \frac{40^{\frac{1}{39}}-1}{41\log 40}\left(40+\frac{40^{\frac{40}{39}}}{\left(1+\frac{1}{39}\right)^2}\right) \leqslant 0.05372. \tag{50}$$

Since $e^x - 1 \leqslant xe^x$ for $x>0$, we have

$$1.001102 \frac{\log(k-1)}{k} \int_{(k-1)^{\frac{11}{12}}}^{k-1} \frac{dt}{\log^2 t}$$

$$\leqslant 1.001102 \cdot \frac{(k-1)^{\frac{11}{12}}}{\left(\frac{11}{12}\right)^2 k \log(k-1)}\left((k-1)^{\frac{1}{12}}-1\right)$$

$$\leqslant 1.001102 \cdot \left(\frac{12}{11}\right)^2 \cdot \frac{1}{12} \cdot \frac{k-1}{k} \leqslant 0.09929. \tag{51}$$

Since $\frac{(k-1)^{\frac{1}{2}+\frac{i}{12}}}{k\log(k-1)}((k-1)^{\frac{1}{12}}-1)$ is decreasing for $k\geqslant 41$ and $0\leqslant i\leqslant 4$, we have for $k\geqslant 41$

$$1.001102\cdot\frac{\log(k-1)}{k}\cdot\sum_{i=0}^{4}\int_{(k-1)^{\frac{1}{2}+\frac{i}{12}}}^{(k-1)^{\frac{1}{2}+\frac{i+1}{12}}}\frac{dt}{\log^2 t}$$

$$\leqslant 1.001102\cdot\frac{\log(k-1)}{k}\cdot\sum_{i=0}^{4}\frac{(k-1)^{\frac{1}{2}+\frac{i}{12}}}{\left(\frac{1}{2}+\frac{i}{12}\right)^2\log^2(k-1)}((k-1)^{\frac{1}{12}}-1)$$

$$\leqslant 1.001102\cdot\frac{\sqrt{40}(40^{\frac{1}{12}}-1)}{41\log 40}\sum_{i=0}^{4}\frac{40^{\frac{i}{12}}}{\left(\frac{1}{2}+\frac{i}{12}\right)^2}\leqslant 0.32473. \tag{52}$$

It remains to estimate the last term on the right hand side of (49). We know

$$\left(\frac{1}{x}\int_2^x\frac{dt}{\log^2 t}\right)'=-\frac{1}{x^2}\int_2^x\frac{dt}{\log^2 t}+\frac{1}{x\log^2 x}=-\frac{2}{x^2}\left(\int_2^x\frac{dt}{\log^3 t}-\frac{1}{\log^2 2}\right) \tag{53}$$

and

$$\log t\leqslant c\,\frac{t-1}{t+1},$$

where

$$c=\begin{cases}2\log 3, & 2\leqslant t\leqslant 3,\\ \frac{4}{3}\log 7, & 3\leqslant t\leqslant 7.\end{cases}$$

Put $g(t)=t+6\log(t-1)-\frac{12}{t-1}-4\frac{1}{(t-1)^2}$, then $g'(t)=\left(\frac{t+1}{t-1}\right)^3$. Thus for $x\geqslant 6.3$ we have

$$\int_2^x\frac{dt}{\log^3 t}\geqslant\int_2^{6.3}\frac{dt}{\log^3 t}\geqslant\frac{1}{(2\log 3)^3}\int_2^3 g'(t)\,dt+\frac{27}{(4\log 7)^3}\int_3^{6.3}g'(t)\,dt$$

$$=\frac{1}{(2\log 3)^3}(g(3)-g(2))+\frac{27}{(4\log 7)^3}(g(6.3)-g(3))$$

$$\geqslant 1.334+0.786=2.120>\frac{1}{\log^2 2}.$$

Hence it follows from (53) that $\frac{1}{x}\int_2^x\frac{dt}{\log^2 t}$ is decreasing for $x\geqslant 6.3$. When $k\geqslant 41$, it is easily to prove that $\frac{(k-1)^{\frac{1}{2}}}{k}\log(k-1)$ is decreasing and $(k-1)^{\frac{1}{2}}\geqslant 6.3$. Thus for $k\geqslant 41$, we have

$$1.001102\,\frac{\log(k-1)}{k}\int_2^{(k-1)^{\frac{1}{2}}}\frac{dt}{\log^2 t}\leqslant 1.001102\cdot\frac{\log 40}{41}\int_2^{6.33}\frac{dt}{\log^2 t}. \tag{54}$$

We note that

$$\log t\geqslant D\,\frac{t-1}{t+1},$$

where

$$D=\begin{cases}3\log 2, & t\geqslant 2,\\ 2\log 3, & t\geqslant 3.\end{cases}$$

Put $h(t)=t+4\log(t-1)-\dfrac{4}{t-1}$, then $h'(t)=\left(\dfrac{t+1}{t-1}\right)^2$. So

$$\begin{aligned}\int_2^{6.33}\frac{dt}{\log^2 t}&\leqslant\frac{1}{9\log^2 2}\int_2^3 h'(t)\,dt+\frac{1}{4\log^2 3}\int_3^{6.33}h'(t)\,dt\\ &=\frac{1}{9\log^2 2}(h(3)-h(2))+\frac{1}{4\log^2 3}(h(6.33)-h(3))\\ &\leqslant 1.33499+1.76071=3.09570.\end{aligned}$$

From this and (54), it follows that for $k\geqslant 41$

$$1.001102\,\frac{\log(k-1)}{k}\int_2^{(k-1)^2}\frac{dt}{\log^2 t}\leqslant 1.001102\cdot\frac{\log 40}{41}\cdot(3.09570)\leqslant 0.27884. \tag{55}$$

In view of (49), (50), (51), (52) and (55), for $k\geqslant 41$ we obtain

$$I_1(k)\leqslant 0.05372+0.09929+0.32473+0.27884=0.75658. \tag{56}$$

Also, it follows from (10) that for $k\geqslant 41$ we have

$$I_3(k)\leqslant\frac{2}{k^2}\pi(k)\log k\leqslant\frac{2}{k}\cdot 1.2551\leqslant\frac{2}{41}\cdot 1.2551\leqslant 0.06123. \tag{57}$$

From this and (37), (48), (56), we see that for $k\geqslant 41$

$$\frac{F(k)}{k}\leqslant 0.75658+1.16929+0.06123=1.98710. \tag{58}$$

By (40)—(47) and (58) we see that $F(k)\leqslant 2k$ is true for all integers $k\geqslant 12$. Thus all that remains is to consider $3\leqslant k\leqslant 11$. By (35), we calculate $\dfrac{F(k)}{k}$ directly.

$$[x_{11}]=16,\quad [x_{11}^2]=[10^{\frac{22}{9}}]=278.$$

$$\begin{aligned}\frac{F(11)}{11}&=\frac{1}{11}\Big[\frac{9}{11}\Big(\pi(16)\cdot\frac{11}{9}\log 10-\vartheta(16)\Big)+\frac{2}{11}\pi(11)\log 11\\ &\quad+\frac{1}{11}(\vartheta(278)-\vartheta(11))+\log 10(\pi(278)-\pi(100))\\ &\quad-\frac{1}{2}(\vartheta(278)-\vartheta(100))\Big]\\ &\leqslant\frac{1}{11}\Big[\frac{9}{11}\Big(6\times\frac{11}{9}\log 10-10.3099\Big)+\frac{2}{11}\times 5\log 11\\ &\quad+\frac{1}{11}(260.0611-7.7450)+(59-25)\log 10\\ &\quad-\frac{1}{2}(260.0610-83.7284)\Big]\leqslant 1.8745.\end{aligned}$$

Similarly, we can obtain

$$\frac{F(10)}{10}\leqslant 1.8843,\quad \frac{F(9)}{9}\leqslant 1.9190,\quad \frac{F(8)}{8}\leqslant 1.9470,\quad \frac{F(7)}{7}\leqslant 1.9835,$$
$$\frac{F(6)}{6}\leqslant 1.9715,\quad \frac{F(5)}{5}\leqslant 1.9932,\quad \frac{F(4)}{4}\leqslant 1.8866.$$

Finally, we consider $k=3$. In this case $(k-1)^{\frac{k}{k-2}}>(k-1)^2$. Hence, from (7), we see that (8) must be replaced by

$$|S(p^l, f(x))|\cdot p^{-l\left(1-\frac{1}{k}\right)} \leqslant \begin{cases} 1, & p>(k-1)^{\frac{2k}{k-2}}, \\ (k-1)p^{-\frac{1}{2}+\frac{1}{k}}, & (k-1)^{\frac{k}{k-2}}<p\leqslant(k-1)^{\frac{2k}{k-2}}, \\ (k-1)p^{\frac{3}{k}-1}, & k<p\leqslant(k-1)^{\frac{k}{k-2}}. \end{cases}$$

By computation as before, we get

$$F(k)=\left(1-\frac{2}{k}\right)(\pi(x_k)\log x_k-\vartheta(x_k))+\frac{2}{k}\pi(k)\log k+\frac{1}{k}(\vartheta(x_k^2)-\vartheta(k))$$
$$+\log(k-1)(\pi(x_k^2)-\pi(x_k))-\frac{1}{2}(\vartheta(x_k^2)-\vartheta(x_k)).$$

From this

$$[x_3]=8,\quad [x_3^2]=[2^6]=64.$$

$$\frac{F(3)}{3}=\frac{1}{3}\Big[\frac{1}{3}(\pi(8)\times 3\log 2-\vartheta(8))+\frac{2}{3}\pi(3)\log 3+\frac{1}{3}(\vartheta(64)-\vartheta(3))$$
$$+\log 2(\pi(64)-\pi(8))-\frac{1}{2}(\vartheta(64)-\vartheta(8))\Big]$$
$$=\frac{1}{3}\Big[\pi(64)\log 2+\frac{2}{3}\pi(3)\log 3-\frac{1}{3}\vartheta(3)-\frac{1}{6}(\vartheta(64)-\vartheta(8))\Big]$$
$$\leqslant\frac{1}{3}\Big[18\log 2+\frac{2}{3}\times 2\log 3-\frac{1}{3}\times 1.7917-\frac{1}{6}(53.1189-5.3472)\Big]$$
$$\leqslant 1.7941.$$

Hence we complete the proof of the Theorem.

References

[1] Hua Lookeng, On an exponential sums, *J. Chinese Math. Soc.*, 2 (1940), 301—312.

[2] Chen Jingrun, On professor Hua's estimate of exponential sums, *Sci. Sinica*, **20**: 6 (1977), 711—719.

[3] Стечкин, С. Б., Оценка полной рациональной тригонометрической суммы, *Тр. Мат. ин-та, АН. СССР*, **143**, (1977), 188—207.

[4] Lu Minggao, On a Note of Complete Trigonometric Suns, (toappear) (in chinese).

[5] Rosser, J. B. and Schoenfeld, L., Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$, *Math. Comp.*, **29** (1975), 243—269.

[6] Rosser, J. B. and Schoenfeld, L., Approximate fomulas for some functions of prime numbers, *Illinois J. Math.*, **6** (1962), 64—94.

[7] Hua Lookeng, *Introduction to Number Theory*, Vol. 7, Science Press, (in chinese).