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EXISTENCE AND UNIQUENESS OFWEAK SOLUTIONS OF UNIFORMLY DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

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Abstract

In this paper we deal with the quasilinear parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left[a_{ij}(x,\ t,\ u)\frac{\partial u}{\partial x_j}\right]+b_i(x,\ t,\ u)\frac{\partial u}{\partial x_i}+c(x,\ t,\ u),$$

which is uniformly degenerate at $u=0$. Under some assumptions we prove existence and uniqueness of nonnegative weak solutions to the Cauchy problem and the first boundary value problem for this equation. Furthermore, the weak solutions are globally Hölder continuous.

§ 1. Introduction

In 1961, E. S. Sabinina[3] studied existence and uniqueness of weak solutions to the Cauchy problem for the N-dimensional porous medium equation

$$\frac{\partial u}{\partial t}=\Delta\varphi(u),$$

where $\varphi(0)=0$ and H. Brezis & M. G. Grandall made a wonderful work[4] on uniqueness for the above problem. Furthermore, L. A. Caffarelli & A. Friedman studied Hölder continuity of weak solutions in [5, 6].

In this paper we establish the existence and uniqueness and Hölder Continuity of weak solutions to the Cauchy problem and the first boundary value problem for uniformly degenerate equations. Since we have found Hölder estimates for positive classical solutions of these equations in [1], weak solutions can be found easily. As to uniqueness we shall apply the method used in [2], but there are substantial differences because we have no estimates for any derivatives of solutions which are required for one-dimensional equations in [2].

Let $\mathbf{R}^N$ be the N-dimensional euclidean space, Ω a bounded open domain in $\mathbf{R}^N$ and $\partial\Omega$ the boundary of Ω. Let

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$$Q_1^T=\{(x, t)\,|\,x\in\Omega,\ 0<t\leqslant T\},$$

and Γ^T be the lateral surface of Q_1^T and Ω_t the section of Q_1^T at time t.

In Q_1^T, we consider the following quasilinear degenerate parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left[a_{ij}(x, t, u)\frac{\partial u}{\partial x_j}\right]+b_i(x, t, u)\frac{\partial u}{\partial x_i}+c(x, t, u) \tag{1.1}$$

with initial and boundary conditions

$$u(x, 0)=u_0(x),\quad x\in\Omega_0, \tag{1.2}$$

$$u|_{\Gamma^T}=g(x, t),\ (x, t)\in\Gamma^T. \tag{1.3}$$

Definition 1.1. *The function $u(x, t)$ defined in $\bar{Q}_1^T$ is called a weak solution to the first boundary value problem* (1.1), (1.2), (1.3), *if*

(i) $u(x, t)\in C(\bar{Q}_1^T)$ *and u is nonnegative in Q_1^T.*

(ii) *for any $\psi(x, t)\in C_{x,t}^{2,1}(\bar{Q}_1^T)$ with $\psi(x, t)|_{\Gamma^T}=0$ and $\psi(x, t)|_{t=T}=0$, $u(x, t)$ satisfies*

$$\iint_{Q_1^T}\left[u\psi_t+A_{ij}(x, t, u)\frac{\partial^2\psi}{\partial x_i\partial x_j}-B_i(x, t, u)\frac{\partial\psi}{\partial x_i}+C(x, t, u)\psi\right]dx\,dt$$

$$=-\int_{\Omega_0}u_0(x)\psi(x, 0)dx+\int_{\Gamma^T}A_{ij}(x, t, g)\frac{\partial\psi}{\partial x_i}\cos(\boldsymbol{n}, x_j)dx\,dt, \tag{1.4}$$

where $\boldsymbol{n}$ is the outward normal on Γ^T and

$$\begin{aligned}A_{ij}(x, t, u)&=\int_0^u a_{ij}(x, t, \tau)d\tau,\\ B_i(x, t, u)&=\int_0^u\left[b_i(x, t, \tau)-\frac{\partial a_{ij}(x, t, \tau)}{\partial x_j}\right]d\tau,\\ C(x, t, u)&=c(x, t, u)+\int_0^u\frac{\partial b_i(x, t, \tau)}{\partial x_i}d\tau.\end{aligned} \tag{1.5}$$

Similarly in the domain

$$Q^T=\{(x, t)\,|\,x\in\mathbf{R}^N,\ 0<t\leqslant T\}$$

we discuss the Cauchy problem for equation (1.1) with initial condition

$$u|_{t=0}=u_0(x),\quad x\in\mathbf{R}^N. \tag{1.6}$$

Definition 1.2. *The function $u(x, t)$ defined in $\bar{Q}^T$ is called a weak solution to the Cauchy problem* (1.1), (1.6) *if*

(i) $u(x, t)\in C(\bar{Q}^T)$ *and u is bounded and nonnegative in Q^T;*

(ii) *for any bounded domain $\Omega\subset\mathbf{R}^n$ with smooth boundary and any*

$$\psi(x, t)\in C_{x,t}^{2,1}(\bar{\Omega}\times[0, T])$$

with $\psi(x, t)|_{\partial\Omega\times[0,T]}=0$, $\psi(x, t)|_{t=T}=0$, $u(x, t)$ satisfies

$$\int_0^T\int_\Omega\left[u\psi_t+A_{ij}(x, t, u)\frac{\partial^2\psi}{\partial x_i\partial x_j}-B_i(x, t, u)\frac{\partial\psi}{\partial x_i}+C(x, t, u)\psi\right]dx\,dt$$

$$=-\int_\Omega u_0(x)\psi(x, 0)dx+\int_0^T\int_{\partial\Omega}A_{ij}(x, t, u)\frac{\partial\psi}{\partial x_i}\cos(\boldsymbol{n}, x_j)dx\,dt. \tag{1.7}$$

Assume that the coefficients of equation (1.1) for the first boundary value problem satisfy the following conditions:

(EA) for any $(x, t)\in Q_1^T$, $0<u<\infty$, $\xi\in\mathbf{R}^N$

$$\nu(|u|)|\xi|^2\leqslant a_{ij}(x, t, u)\xi_i\xi_j\leqslant \Lambda\nu(|u|)|\xi|^2, \tag{1.8}$$

where Λ is a constant and $\nu(s)$ has the following properties:

(a) $\nu(s)\in C[0, \infty)$, $\nu(0)=0$ and $\nu(s)>0$ if $s>0$. (1.9)

(b) Let $\varphi(v)=\int_0^v \nu(s)ds$ and its inverse be $\Phi(W)$. There exist $\delta>0$ and $m>1$ such that, for any W_1 and W_2 satisfying $0<W_1<W_2\leqslant\delta$,

$$\frac{1}{m}\left(\frac{W_1}{W_2}\right)^{1-\frac{1}{m}}\leqslant\frac{\Phi'(W_2)}{\Phi'(W_1)}\leqslant m. \tag{1.10}$$

(EB) for any $(x, t)\in Q_1$, $u\in[0, \infty)$,

$$C'_u(x, t, u)\leqslant\Lambda,\ C(x, t, 0)=0; \tag{1.11}$$

(EC) $a_{ij}(x, t, u)$, $b_i(x, t, u)$, $\dfrac{\partial b_i(x, t, u)}{\partial x_i}$, $\dfrac{\partial a_{ik}(x, t, u)}{\partial x_k}$, $C'_u(x, t, u)$ $(i, j, k=1, 2, \cdots, N)$ belong to $C^\alpha(\overline{Q}_1^T\times(0, \infty))$ $(\alpha>0)$ and satisfy

$$\frac{1}{\nu(|\mu|)}\left\{\sum_{i=1}^N b_i^2(x, t, u)\right\}+\sum_{i=1}^N\left|\sum_{k=1}^N\frac{\partial a_{ik}(x, t, u)}{\partial x_k}\right| +\sum_{i=1}^n\left|\frac{\partial b_i(x, t, u)}{\partial x_i}\right|+|C'_u(x, t, u)|\leqslant\eta(|u|) \tag{1.12}$$

in $\overline{Q}_1^T\times(0, \infty)$ where $\eta(\tau)$ is an increasing function.

In addition, assume that the initial and boundary data satisfy that

(DA) $u_0(x)\in C^\alpha(\overline{\Omega})$, $g(x, t)\in C^\alpha(\Gamma^T)$ $(\alpha>0)$;

(DB) $u_0(x)\geqslant0$ in $\overline{\Omega}_0$ and $g(x, t)\geqslant0$ in Γ^T;

(DC) $u_0(x)=g(x, 0)$ if $x\in\partial\Omega_0$.

On the boundary $\partial\Omega$ of Ω we suppose that

(BA) $\partial\Omega$ belongs to $C^{2+\alpha}$.

By assumption (BA), we have the following corollaries:

$(\mathrm{BA})_1$ $\partial\Omega$ satisfies the uniformly outer spherical condition, that is, there exists $\delta_0>0$ such that for any $x_0\in\partial\Omega$ an outer sphere $K(\delta_0)$ with radius δ_0 satisfying $K(\delta_0)\subset\mathbf{R}^N\backslash\overline{\Omega}$ and $x_0\in\overline{K}(\delta_0)$ can be found.

$(\mathrm{BA})_2$ there exist $a_0>0$ and $\theta_0>0$ such that for any $x_0\in\partial\Omega$ the sphere $K(\rho; x_0)$ with its centre at x_0 and radius $\rho\leqslant a_0$ satisfies

$$\operatorname{mes}\{K(\rho; x_0)\backslash(K(\rho; x_0)\cap\Omega)\}\geqslant\theta_0\operatorname{mes}\{K(\rho; x_0)\},$$

where mes $\{\cdot\}$ is the measure of a set in $\mathbf{R}^N$.

In what follows we shall describe the main results in this paper.

Theorem 1.1. *Suppose that* (EA), (EB), (EC), (DA), (DB), (DC) *and* (BA) *hold. Then, problem* (1.1), (1.2), (1.3) *has at least a weak solution which is Hölder continuous in* $\overline{Q}_1^T$.

In order to establish the uniqueness theorem, we have to give additional assumptions:

(ED) for any s_1, s_2 satisfying $0<s_1<s_2\leqslant\delta$

$$\frac{\nu(s_1)}{\nu(s_2)}\leqslant\lambda\left(\frac{s_1}{s_2}\right),$$

where $\lambda(\tau)$ is a function which tends to zero as $\tau\to 0^+$.

(DD) $\nu(u_0(x))$ is Lipschitz continuous on $\overline{\Omega}_0$ and $\nu(g(x, t))$ Lipschitz continuous with respect to x and t on $\overline{\Gamma}^T$.

Theorem 1.2. *Suppose that (ED) and (DD) hold in addition to the assumptions of Theorem* 1.1. *Then, the weak solution of problem* (1.1), (1.2), (1.3) *is unique.*

For the Cauchy problem we still suppose that (EA), (EB), (EC) are satisfied, but instead of $\overline{Q}_1^T$ and Ω we use Q^T and $\mathbf{R}^N$ respectively.

(DA)′ $u_0(x)$ is uniformly Hölder continuous in $\mathbf{R}^N$.

(DB)′ $u_0(x)\geqslant 0$ in $\mathbf{R}^N$.

Theorem 1.3. *Suppose that (EA), (EB), (EC), (DA)′ and (DB)′ hold. Then, problem* (1.1), (1.6) *admits a unique weak solution. Furthermore, the weak solution is uniformly Hölder continuous in $\overline{Q}^T$.*

§ 2. The First Boundary Value Problem

By assumptions (DA), (DB) and (DC), we can establish the sequences $\{u_{0n}(x)\}$ and $\{g_n(x, t)\}$ of sufficiently smooth functions satisfying

$$\frac{1}{n}\leqslant u_{0n}(x)-u_0(x)\leqslant\frac{2}{n},\quad x\in\Omega,$$

$$\frac{1}{n}\leqslant g_n(x, t)-g(x, t)\leqslant\frac{2}{n},\quad (x, t)\in\Gamma^T, \tag{2.1}$$

$$\|u_{0n}(x)\|_{C^\alpha(\overline{\Omega})}\leqslant L_\alpha,$$

$$\|g_n(x, t)\|_{C^{\alpha,\frac{\alpha}{2}}_{x,t}(\Gamma)}\leqslant L_\alpha, \tag{2.2}$$

and the corresponding consistency conditions, where L_α is independent of n.

If (DD) holds, we still require that

$$|\nu(u_{0n}(x_1))-\nu(u_{0n}(x_2))|\leqslant L|x_1-x_2|,\ \forall x_1,\ x_2\in\Omega_0, \tag{2.3}$$

$$|\nu(g_n(x_1, t_1))-\nu(g_n(x_2, t_2))|\leqslant L(|x_1-x_2|+|t_1-t_2|),$$

$$\forall(x_1, t_1),\ (x_2, t_2)\in\Gamma^T.$$

Consider equation (1.1) with the initial and boundary conditions

$$u|_{t=0}=u_{0n}(x),\ x\in\Omega_0, \tag{2.4}$$

$$u|_{\Gamma^T}=g_n(x, t). \tag{2.5}$$

Here we extend $a_{ij}(x, t, u)$, $b_i(x, t, u)$, $c(x, t, u)$ from $Q_1^T\times(0, +\infty)$ to $Q_1^T\times(-\infty, +\infty)$ properly so that (EA), (EB) and (EC) still hold in $Q_1^T\times(-\infty, +\infty)$.

Lemma 2.1. *If $u_n(x, t)$ is a solution to problem* (1.1), (2.4), (2.5), *we have*

$$\frac{1}{n}e^{-\eta(M)T}\leqslant u_n(x, t)\leqslant M, \tag{2.6}$$

where $M = e^{AT}\max\{\max\limits_{\Omega} u_0(x),\ \max\limits_{\Gamma^T} g(x, t)\} + 2.$

Proof Set $u_n = e^{At}v$. By means of the maximum principle, it follows immediately that

$$0 \leqslant u_n(x, t) \leqslant M.$$

In order to obtain a positive lower bound of $u_n(x, t)$, we set, for any $\varepsilon > 0$,

$$W(x, t) = e^{\eta(M)t}u_n(x, t) + \varepsilon e^t.$$

$W(x, t)$ will satisfy the equation

$$\frac{\partial W}{\partial t} - \frac{\partial}{\partial x_i}\left[a_{ij}\frac{\partial W}{\partial x_j}\right] - b_i\frac{\partial W}{\partial x_i} - [c(x, t, u) + \eta(M)](W - \varepsilon e^t) - \varepsilon e^t = 0.$$

Owing to $W - \varepsilon e^t \geqslant 0$, $W(x, t)$ will not reach its positive minimum in the interior of Q_1^T and so

$$W(x, t) \geqslant \frac{1}{n} + \varepsilon,$$

i. e.

$$u_n \geqslant \left[\frac{1}{n} + \varepsilon - \varepsilon e^t\right]e^{-\eta(M)t}.$$

Setting $\varepsilon \to 0$, we obtain (2.6). The proof is completed.

Remark. We can suppose that condition (1.10) holds for any W_1, W_2 satisfying $0 < W_1 < W_2 \leqslant \varphi(Me^{\eta(M)T})$ if we change the constant m properly.

By means of the a priori estimates in Lemma 2.1, it is easy to show that the classical solution to problem (1.1), (2.4), (2.5) exists. Applying Hölder estimates for solutions in [1], we have

Lemma 2.2. *The solution* $u_n(x, t)$ *to problem* (1.1), (2.4), (2.5) *has the following estimate*

$$|u_n|_{C_{x,t}^{\alpha_1, \frac{\alpha_1}{2}}(\bar{Q}_1^T)} \leqslant M_{\alpha_1}, \tag{2.7}$$

where $\alpha_1 \in (0, 1)$ and M_{α_1} are independent of n.

The proof of Theorem 1.1 By Lemma 2.1 and 2.2, the sequence $\{u_n\}$ of solutions is compact in $C(\bar{Q}_1^T)$ and so we can extract a subsequence $\{u_{n_k}\}$ which converges to some function $u(x, t)$ in $C(\bar{Q}_T^1)$. Without any difficulty, we can verify that $u(x, t)$ is a weak solution to problem (1.1), (1.2), (1.3). These prove Theorem 1.1.

To discuss the uniqueness theorem, we need other properties of $u_n(x, t)$.

Let $(x^0, t^0) \in \bar{Q}_1^T$ and $K(\rho, x^0)$ (for simplicity, usually denoted by $K(\rho)$) be a sphere with its centre at x^0 and radius ρ. We denote that

$$B_{k,\rho}(t) = \{x \in K(\rho) \cap \Omega \mid W(x, t) < k\} \quad (0 \leqslant t \leqslant T), \tag{2.8}$$

where

$$W(x, t) = \int_0^{u_n(x,t)e^{\eta(M)t}} \nu(s)ds. \tag{2.9}$$

Lemma 2.3. *Let* $u_n(x, t)$ *be a solution to problem* (1.1), (2.4), (2.5), *and*

$\zeta(x)$ a cut-off function in $K(\rho)$. If

$$0<k\leqslant \min_{K(\rho)\cap\partial\Omega} W(x, t),$$

then we have

$$\frac{\partial}{\partial t}\left[e^{-\gamma t}\int_{B_{k,\rho}(t)}\zeta^2\tilde{\chi}_k(k-W)dx\right]+\frac{1}{2m^m}e^{-\gamma t-(m-1)\eta(M)T}\int_{B_{k,\rho}(t)}\zeta^2|\nabla W|^2dx$$
$$\leqslant\gamma\int_{B_{k,\rho}(t)}|\nabla\zeta|^2(W-k)^2dx, \tag{2.10}$$

where γ depends only on N, m, T, M and Λ,

$$\tilde{\chi}_k(s)=\int_0^s \Phi'(k-\tau)\tau\, ds, \tag{2.11}$$

and $v=\Phi(W)$ is the inversion of

$$W=\int_0^v \nu(s)ds=\varphi(v).$$

Proof Set $v=u_n(x, t)e^{\eta(M)t}$. $v(x, t)$ will satisfy the equation

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x_i}\left[\tilde{a}_{ij}(x, t, v)\frac{\partial v}{\partial x_j}\right]+\tilde{b}_i(x, t, v)\frac{\partial v}{\partial x_j}+\tilde{c}(x, t, v)v, \tag{2.12}$$

where

$$\tilde{a}_{ij}(x, t, v)=a_{ij}(x, t, e^{-\eta(M)t}v),$$
$$\tilde{b}_i(x, t, v)=b_i(x, t, e^{-\nu(M)t}v),$$
$$\tilde{c}(x, t, v)=\int_0^1 c'_u(x, t, \tau u)d\tau+\eta(M).$$

By conditions (EA), (EB), (EC) and (2.13) which will be proved below, we have

$$\frac{1}{m^m}e^{-(m-1)\eta(M)T}\nu(v)|\xi|^2\leqslant\tilde{a}_{ij}(x, t, v)\xi_i\xi_j\leqslant m\Lambda\nu(v)|\xi|^2,$$

$$\sum_{i=1}^n\tilde{b}_i(x, t, v)\leqslant m\eta(M)\nu(v),$$

$$0\leqslant\tilde{c}(x, t, v)\leqslant 2\eta(M).$$

The rest of proof will be similar to the proof of Lemma 2.1 in [1].

By condition (1.10), for $0<s_1<s_2\leqslant Me^{\eta(M)T}$, we have

$$\frac{1}{m^m}\left(\frac{s_1}{s_2}\right)^{m-1}\leqslant\frac{\nu(s_1)}{\nu(s_2)}\leqslant m. \tag{2.13}$$

In fact, if denoting $\theta=W_2/W_1$, we have, for $0<W_1<W_2$,

$$\frac{\Phi(W_1)}{\Phi(W_2)}=\frac{\Phi(W_1)-\Phi(0)}{\Phi(\theta W_1)-\Phi(0)}=\frac{\Phi'(\widetilde{W})}{\theta\Phi'(\theta\widetilde{W})},\quad 0<\widetilde{W}<W_1.$$

Using (1.10) we find that, for $0<W_1<W_2<\varphi(Me^{\eta(M)T})$,

$$\frac{1}{m}\frac{W_1}{W_2}\leqslant\frac{\Phi(W_1)}{\Phi(W_2)}\leqslant m\left(\frac{W_1}{W_2}\right)^{\frac{1}{m}},$$

which implies that, for $0<s_1<s_2<Me^{\eta(M)T}$,

$$\frac{1}{m^m}\left(\frac{s_1}{s_2}\right)^m\leqslant\frac{\varphi(s_1)}{\varphi(s_2)}\leqslant m\left(\frac{s_1}{s_2}\right). \tag{2.14}$$

Combining (1.10) with (2.14), we can obtain (2.13).

Lemma 2.4. *Suppose that for* $\rho\in(0,1]$,

$$\operatorname{mes}(K(\rho)\setminus\{K(\rho)\cap\Omega\})\geqslant b\chi_n\rho^N, \tag{2.15}$$

where b *is a positive constant. Then, there exists a constant* s *which depends only on* N, m, T, γ *and* b *such that if* k *satisfies*

$$k\leqslant \min_{\substack{x\in K(\rho)\cap\partial\Omega\\ t^0-\Phi'\left(\frac{k}{2^{s+2}}\right)\rho^2<t<t^0}}\{W(x,t)\}, \tag{2.16}$$

then we have

$$\operatorname{mes} B_{\frac{k}{2^{s+2}},\frac{\rho}{8}}(t)=0 \quad \textit{for}\ t\in\left[t^0-\frac{1}{16}\Phi'\left(\frac{k}{2^{s+2}}\right)\rho^2,\ t^0\right]. \tag{2.17}$$

Moreover if $\operatorname{mes} B_{k,\rho}\left(t^0-\Phi'\left(\frac{k}{2^{s+2}}\right)\rho^2\right)=0$, *then* (2.17) *holds for*

$$t\in\left[t^0-\Phi'\left(\frac{k}{2^{s+2}}\right)\rho^2,\ t^0\right].$$

Proof Similarly to the proof of Lemma 3.5 in the paper [1], we can get that for any $\theta_1>0$ there exists $S=S(\theta_1)$ such that

$$\int_{t^0-\Phi'\left(\frac{k}{2^{s+2}}\right)\rho^2}^{t^0}\operatorname{mes} B_{\frac{k}{2^s},\frac{\rho}{2}}(t)\,dt\leqslant\theta_1\Phi'\left(\frac{k}{2^{s+2}}\right)\rho^{N+2},$$

but now there will not come up the case: $k\leqslant 2^{s+2}\rho^{\varepsilon}$ because there is no term $\operatorname{mes} B_{k,\rho}(t)$ on the right side of inequality (2.10).

Then, for any $\theta_2>0$, there exists $\theta_1=\theta_1(\theta_2)$ such that

$$\operatorname{mes} B_{\frac{k}{2^{s+1}},\frac{\rho}{4}}(t)\leqslant\theta_2\rho^N \text{ for } t\in\left[t^0-\frac{1}{4}\Phi'\left(\frac{k}{2^{s+2}}\right)\rho^2,\ t^0\right]$$

and finally there exists θ_2 such that

$$\operatorname{mes} B_{\frac{k}{2^{s+2}},\frac{\rho}{8}}(t)=0 \quad \text{for}\quad t\in\left[t^0-\frac{1}{16}\Phi'\left(\frac{k}{2^{s+2}}\right)\rho^2,\ t^0\right],$$

which are analogous to Lemmas 3.6 and 3.7 in [1]. We can determine θ_1 by θ_2 and then s by θ_1, which is the required constant.

Lemma 2.5. *Suppose that the assumptions of Theorem 2.2 hold. Then, there exist constants* C_1 *and* C_2 *independent of* n *such that for any* $(x^0,t^0)\in\Gamma$ *we have*

$$\min_{N_\rho} u_n(x,t)\geqslant\frac{1}{C_2}g_n(x^0,t^0), \tag{2.18}$$

where

$$\rho=C_1^{-1}\min\{\nu(g_n(x^0,t^0)),\ a_0\}, \tag{2.19}$$

$$N_\rho=\{(x,t)\in Q_1^T\,|\,|x-x^0|<\rho,\ t^0\leqslant t\leqslant\min\{t^0+\rho,\ T\}\}, \tag{2.20}$$

and a_0 *is defined in* $(BA)_2$.

Proof Denote that

$$\tilde g_n(x,t)=\begin{cases} g_n(x,t)e^{\eta(M)t}, & (x,t)\in\Gamma,\\ u_{0n}(x), & (x,t)\in\Omega_0.\end{cases} \tag{2.21}$$

By (2.3), for any (x^0,t^0), $(x,t)\in\Gamma\cup\Omega_0$

$$|\nu(g_n(x,\ t))-\nu(g_n(x^0,\ t^0))|\leqslant L(|x-x^0|+|t-t^0|), \tag{2.22}$$

where $g_n(x,\ t)$ is defined as $u_{0n}(x)$ on Ω_0. Denote that

$$\rho_0=\frac{1}{4L}\nu(g_n(x^0,\ t^0)), \tag{2.23}$$

$$Q_{\rho_0}=\{(x,\ t)\mid |x-x^0|<\rho_0,\ |t-t^0|<\rho_0\}, \tag{2.24}$$

then we have

$$\frac{1}{2}\nu(g_n(x^0,\ t^0))\leqslant\nu(g_n(x,\ t))\leqslant\frac{2}{3}\nu(g_n(x^0,\ t^0)),\ (x,\ t)\in Q_{\rho_0}\cap\Gamma. \tag{2.25}$$

Since $\lambda(s)\to 0$ as $s\to 0^+$, we can find $\delta_0>0$ such that

$$\lambda(s)<\frac{1}{2}\quad\text{if}\quad 0<s\leqslant\delta_0. \tag{2.26}$$

If $g_n(x,\ t)\leqslant g_n(x^0,\ t^0)$, by (ED) and (2.25)

$$\frac{1}{2}\leqslant\frac{\nu(g_n(x,\ t))}{\nu(g_n(x^0,\ t^0))}\leqslant\lambda\Big(\frac{g_n(x,\ t)}{g_n(x^0,\ t^0)}\Big),\ (x,\ t)\in Q_{\rho_0}\cap\Gamma.$$

From (2.26) it follows that

$$\delta_0\leqslant\frac{g_n(x,\ t)}{g_n(x^0,\ t^0)}\leqslant 1\quad\text{for }(x,\ t)\in Q_{\rho_0}\cap\Gamma.$$

If $g_n(x,\ t)\geqslant g_n(x^0,\ t^0)$, by (ED) and (2.25)

$$\frac{2}{3}\leqslant\frac{\nu(g_n(x^0,\ t^0))}{\nu(g_n(x,\ t))}\leqslant\lambda\Big(\frac{g_n(x^0,\ t^0)}{g_n(x,\ t)}\Big),\ (x,\ t)\in Q_{\rho_0}\cap\Gamma.$$

Using (2.26) again, we have

$$\delta_0\leqslant\frac{g_n(x^0,\ t^0)}{g_n(x,\ t)}\leqslant 1\quad\text{for }(x,\ t)\in Q_{\rho_0}\cap\Gamma.$$

Thus, no matter which case it is, we always have

$$\delta_0 g_n(x^0,\ t^0)\leqslant g_n(x,\ t)\leqslant\frac{1}{\delta_0}g_n(x^0,\ t^0)\ \text{for }(x,\ t)\in Q_{\rho_0}\cap\Gamma. \tag{2.27}$$

By means of (2.14), we can find that

$$C_0^{-1}\varphi(\tilde{g}_n(x^0,\ t^0))\leqslant\varphi(\tilde{g}_n(x,\ t))\leqslant C_0\varphi(\tilde{g}_n(x^0,\ t^0)),\ (x,\ t)\in Q_{\rho_0}\cap\Gamma, \tag{2.28}$$

where $C_0=C_0(m,\ \delta_0,\ T)$.

We shall apply Lemma 2.4 to $W(x,\ t)=\varphi(u_n e^{\eta(M)t})$. Clearly, we have

$$W(x,\ t)|_{\Gamma\cup\Omega_0}=\varphi(\tilde{g}_n(x,\ t)). \tag{2.29}$$

Now take

$$k=(2C_0)^{-1}\varphi(\tilde{g}_n(x^0,\ t^0)) \tag{2.30}$$

and then take $\rho_1>0$ such that

$$\Phi'\Big(\frac{k}{2^{s+2}}\Big)\rho_1^2\leqslant\rho_0, \tag{2.31}$$

where s is defined in Lemma 2.4. By the definition of ρ_0 and k, it follows that

$$\Phi'\Big(\frac{k}{2^{s+2}}\Big)\rho_0=\frac{1}{4L}\Phi'\Big(\frac{1}{2^{s+3}C_0}\varphi(\tilde{g}_n(x^0,\ t^0))\Big)\Big/\Phi'(\varphi(g_n(x^0,\ t^0))),$$

and so, by (1.10)

$$\tilde{C}_0^{-1}\leqslant\Phi'\Big(\frac{k}{2^{s+2}}\Big)\rho_0\leqslant\tilde{C}_0,$$

where $\widetilde{C}_0=\widetilde{C}_0(m, L, S, T)$. To guarantee the inequality (2.31), it suffices to take

$$\rho_1=\min\{\widetilde{C}_0^{-\frac{1}{2}}\rho_0,\ a_0\}, \tag{2.32}$$

Denote that $t^1=\min\left\{t^0+\Phi'\left(\frac{k}{2^{s+2}}\right)\frac{\rho_1^2}{16},\ T\right\}$ and consider the domain

$$\widetilde{Q}_{\rho_1}=\left\{(x,\ t)\,|\,|x-x^0|<\rho_1,\ \max\left\{0,\ t^1-\Phi'\left(\frac{k}{2^{s+2}}\right)\rho_1^2\right\}\leqslant t\leqslant t^1\right\},$$

which is contained in Q_{ρ_0}. By the selection of k, (2.28) and (2.29), we have

$$k\leqslant \min_{\widetilde{Q}_{\rho_1}\cap(\Gamma\cup\Omega_0)}\{W(x,\ t)\}. \tag{2.33}$$

This inequality and the condition $(BA)_2$ yield

$$\operatorname{mes} B_{\frac{k}{2^{s+2}},\ \frac{\rho_1}{8}}(t)=0 \quad \text{for } t\in[t^0,\ t^1]$$

by Lemma 2.4. This implies that if we take

$$\rho=\min\left\{\frac{\rho_1}{8},\ \frac{1}{16}\widetilde{C}_0^{-2}\rho_0,\right\},$$

then

$$\min_{N_\rho} W(x,\ t)\geqslant\frac{k}{2^{s+2}}\geqslant\frac{1}{2^{s+3}C_0}\ \varphi(\tilde{g}_n(x^0,\ t^0)). \tag{2.34}$$

In virtue of the definition of $W(x, t)$ we have

$$\min_{N_\rho} u_n(x,\ t)\geqslant\frac{1}{2^{s+3}mC_0}\ e^{-\eta(M)T}g_n(x^0,\ t^0),$$

which is required.

Now pass to the proof of uniqueness. Let $u(x, t)$ be the weak solution to problem (1.1), (1.2), (1.3) established in the proof of Theorem 1.1. There exists a subsequence $\{u_{n_k}(x, t)\}$ of weak solutions to problems (1.1), (2.4), (2.5), which converges to $u(x, t)$. Assume that there is another solution $\tilde{u}(x, t)$ to problem (1.1), (1.2), (1.3) such that $u(x, t)\not\equiv\tilde{u}(x, t)$. Then, there exists a point $(x', t')\in Q_1^T$ and its neighborhood $G_\delta(x', t')\subset Q_1^T$ in which $u(x, t)\neq\tilde{u}(x, t)$ every where. Choose a sufficiently smooth function $U(x, t)$ such that $U(x', t')>0$ and

$$\operatorname{supp}\{U(x,\ t)\}\subset G_\delta(x',\ t').$$

It is clear that

$$\iint_{Q_1^T} U(x,\ t)(u(x,\ t)-\tilde{u}(x,\ t))\,dx\,dt\neq 0. \tag{2.35}$$

By the definition of weak solutions, it follows that for any $\psi(x, t)\in C_{x,t}^{2,1}(\overline{Q}_1^T)$ with

$$\psi|_{t=T}=0, \quad \psi|_{\Gamma^T}=0,$$

$$\iint_{Q_1^T}(u_n-\tilde{u})\left[\psi_t+\tilde{a}_{ij}^{(n)}\frac{\partial^2\psi}{\partial x_i\partial x_j}-\tilde{b}_i^{(n)}(x,\ t)\frac{\partial\psi}{\partial x_i}+\tilde{c}^{(n)}(x,\ t)\psi\right]dx\,dt$$
$$=-\int_{\Omega_0}(u_{0n}-u_0)\psi(x,\ 0)\,dx+\int_\Gamma[A_{ij}(x,\ t,\ g_n)-A_{ij}(x,\ t,\ g)]\frac{\partial\psi}{\partial x_i}\cos(\mathbf{n},\ x_j)\,ds, \tag{2.36}$$

where

$$\tilde{a}_{ij}^{(n)}(x,\ t)=\frac{A_{ij}(x,\ t,\ u_n)-A_{ij}(x,\ t,\ \tilde{u})}{u_n-\tilde{u}}=\int_0^1 a_{ij}(x,\ t,\ \tau u_n+(1-\tau)\tilde{u})d\tau,$$

$$\begin{aligned}\tilde{b}_k^{(n)}(x,\ t)&=\frac{B_i(x,\ t,\ u_n)-B_i(x,\ t,\ \tilde{u})}{u_n-\tilde{u}}\\&=\int_0^1 b_i(x,\ t,\ \tau u_n+(1-\tau)\tilde{u})+\frac{\partial a_{ij}}{\partial x_j}(x,\ t,\ \tau u_n+(1-\tau)\tilde{u})d\tau,\end{aligned}\tag{2.37}$$

$$\begin{aligned}\tilde{c}^{(n)}(x,\ t)&=\frac{c(x,\ t,\ u_n)-c(x,\ t,\ \tilde{u})}{u_n-\tilde{u}}\\&=\int_0^1\Big[-\frac{\partial b_i(x,\ t,\ \tau u_n+(1-\tau)\tilde{u})}{\partial x_i}+c_u'(x,\ t,\ \tau u_n+(1-\tau)\tilde{u})\Big]d\tau.\end{aligned}$$

Denote that

$$\begin{aligned}&Q_\rho^{x^0,t^0}=\{(x,\ t)\in\bar{Q}_1^T\,|\,|x-x^0|\leqslant\rho,\ |t-t^0|\leqslant\rho^2\},\\&\omega(\rho)=\max_{(x^0,t^0)\in\bar{Q}_1^T}\operatorname{osc}\{\tilde{u};\ Q_\rho^{x^0,t^0}\}.\end{aligned}\tag{2.38}$$

By the continuity of $\tilde{u}(x,\ t)$, we have $\omega(\rho)\to 0$ as $\rho\to 0$. For any integer $k>0$, we can select asufficiently smooth function $\tilde{u}_k(x,\ t)$ such that

$$0\leqslant\tilde{u}_k-\tilde{u}\leqslant\frac{1}{k},$$

$$\omega_k(\rho)=\max_{(x^0,\ t^0)\in\bar{Q}_T^1}\operatorname{osc}\{\tilde{u}_k(x,\ t);\ Q_\rho^{x^0,t^0}\}\leqslant\omega\Big(\rho+\frac{1}{k}\Big),\tag{2.39}$$

and then define

$$a_{ij}^{(n,k)}(x,\ t)=\int_0^1 a_{ij}(x,\ t,\ \tau u_n+(1-\tau)\tilde{u}_k)d\tau,$$

$$\tilde{b}_i^{(n,k)}(x,\ t)=\int_0^1\Big[b_i(x,\ t,\ \tau u_n+(1-\tau)\tilde{u}_k)-\frac{\partial a_{ij}}{\partial x_j}(x,\ t,\ \tau u_n+(1-\tau)\tilde{u}_k)\Big]d\tau,\tag{2.40}$$

$$\tilde{c}^{(n,k)}(x,\ t)=\int_0^1\Big[-\frac{\partial b_i(x,\ t,\ \tau u_n+(1-\tau)\tilde{u}_k)}{\partial x_i}+c_u^1(x,\ t,\ \tau u_n+(1-\tau)\tilde{u}_k)\Big]d\tau.$$

Now consider the problem

$$\begin{cases}\tilde{a}_{ij}^{(n,k)}(x,\ t)\dfrac{\partial^2\psi}{\partial x_i\partial x_j}-\tilde{b}_i^{(n,k)}(x,\ t)\dfrac{\partial\psi}{\partial x_i}+\tilde{c}^{(n,k)}(x,\ t)\psi+\psi_t=U(x,\ t),\\ \psi|_{t=T}=0,\quad\psi|_{\Gamma^T}=0.\end{cases}\tag{2.41}$$

For any $\xi\in\mathbf{R}^n$, we have

$$|\xi|^2\int_0^1\nu(\tau u_n+(1-\tau)\tilde{u}_k)d\tau\leqslant\tilde{a}_{ij}^{(n,k)}(x,\ t)\xi_i\xi_j\leqslant\Lambda|\xi|^2\int_0^1\nu(\tau u_n+(1-\tau)\tilde{u}_k)d\tau$$

and

$$\begin{aligned}\tilde{\nu}(x,\ t)&=\int_0^1\nu(\tau u_n+(1-\tau)\tilde{u}_k)d\tau\geqslant\int_{\frac{1}{2}}^1\nu(\tau u_n+(1-\tau)\tilde{u}_k)d\tau\\&\geqslant\frac{1}{m^m}\,\nu\Big(\frac{e^{-\eta(M)T}}{2n}\Big),\end{aligned}\tag{2.42}$$

which implies that the equation is nondegenerate. Under the assumptions of Theorem 1.2, the solution $\psi_{nk}(x,\ t)\in C_{x,t}^{2+\alpha,\,1+\frac{\alpha}{2}}(\bar{Q}_1^T)$ to problem (2.41) exists. We shall obtain some estimations of $\psi_{nk}(x,\ t)$. By means of the maximum principle, it follows that

Lemma 2.6. *For the solutions $\psi_{nk}(x, t)$ to problems (2.41), we have*

$$|\psi_{nk}(x, t)| \leqslant \widetilde{M}_1, \tag{2.43}$$

where $\widetilde{M}_1$ is independent of n and k.

Lemma 2.7. *For the solutions $\psi_{nk}(x, t)$ to problems (2.41), we have*

$$\left| |\nu(g_n(x, t)) \frac{\partial \psi_{nk}}{\partial \mathbf{n}} \right|_{\Gamma^T} \leqslant \widetilde{M}_2, \tag{2.44}$$

where $\mathbf{n}$ is the outward normal on Γ^T and $\widetilde{M}_2$ is independent of n, k.

Proof Let $(x^0, t_0) \in \Gamma$. Denote that $R = \min\{\rho, \delta_0\}$, where ρ is defined in Lemma 2.5 and δ_0 in $(BA)_1$. By condition $(BA)_1$, we can make a sphere $K(R, y)$ such that $K(R, y) \subset \mathbf{R}^N \setminus \overline{\Omega}$ and $x^0 \in \overline{K}(R, y)$. Denote that

$$d = |x-y| - R,$$

$$S(r) = \frac{\widetilde{M}_1 e^{\eta(M)K}}{1-e^{-k}}(1-e^{-Kr}),$$

where K is a constant to be determined. Consider the function

$$p(x, t) = \pm \psi_{nk}(x, t) e^{\eta(M)t} + S\left(\frac{d+t-t^0}{\rho}\right)$$

in the neighborhood of (x^0, t^0)

$$N_\rho = \{(x, t) \in Q_1^T \mid d < \rho,\ t^0 \leqslant t < t^0 + \rho\}.$$

Without loss of generality, we can assume that $\operatorname{supp}\{U(x, t)\}$ is outside N_ρ, other wise it suffices to let ρ be small enough.

By Lemma 2.6, it is easy to show $p(x, t) \geqslant 0$ on the lateral surface and the top of N_ρ. In N_ρ we consider (drop the superscripts k, n)

$$\begin{aligned} Lp &= \tilde{a}_{ij}(x, t) \frac{\partial^2 p}{\partial x_i \partial x_j} + \tilde{b}_i(x, t) \frac{\partial p}{\partial x_i} + [\tilde{c}(x, t) - \eta(M)]p + p_t \\ &= \frac{S'}{\rho}\left\{\tilde{a}_{ij}\left[\frac{1}{|x-y|}\delta_{ij} - \frac{(x_i-y_i)(x_j-x_j)}{|x-y|^3}\right] + \tilde{a}_{ij}\frac{1}{\rho}\frac{S''}{S'}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right. \\ &\quad \left. + \tilde{b}_i \frac{x_i-y_i}{|x-y|} + 1\right\} + [\tilde{c} - \eta(M)]S. \end{aligned}$$

Noting that $S''(r) = -KS'(r)$ and using conditions (EA), (EC) and the expression (2.42), we have

$$Lp \leqslant \frac{1}{\rho} S'\left[\frac{nA\tilde{\nu}}{\rho} - K\frac{\tilde{\nu}}{\rho} + A + 1\right] + [\tilde{c} - \eta(M)]S.$$

By the definition of ρ in Lemma 2.5, it follows that

$$\begin{aligned} \frac{\tilde{\nu}}{\rho} &\geqslant \frac{C_1 \int_0^1 \nu(\tau u_n + (1-\tau)\tilde{u}_k) d\tau}{\nu(g_n(x^0, t^0))} \\ &\geqslant \frac{C_1}{2m} \frac{\nu\left(\frac{1}{2}u_n\right)}{\nu(g_n(x^0, t^0))} \geqslant \frac{C_1}{2m} \min\left\{\frac{1}{m}\left(\frac{\frac{1}{2}u_n}{g_n(x^0, t^0)}\right)^{m-1}, \frac{1}{m}\right\} \\ &\geqslant \frac{C_1}{2m^2} \min\left\{\left(\frac{1}{2C_2}\right)^{m-1}, 1\right\} = \frac{1}{C_3}. \end{aligned}$$

Taking $K \geqslant 2\Lambda n + (\Lambda+2)C_3$, we have

$$Lp<0 \text{ in } N_\rho.$$

Thus, $p(x, t)$ will reach its minimum in N_ρ at (x^0, t^0) and so

$$\left.\frac{\partial p}{\partial \mathbf{n}}\right|_{(x^0,t^0)} \leqslant 0,$$

which implies (2.44).

Lemma 2.8. *For appropriately large $k>0$, we have*

$$\iint_{Q_1^T} \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right)^2 dx\,dt \leqslant \widetilde{M}_3(n), \quad \iint_{Q_1^T} \left(\frac{\partial \psi}{\partial x_i}\right)^2 dx\,dt \leqslant \widetilde{M}_3(n),$$

where $\widetilde{M}_3(n)$ is independent of k.

Proof By means of Theorem 17 in the paper [7] and the condition (2.39) on $\tilde{u}_k$, we can find these conclusions.

Now return to the proof of Theorem 1.2. Substituting the solutions $\psi_{nk}(x, t)$ to problems (2.41) into (2.36), we obtain

$$\iint_{Q_1^T} (u_n - \tilde{u}) U(x, t)\,dx\,dt = -\int_{\Omega_0} (u_{0n} - u_0)\psi(x, 0)\,dx$$

$$+\int_{\Gamma^T} (A_{ij}(x, t, g_n) - A_{ij}(x, t, g)) \frac{\partial \psi_{nk}}{\partial x_i} \cos(\mathbf{n}, x_j)\,ds$$

$$+\iint_{Q_1^T} (u_n - \tilde{u}) \Big[(\tilde{a}_{ij}^{(n,k)} - \tilde{a}_{ij}^{(n)}) \frac{\partial^2 \psi_{nk}}{\partial x_i \partial x_j} + (\tilde{b}_i^{(n,k)} - b_i^{(n)}) \frac{\partial \psi_{nk}}{\partial x_i}$$

$$+ (\tilde{c}^{(n,k)} - \tilde{c}^{(n)})\psi \Big]\,dx\,dt.$$

With the help of Lemmas 3.6—3.8, it follows that

$$\left|\iint_{Q_1^T} (u_n - \tilde{u}) U(x, t)\,dx\,dt\right| \leqslant \widetilde{M}_1 \max_{\Omega_0} |u_{0n} - u_0| + \max_{\Gamma} \frac{\Lambda \tilde{\nu}(x, t) \widetilde{M}_2}{\nu(g_n(x, t))} |g_n(x, t) - g(x, t)|$$

$$+ C(n) \max_{Q_1^T} [\,|\tilde{a}_{ij}^{(n,k)} - \tilde{a}_{ij}^{(n)}| + |\tilde{b}_i^{(n,k)} - \tilde{b}_i^{(n)}| + |\tilde{c}^{(n,k)} - \tilde{c}^{(n)}|\,].$$

Setting $k \to \infty$, we get

$$\left|\iint_{Q_1^T} (u_n - \tilde{u}) U(x, t)\,dx\,dt\right| \leqslant \widetilde{M}_1 \max_{\Omega_0} |u_{0n} - u_0| + \max_{\Gamma} \frac{2\Lambda \widetilde{M}_2}{n} \frac{\int_0^1 \nu(\tau g_n + (1-\tau) g)\,d\tau}{\nu(g_n(x, t))}$$

$$\leqslant \frac{2\widetilde{M}_1}{n} + \frac{2\Lambda \widetilde{M}_2 m}{n},$$

and then setting $n \to \infty\ (n = n_k)$, we have

$$\iint_{Q_1^T} (u(x, t) - \tilde{u}(x, t)) U(x, t)\,dx\,dt = 0,$$

which is contrary to (2.35). Theorem 1.2 has been proved.

§ 3. Cauchy Problem

For each integer $n>0$, let Ω_n be a sphere in $\mathbf{R}^N$ with its centre at the origin and radius n, Q_n^T the $(N+1)$-dimensional cylinder, $\Omega_n\times(0, T]$, and Γ_n the lateral surface of Q_n^T, and select a smooth function $u_{0n}(x)$ defined in Ω_n satisfying

$$\frac{1}{n}\leqslant u_{0n}(x)-u_0(x)\leqslant\frac{2}{n},$$

$$\|u_{0n}(x)\|_{C^\alpha(\Omega_n)}\leqslant L, \tag{3.1}$$

where L is a constant independent of n. We define a smooth function $g_n(x, t)$ on Γ_n such that

$$\|g_n(x, t)\|_{C^\alpha(\Gamma_n)}\leqslant L,$$

$$\frac{1}{n}\leqslant g_n(x, t)\leqslant\max|u_0(x)|+2 \tag{3.2}$$

and $g_n(x, t)$ and $u_{0n}(x)$ satisfy the consistency conditions of order one.

Consider equation (1.1) with initial and boundary conditions

$$u|_{t=0}=u_{0n}(x),\ x\in\Omega_n, \tag{3.3}$$

$$u|_{\Gamma_n}=g_n(x, t). \tag{3.4}$$

It is similar to the above section that the problems (1.1), (3.3), (3.4) have classical solutions $u_n(x, t)$ which have the following estimates

$$\max_{\overline{Q}_n}|u_n(x, t)|\leqslant M=e^{AT}(\max_{\mathbf{R}_n}u_0(x)+2), \tag{3.5}$$

$$\|u_n\|_{C_{x,t}^{\alpha_1,\frac{\alpha_1}{2}}(\overline{Q}_n)}\leqslant M_1, \tag{3.6}$$

where M_1, $\alpha_1>0$ are independent of n.

From this it follows at once that we can extract a subsequence $\{u_{n_k}(x, t)\}$ which uniformly converges to a function $u(x, t)$ in any compact subset in Q^T. It is easy to verify that $u(x, t)$ is a weak solution to the Cauchy problem (1.1), (1.6) which implies the existence of weak solutions.

If problem (1.1), (1.6) has another weak solution $\tilde{u}(x, t)$ such that $\tilde{u}(x, t)\not\equiv u(x, t)$, then there exists a sufficiently smooth function $U(x, t)$ with a compact support in Q^T such that

$$\iint_{Q^T}U(x, t)(u(x, t)-\tilde{u}(x, t))\,dx\,dt\neq0. \tag{3.7}$$

Similarly to (2.41), we consider the following problems

$$\begin{cases}\tilde{a}_{ij}^{(n,k)}(x, t)\dfrac{\partial^2\psi}{\partial x_i\partial x_j}+\tilde{b}_i^{(n,k)}(x, t)\dfrac{\partial\psi}{\partial x_i}+\tilde{c}^{(n,k)}(x, t)\psi+\psi_t=U(x, t)\ \text{in}\ Q_n^T,\\ \psi|_{t=T}=0,\ \psi|_{\Gamma_n}=0.\end{cases} \tag{3.8}$$

These problems will have classical solutions $\psi_{nk}(x, t)$ which have the following estimates.

Lemma 3.1. *For the solutions $\psi_{nk}(x, t)$ to problems (3.8), we have*

$$\max_{\overline{Q}_T^n} |\psi_{nk}(x, t)| \leqslant \overline{M}_1, \tag{3.9}$$

where $\overline{M}_1$ is independent of n and k.

Lemma 3.2. *For the solutions $\psi_{nk}(x, t)$ to problems (3.8), we have*

$$\max_{\overline{Q}_n^T} |\psi_{nk}(x, t)| \leqslant \overline{M}_2 e^{-\frac{1}{\sqrt{N}}|x|} \text{ in } Q_n, \tag{3.10}$$

where $\overline{M}_2$ is independent of n and k.

Proof In the domain

$$Q_n^+ = \{(x, t) \in Q_n^T \mid x_1 > 0\}, \tag{3.11}$$

consider the function

$$p(x, t) = Ke^{-|x_1| + \beta(T-t)} \pm \psi_{nk}(x, t) e^{\eta(M)t}. \tag{3.12}$$

Clearly, we have $p(x, T) \geqslant 0$, $p(x, t)|_{\Gamma_n} \geqslant 0$ and

$$p(x, t)|_{x_1=0} \geqslant K \pm \psi_{nk}(x, t) e^{\eta(M)t} \geqslant 0,$$

if we take $K \geqslant \overline{M}_1 e^{\eta(M)T}$. In Q_n^+, we find

$$\begin{aligned} Lp &= \tilde{a}_{ij}^{(n,k)}(x, t) \frac{\partial^2 p}{\partial x_i \partial x_j} + \tilde{b}_i^{(n,k)}(x, t) \frac{\partial p}{\partial x_i} + [\tilde{c}^{(n,k)}(x, t) - \eta(M)] p + p_t \\ &= [\tilde{a}_{11}^{(n,k)} - \tilde{b}_1^{(n,k)} + (\tilde{c}^{(n,k)} - \eta(M)) - \beta] \cdot Ke^{-|x_1| + \beta(T-t)} + U(x, t). \end{aligned}$$

It is sure to be able to find n_0 such that

$$\operatorname{supp}\{U(x, t)\} \subset Q_{n_0}^T.$$

Taking

$$K \geqslant \max |U(x, t)| \cdot e^{n_0},$$
$$\beta \geqslant 3\eta(M) + 1,$$

we have

$$Lp < 0$$

and

$$p(x, t) \geqslant 0 \quad \text{in } Q_n^+,$$

which implies that

$$|\psi_{nk}(x, t)| \leqslant Ke^{-|x_1| + \beta(T-t)} \text{ in } Q_n^+.$$

In the same way, we can obtain

$$|\psi_{nk}(x, t)| \leqslant Ke^{-\max_i |x_i| + \beta(T-t)} \leqslant Ke^{\beta T} e^{-\frac{1}{\sqrt{N}}|x|}$$

as claimed if we take $\overline{M}_2 = Ke^{\beta T}$.

Lemma 3.3. *For the solutions ψ_{nk} to problems (3.8), we have*

$$\left| \nu\left(\frac{1}{n}\right) \frac{\partial \psi_{nk}}{\partial x} \right|_{\Gamma_n} \leqslant \overline{M}_3 \exp\left\{-\frac{n}{\sqrt{N}}\right\}, \tag{3.13}$$

if n is sufficiently large, where $\overline{M}_3$ is independent of n.

Proof Let $\rho = \min\left\{\nu\left(\frac{1}{n}\right), 1\right\}$. For any $(x^0, t^0) \in \Gamma_n$ we make an outer sphere $K(\rho; y)$ such that $K(\rho; y) \subset \mathbf{R}^n \setminus \Omega_n$ and $x_0 \in \overline{K}(\rho; y)$. Denote that $d = |x - y| - \rho$ and

$$N_\rho = \{(x, t) \in Q_n^T \mid d < \rho, \ 0 \leqslant t < T\}. \tag{3.14}$$

In N_ρ we consider the function

$$p(x,\ t)=\pm\psi_{nk}(x,\ t)e^{\eta(M)t}+M_3 e^{-\frac{n}{\sqrt{N}}}(1-e^{-\frac{Kd}{\rho}}). \tag{3.15}$$

Take $K>1$ and $M_3=2\overline{M}_2\exp\left\{\frac{1}{\sqrt{N}}\right\}$. It is easy to verify that $p(x,\ t)\geqslant 0$ on the lateral surface and the top of N_ρ. Let n be sufficiently large so that N_ρ and supp $\{U(x,\ t)\}$ do not intersect. It is similar to Lemma 2.7 that we can obtain

$$Lp<0$$

if K is sufficiently large. We shall obtain the estimate (3.13) from this.

Lemma 3.4. *For appropriatety large $k>0$, we have*

$$\sum_{ij=1}^{N}\iint\limits_{Q_n^T}\left(\frac{\partial^2\psi_{nk}}{\partial x_i\,\partial x_j}\right)^2 dx\,dt\leqslant\overline{M}_4(n).\quad \sum_{i=1}^{N}\iint\limits_{Q_n^T}\left(\frac{\partial\psi}{\partial x_i}\right)^2 dx\,dt\leqslant\overline{M}_4(n), \tag{3.16}$$

where $\overline{M}_4(n)$ is independent of k.

Applying Lemma 3.1 to Lemma 3.4, we can prove the uniqueness for the Cauchy problem. At this time, it is easier than that for the first boundary value problem since

$$\left|\frac{\partial\psi_{nk}(x,\ t)}{\partial x}\right|_{\Gamma_n}\leqslant\frac{\overline{M}_3 e^{-\frac{n}{\sqrt{N}}}}{\nu\left(\frac{1}{n}\right)}\to 0\quad\text{as}\quad n\to+\infty$$

by Lemma 3.3. These will prove Theorem 1.3.

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