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THE ISOMORPHISMS OF SYMPLECTIC GROUPS OVER Φ-SURJECTIVE RINGS

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Abstract

Let R and R_1 are Φ-surjective rings with 2 as a unit. Then Λ: $S_{p_n}(R)\to S_{p_{n_1}}(R_1)$ is a group isomorphism if and only if $n=n_1$, there exists a ring homomorphism σ: $R\to R_1$ and Λ has the standard form $\Lambda X=PX^\sigma P^{-1}$, for all X in $S_{p_n}(R)$, where P is a generalized symplectic matrix.

Let R be a commutative ring with 1, Max (R) be the set of all its maximal ideals. For any ideal A in R, we have a natural ring homomorphism of R onto R/A which is denoted by λ_A. As is well known

$$R\to \prod_{M\in\max(R)} R/M \text{ by } x\mapsto(\cdots,\ \lambda_M x,\ \cdots)$$

is a ring homomorphism. This homomorphism, in general, is not surjective. If there exists $\{M_t\}_{t\in T}\subseteq \mathrm{Max}(R)$ such that

$$\Phi\colon R\to \prod_{t\in T} R/M_t$$

is a surjective homomorphism, and $M\subseteq \bigcup_{t\in T} M_t$ for each M in Max (R), then we call R a Φ-surjective ring. Clearly, the semi-local rings, the infinite complete direct product of fields are all Φ-surjective rings.

The set $\{M_t\}_{t\in T}$ in above definition is called the defining system of ideals of the Φ-surjective ring R.

It is easy to see that, Ker $\Phi=J(R)$ and $x\in R^*\Leftrightarrow\Phi(x)\in(\prod_{t\in T} R/M_t)^*\Leftrightarrow\lambda_t x\neq 0$ for all $t\in T$, where $J(R)$ is the Jacobson radical of R, R^* the set of units in R.

In this note, we always assume that R and R_1 are Φ-surjective rings with 2 as a unit, $S_{p_n}(R)$ and $S_{p_{n_1}}(R_1)$ are symplectic groups with n, $n_1\geqslant 4$, and $\{M_t\}_{t\in T}$, $\{N_s\}_{s\in S}$ are the defining system of ideals of R and R_1 respectively.

We shall determine the isomorphisms between two symplectic groups. The main result is the following

Theorem *Under the above assumption,* Λ: $S_{p_n}(R)-S_{p_{n_1}}(R)$ *is a group isomor-*

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phism if and only if $n=n_1$, there exists a ring homomorphism $\sigma: R\to R_1$ and Λ has the form

$$\Lambda X=PX^{\sigma}P^{-1},\quad \text{for all}\quad X\in S_{p_n}(R),$$

where P is some generalized symplectic matrix.

Prerequisites and Notations

We use $E_{ij}(a)$ $(a\in R)$ to denote the $n\times n$ matrix with the element a in the (i, j)position and 0, other wise.

$$Q_{ij}=I+E_{ii}(-1)+E_{jj}(-1)+E_{ij}(1)+E_{ij}(-1),$$

$$T_{ij}(a)=I+E_{ij}(a),\ i\neq j,\ a\in R.$$

Let $n=2\nu$ and

$$T_i=\begin{array}{l}\left(\begin{array}{cccccc} I &&&&& \\ & -1 &&&& \\ && I &&& \\ &&& I && \\ &&&& -1 & \\ &&&&& I \end{array}\right)\begin{array}{l} \\ (i) \\ \\ \\ (\nu+i) \\ \\ \end{array}\end{array},$$

$$J_{1j}=\begin{array}{c} \begin{array}{ccccccccc} &&(j)&&&&&& \end{array} \\ \left(\begin{array}{ccccccccc} -1 && -1 &&&&&& \\ & I &&&&&&& \\ && 1 &&&&&& \\ &&& I &&&&& \\ &&&& -1 &&&& \\ &&&&& I &&& \\ &&&& -1 && 1 && \\ &&&&&&& I & \end{array}\right)\begin{array}{l} \\ \\ \\ \\ \\ \\ (\nu+j) \\ \\ \end{array}\end{array},$$

$$J_{j1}=J'_{1j},\ i=1,\ \cdots,\ \nu,\ j=2,\ \cdots,\ \nu$$

$$(GS_{p_n}(R)=\left\{X\in GL_n(R)\,|\,X=P\begin{pmatrix} I^{(\nu)} & \\ & aI^{(\nu)}\end{pmatrix},\ P\in S_{p_n}(R),\ a\in R^*\right\}^{*)}$$

$$GS_{p_n}(R,\ A)=\{X\in S_{p_n}(R)\,|\,\lambda_A X=aI,\ a^2=1,\ a\in R/A\},$$

$$SS_{p_n}(R,\ A)=\{X\in S_{p_n}(R)\,|\,\lambda_A X=I\},$$

where A is a proper ideal of R.

Obviousely, T_i, J_{1j}, J_{j1}, $T_{i,\nu+i}(a)$, $T_{\nu+i,i}(a)$ $(a\in R,\ i=1,\ \cdots,\ \nu,\ j=2,\ \cdots,\ \nu)$ are the elcments of $S_{p_n}(R)$. $SS_{p_n}(R,\ A)$ and $G\,S_{p_n}(R,\ A)$ are the normal subgroups of $S_{p_n}(R)$. $GS_{p_n}(R)$ is the subgroup of $GL_n(R)$, and $PXP^{-1}\in S_{p_n}(R)$ for all $P\in GS_{p_n}(R)$, $X\in S_{p_n}(R)$.

§ 1. Lemmas

Lemma 1.[2] *The $S_{p_n}(R)$ is generated by the following matrices:*

*) The matrices in $GR_{p_n}(R)$ are callad generalized symplectic.

$$T_1,\ J_{1j},\ J_{j1},\ (j=2,\ \cdots,\ \nu),\ T_{1,\nu+1}(a)\,(a\in R),\begin{pmatrix} & I^{(\nu)} \\ -I^{(\nu)} & \end{pmatrix}.$$

Lemma 2. *Suppose that G is a normal subgroup of $S_{p_n}(R)$, and $O(G)=R$. Then $G=S_{p_n}(R)$.*

Proof (1) If there is a matrix $X\in G$ such that $O(X)=R$, then the normal subgroup generated by X coincides with $S_{p_n}(R)$ by [2]. Hence $G=S_{p_n}(R)$.

(2) If $X_1,\ X_2\in G,\ T_i=\{t\in T\mid X_i\notin GS_{p_n}(R,\ M_t)\},\ T_i\neq\emptyset,\ T_1\cap T_2=\emptyset$, then there exists $Y\in G$ such that $Y\notin GS_{p_n}(R,\ M_t)$ if $t\in T_1\cup T_2$ and $Y\in GS_{p_n}(R,\ M_t)$ if $t\in T-(T_1\cup T_2)$.

Pick $a\in R$ as follows: $\lambda_t\ a=0$ for $t\in T_1$, and $\lambda_t a=1$ for $t\in T-T_1$. Then we can see that $O\left(X_1\begin{pmatrix} I^{(\nu)} & aE_{11} \\ & I^{(\nu)}\end{pmatrix}\right)=R$. Thus, by step (1), there exist $P_i\in S_{p_n}(R)$ and $m\in Z^+$ such that

$$\begin{pmatrix} I & E_{11} \\ & I\end{pmatrix}=\prod_{i=1}^{m}P_i\left(X_1\begin{pmatrix} I & aE_{11} \\ & I\end{pmatrix}\right)^{\pm1}P_i^{-1}.$$

Let $Y_1=\prod\limits_{i=1}^{m}P_iX_1^{\pm1}P_j^{-1}$. Then $\lambda_t,\ Y_1=\begin{pmatrix} I & E_{11} \\ & I\end{pmatrix}$ for t in T_1 and λ_tY_1 is in the center of $S_{p_n}(R/M_t)$ for t in $T-T_1$, that is, $Y_1\notin GS_{p_n}(R,\ M_t),\ t\in T_1$ and $Y_1\in GS_{p_n}(R,\ M_t),\ t\in T-T_1$.

Consider X_2. By the same reason, we can find $Y_2\in G$ such that $Y_2\notin GS_{p_n}(R,\ M_t)$ for t in T_2 and $Y_2\in GS_{p_m}(R,\ M_t)$ for t in $T-T_2$. Let $Y=Y_1Y_2$. Then $Y\in G$ and $Y\notin GS_{p_n}(R,\ M_t)$ for $t\in T_1\cup T_2$ and $Y\in GS_{p_n}(R,\ M_t)$ for $t\in T-(U_1\cup T_2)$.

(3) By $O(G)=R$, there exist $X_i\in G,\ i=1,\ \cdots,\ k$ such that

$$\sum_{i=1}^{k}O(X_i)=R.$$

Let $T=\{t\in T\mid X_i\notin GS_{p_n}(R,\ M_t),\ i=1,\ \cdots,\ k.$ Without loss of generality, we can assume that $T_i\neq\emptyset,\ T_i\cap(T_1\cup\cdots\cup T_{i-1}\cup T_{i+1}\cup\cdots\cap_{T_k})=\emptyset,\ 1\leqslant i\leqslant k$ and $T=T_1\cup\cdots\cup T_k$.

Since $X_1,\ X_2$ satisfy the condition in (2), there is $Y\in G$ such that $Y\notin GS_{p_n}(R,\ M_t)$ for $t\in T_1\cup T_2$ and $Y\in GS_{p_n}(R,\ M_t)$ for $t\in T-(T_1\cup T_2)$, It is clear that Y and X_3 also satisfy the condition which X_1 and X_2 satisfy. So we have $Z\in G$ such that $Z\notin GS_{p_n}(R,\ M_t)$ if $t\in T_1\cup T_2\cup T_3$ and $Z\in GS_{p_n}(R,\ M_t)$ if $t\in T-(T_1\cup T_2\cup T_3)$. Repeat step (2), we shall have our lemma.

Lemma 3. *For any ideal A in R, we have*

$$SS_{p_n}(R,\ A)=[SS_{p_n}(R,\ A),\ S_{p_n}(R)]=[GS_{p_n}(R,\ A),\ S_{p_n}(R)].$$

Proof Let $G=[SS_{p_n}(R,\ A),\ S_{p_n}(R)]$.

(1) Suppose $n\geqslant6$. Take $a\in A$. Since

$$X_1=\begin{pmatrix} T_{\nu1}^{(\nu)}(1) & \\ & T_{1\nu}^{(\nu)}(-1)\end{pmatrix}\begin{pmatrix} I & aE_{11} \\ & I\end{pmatrix}\begin{pmatrix} T_{\nu1}^{(\nu)}(1) & \\ & T_{1\nu}^{(\nu)}(-1)\end{pmatrix}^{-1}\begin{pmatrix} I & aE_{11} \\ & I\end{pmatrix}^{-1}$$

$$=\begin{pmatrix} I & aE_{\nu\nu}+aE_{1\nu}+aE_{\nu 1} \\ & I \end{pmatrix}\in G,$$

$$X_2=\begin{pmatrix} T_{\nu 2}^{(\nu)}(1) & \\ & T_{2\nu}^{(\nu)}(-1) \end{pmatrix}\begin{pmatrix} I & aE_{12}+aE_{21} \\ & I \end{pmatrix}\begin{pmatrix} T_{\nu 2}^{(\nu)}(1) & \\ & T_{2\nu}^{(\nu)}(-1) \end{pmatrix}^{-1}$$

$$\times\begin{pmatrix} I & aE_{12}+aE_{21} \\ & I \end{pmatrix}^{-1}=\begin{pmatrix} I & aE_{1\nu}+aE_{\nu 1} \\ & I \end{pmatrix}\in G.$$

We have

$$\begin{pmatrix} I & aE_{11} \\ & I \end{pmatrix}=\begin{pmatrix} & & 1 & & & \\ & I & & & & \\ 1 & & & & & \\ & & & & & 1 \\ & & & & I & \\ & & & 1 & & \end{pmatrix}X_1X_2^{-1}\begin{pmatrix} & & 1 & & & \\ & I & & & & \\ 1 & & & & & \\ & & & & & 1 \\ & & & & I & \\ & & & 1 & & \end{pmatrix}^{-1}\in G.$$

Therefore, the normal subgroup generated by $\begin{pmatrix} I & aE_{11} \\ & I \end{pmatrix}$ is contained in G. So $S_{p_n}(R, A)\subseteq G$ by [2], and

$$SS_{p_n}(R, A)=[S_{p_n}(R, A), S_{p_n}(R)].$$

(2) If $n=4$, since $R/M\neq F_2$, $\forall t\in T$, we can pick an ε in R^* such that $\varepsilon(\varepsilon-1)\in R^*$. So for any $a\in A$, we have

$$Y_1=\begin{pmatrix} \varepsilon & & & \\ & 1 & & \\ & & \varepsilon^{-1} & \\ & & & 1 \end{pmatrix}\begin{pmatrix} I & \begin{matrix} 0 & a \\ a & 0 \end{matrix} \\ & I \end{pmatrix}\begin{pmatrix} \varepsilon & & & \\ & 1 & & \\ & & \varepsilon^{-1} & \\ & & & 1 \end{pmatrix}^{-1}\begin{pmatrix} I & \begin{matrix} 0 & a \\ a & 0 \end{matrix} \\ & I \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} I & \begin{matrix} 0 & (\varepsilon-1)a \\ (\varepsilon-1)a & 0 \end{matrix} \\ & I \end{pmatrix}\in G,$$

$$Y_2=\begin{pmatrix} \varepsilon-1 & & & \\ & 1 & & \\ & & (\varepsilon-1)^{-1} & \\ & & & 1 \end{pmatrix}\begin{pmatrix} I & \begin{matrix} 0 & a \\ a & 0 \end{matrix} \\ & I \end{pmatrix}\begin{pmatrix} \varepsilon-1 & & & \\ & 1 & & \\ & & (\varepsilon-1)^{-1} & \\ & & & 1 \end{pmatrix}$$

$$=\begin{pmatrix} I & \begin{matrix} 0 & (\varepsilon-1)a \\ (\varepsilon-1)a & a \end{matrix} \\ & I \end{pmatrix}\in G.$$

Hence, we also have

$$\begin{pmatrix} I & aE_{11} \\ & I \end{pmatrix}=\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}Y_2Y_1^{-1}\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}\in G.$$

So $SS_{p_n}(R, A)=[SS_{p_n}(R, A), S_{p_n}(R)]$. $G=[GS_{p_n}(R, A), S_{p_n}(R)]$ is trivial.

Lemma 4. *If Λ is a group isomorphism of $S_{p_n}(R)$ onto $S_{p_{n_1}}(R_1)$, then there is a bijection τ between* $\mathrm{Max}(R)$ *and* $\mathrm{Max}(R_1)$ *such that*

$$\Lambda SS_{p_n}(R, M) = S_{p_{n_1}}(R, \tau(M)), \ \forall M \in \mathrm{Max}(R). \tag{*}$$

Further, the isomorphism Λ induces a set of isomorphisms $\Lambda_{\tau(M)}$ of $S_{p_n}(R/M)$ onto $S_{p_n}(R_1/\tau(M))$ with

$$\Lambda_{\tau(M)}(\lambda_M X) = \lambda_{\tau(M)}(\lambda X), \ \forall X \in S_{p_n}(R), \ M \in \mathrm{Max}(R).$$

Proof For each $M \in \mathrm{Max}(R)$, there is an N in $\mathrm{Max}(R_1)$ such that

$$\Lambda SS_{p_n}(R, M) \subseteq GS_{p_{n_1}}(R_1, N),$$

since $\Lambda SS_{p_n}(R, M)$ is a normal subgroup of $S_{p_{n_1}}(R_1)$. But

$$SS_{p_n}(R, M) = [S_{p_n}(R), GS_{p_n}(R, M)] = [S_{p_n}(R), SS_{p_n}(R, M)]$$

by lemma 3. So we have

$$\Lambda SS_{p_n}(R, M) \subseteq [S_{p_{n_1}}(R_1), GS_{p_{n_1}}(R, N)] = SS_{p_{n_1}}(R_1, N).$$

Similarly, there exists $M_1 \in \mathrm{Max}(R)$ such that

$$\Lambda^{-1} SS_{p_{n_1}}(R_1, N) \subseteq SS_{p_n}(R, M_1).$$

Hence $\Lambda SS_{p_n}(R, M) \subseteq SS_{p_{n_1}}(R_1, N) \subseteq \Lambda SS_{p_n}(R, M_1)$. Consequently $M \subseteq M_1$, $M = M_1$, i. e.

$$\Lambda SS_{p_n}(R, M) = SS_{p_{n_1}}(R_1, N).$$

Now let τ: $M \to N$. It is easy to see that τ is a bijection of Max (R) onto Max (R_1) and

$$S_{p_n}(R)/SS_{p_n}(R, M) \simeq S_{p_{n_1}}(R_1)/SS_{p_{n_1}}(R_1, \tau(M)). \tag{1}$$

Hence

$$S_{p_n}(R/M) \overset{\Lambda_{\tau(M)}}{\sim} S_{p_{n_1}}(R_1/\tau(M)). \tag{2}$$

The $\Lambda_{\tau(M)}(\lambda_M X) = \lambda_{\tau(M)}(\Lambda X)$ can be obtained by (1) and (2) immediately.

Lemma 5. *Suppose that $\Lambda: S_{p_n}(R) \to S_{p_{n_1}}(R_1)$ is an isomorphism, $\{M_t\}_{t\in T}$ is a defining system of ideals of R. Then the $\{\tau(M_t)\}_{t\in T} = \{N_{S_t}\}_{t\in T}$ obtained from τ:* $\mathrm{Max}\ (R) \to \mathrm{Max}(R_1)$ *by restriction is a defining system of ideals of R_1.*

Proof For each $t \in T$, take a b_t in R_1/N_{S_t}. There is an X_t in $S_{p_n}(R/M_t)$ such that $\Lambda_{S_t} X_t = T_{1,\nu+1}(b_t)$. But we have an X in $S_{p_n}(R)$ such that $\lambda_t X = X_t$ for all $t \in T$. Therefore we have $\Lambda_{S_t}(\lambda_t X) = \lambda_{S_t}(\Lambda X) = T_{1,\nu+1}(b_t)$. Let b be the element of ΛX in $(1, \nu+1)$ position. Then we have $\lambda_{S_t} b = b_t$ for all $t \in T$ by

$$\lambda_{S_t}(\Lambda X) = T_{1,\nu+1}(b_t).$$

Thus $b \mapsto (\cdots \lambda_{S_t} b \cdots)$ is a surjective homomorphism of R_1 onto $\prod_{t\in T} R_1/N_{S_t}$, i.e. $\{N_{S_t}\}_{t\in T}$ is a defining system of ideals of R_1.

By this lemma, if $S_{p_n}(R) \simeq S_{p_{n_1}}(R_1)$ and $\{M_t\}_{t\in T}$ is a defining system of ideals of R, then we can assume that the defining system of R_1 is $\{N_{S_t}\}_{t\in T}$, which satisfy

$$\Lambda SS_{p_n}(R, M_t) = SS_{p_{n_1}}(R_1, N_{S_t}), \ \forall t \in T.$$

Besides, by the proof of this lemma, we have $n = n_1$ whenever

$$S_{p_n}(R)\simeq S_{p_{n_1}}(R_1).$$

Lemma 6. (1) *If* $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(R)$, *and* $a-1\in J(R)$, *then* $X=I$.

(2) *If* $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(R)$, $X^2=-I$ *and* $c\in R^*$, then

$$\begin{pmatrix}1 & -c^{-1}a\\ & c^{-1}\end{pmatrix}X\begin{pmatrix}1 & -c^{-1}a\\ & c^{-1}\end{pmatrix}^{-1}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

If $X^3=-I$, *then*

$$\begin{pmatrix}1 & -c^{-1}a\\ & c^{-1}\end{pmatrix}X\begin{pmatrix}1 & -c^{-1}a\\ & c^{-1}\end{pmatrix}^{-1}=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}.$$

(3) *If* $X\in S_{p_n}(R)$, $X^2=I$ *and* $\lambda_0 X=T_1$ *for all* $t\in T$, *then there exists* $P\in SL_n(R)$ *such that*

$$PXP^{-1}=T_1,\ \lambda_t P=I,\ \forall t\in T.$$

Proof the Proof of (3) can be found in [5]. (1) and (2) can be proved by calculation.

For convenience, we denote

$$A_p X=PXP^{-1},\quad \text{for } P\in GL_n(R).$$

Besides, Let, $S_{0,1}$ be the set in $S_{p_n}(R)$ (or in $S_{p_{n_1}}(R_1)$), whose entries just consist of 0, 1 or -1.

Lemma 7. *Suppose that* A: $SL_2(R)\to SL_2(R_1)$ *is an isomorphism.*

(1) *If* $A\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}$, $A\begin{pmatrix}1 & \\ 1 & 1\end{pmatrix}=\begin{pmatrix}1 & \\ 1 & 1\end{pmatrix}$,

then there exists a ring isomorphism σ: $R\to R_1$ *such that*

$$AX=X^\sigma,\ \forall X\in SL_2(R).$$

(2) *If* $A\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}=\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}$ *and* $\lambda_{S_t}(AX)=\lambda_{S_t}X$ *for all* $t\in T$ *and* $X\in S_{0,1}$, *then there exist an isomorphism* σ: $R\to R_1$ *and* $P=\begin{pmatrix}1 & a\\ & 1\end{pmatrix}\in SL_2(R_1)$, $\lambda_{S_t}P=I$, $\forall t\in T$ *such that*

$$A_p AX=X^\sigma,\ \forall X\in SL_2(R).$$

Proof As for the proof of (1), see [6]. We prove (2) as follows. Since $-I$ is in the center of $SL_2(R)$, there is $x\in R^*$ such that $A(-I)=xI$, $x^2=1$. But $-I\in S_{0,1}$. So $\lambda_{S_t}x=-1$ for all $t\in T$. Therefore $x=-1$ by $x^2-1=0$ and $2\in R_1^*$.

Let $A\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}x & y\\ z & u\end{pmatrix}$. We have $\begin{pmatrix}x & y\\ z & u\end{pmatrix}^2=-I$. So $\lambda_{S_t}z=1$ for all $t\in T$ i. e. $z\in R^*$. Therefore we can take $P=\begin{pmatrix}1 & -z^{-1}x\\ & z^{-1}\end{pmatrix}$ such that

$$A_p A\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix} & -1\\ 1 & \end{pmatrix}$$

by lemma 6. $\Lambda_p\Lambda\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}=\begin{pmatrix}1 & z\\ & 1\end{pmatrix}$ by calculation. Clearly

$$\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\right)^3=-I.$$

So we have

$$\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & Z\\ & 1\end{pmatrix}\right)^3=-I.$$

Hence $z=1$, i. e.

$$\Lambda_p\Lambda\left(\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\right)=\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}.$$

We also have

$$\Lambda_p\Lambda\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\quad\text{by}\quad\begin{pmatrix}1 & \\ 1 & 1\end{pmatrix}=\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}^{-1}\begin{pmatrix} & -1\\ 1 & \end{pmatrix}^{-1}.$$

Then (2) follows from (1) in this lemma.

Lemma 8. *Suppose $\Lambda T_i=T_i$ for some $i(1\leqslant i\leqslant\nu)$.*

(1) *The centralizer of T_i in $S_{p_n}(R)$ is*

$$C_{T_i}=\left\{\begin{array}{c}\begin{array}{cl} \begin{array}{c}(\nu+i)\\ \begin{pmatrix}A_1 & 0 & A_2 & B_1 & 0 & B_2\\ 0 & a & 0 & 0 & b & 0\\ A_3 & 0 & A_4 & B_3 & 0 & B_4\\ C_1 & 0 & C_2 & D_1 & 0 & D_2\\ 0 & c & 0 & 0 & d & 0\\ C_3 & 0 & C_4 & D_3 & 0 & D_4\end{pmatrix}\end{array} & \begin{array}{l}\\ \\ (i)\\ \\ \\ (\nu+i)\\ \\ \end{array}\end{array}\middle|\ \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(R),\\ \begin{pmatrix}A_1 & A_2 & B_1 & B_2\\ A_3 & A_4 & B_3 & B_4\\ C_1 & C_2 & D_1 & D_2\\ C_3 & C_4 & D_3 & D_4\end{pmatrix}\in S_{p_{n-2}}(R)\end{array}\right\}$$

(2) *Suppose $\lambda_{S_t}(\Lambda X)=\lambda_{S_t}X$, $\forall t\in T$, $X\in S_{0,1}$ and*

$$\mathfrak{a}_i=\left\{\begin{pmatrix}I & & & 0 & & \\ & a & & & b & \\ & & I & & & 0\\ 0 & & & I & & \\ & c & & & d & \\ & & 0 & & & I\end{pmatrix}\begin{array}{l}\\ (i)\\ \\ \\ (\nu+1)\\ \\ \end{array}\middle|\ \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(R)\right\}$$

$$\mathfrak{b}_i=\left\{\begin{pmatrix}I & & & 0 & & \\ & x & & & y & \\ & & I & & & 0\\ 0 & & & I & & \\ & z & & & u & \\ & & 0 & & & I\end{pmatrix}\begin{array}{l}\\ (i)\\ \\ \\ (\nu+1)\\ \\ \end{array}\middle|\ \begin{pmatrix}x & y\\ z & u\end{pmatrix}\in SL_2(R)\right\}.$$

Then

$$\Lambda \boldsymbol{a}_i=\boldsymbol{b}_i(1\leqslant i\leqslant\nu).$$

Proof Since 2 is a unit, by direct calculation we obtain (1). Then we prove (2). It is clear that $Q_{i,\nu+i}\in C_{T_i}$, $Q_{i,\nu+i}T_{i,\nu+i}(1)\in C_{T_i}$. so both $\Lambda Q_{i,\nu+i}$ and $\Lambda(Q_{i,\nu+i}T_{i,\nu+i}(1))$ are the elements of $C_{\Lambda T_i}=C_{T_i}$. Hence, by (1), we have

$$\Lambda Q_{i,\nu+1}=\begin{pmatrix} E_1 & 0 & E_2 & F_1 & 0 & F_2\\ 0 & e_1 & 0 & 0 & f_1 & 0\\ E_3 & 0 & E_4 & F_3 & 0 & F_4\\ G_1 & 0 & G_2 & H_1 & 0 & H_2\\ 0 & g_1 & 0 & 0 & h_1 & 0\\ G_3 & 0 & G_4 & H_3 & 0 & H_4\end{pmatrix},$$

$$\Lambda Q_{i,\nu+i}T_{i,\nu+i}(1)=\begin{pmatrix} E_5 & 0 & E_6 & F_5 & 0 & F_6\\ 0 & e_2 & 0 & 0 & f_2 & 0\\ E_7 & 0 & E_8 & F_7 & 0 & F_8\\ G_5 & 0 & G_6 & H_5 & 0 & H_6\\ 0 & g_2 & 0 & 0 & h_2 & 0\\ G_7 & 0 & G_8 & H_7 & 0 & H_8\end{pmatrix},$$

where $\lambda_{S_t}g_k=1$, $\lambda_{s_t}e_k=0$ for all $t\in T$, $k=1$, 2. Since $Q^2_{i,\nu+i}=T_i(Q_{i,\nu+1}\ T_{i,\nu+i}(1))^3=T_i$, we have

$$\begin{pmatrix} e_1 & g_1\\ f_1 & h_1\end{pmatrix}^2=-I,\quad \begin{pmatrix} e_2 & f_2\\ g_2 & h_2\end{pmatrix}^2=-I.$$

So, by lemma 6, there are matrices

$$P_k=\begin{pmatrix} I & & 0 & \\ & & & e_k\\ & & & 0\\ & & g_k & I\end{pmatrix},\quad k=1,\ 2,$$

where $P_k\in GSp_{2n}(R_1)$, $\lambda_{S_t}P_k=I$, $\forall t\in T$, such that

$$\Lambda Q_{i,\nu+i}=P_1\begin{pmatrix} E_1 & 0 & E_2 & g_1F_1 & 0 & g_1F_2\\ 0 & 0 & 0 & 0 & -1 & 0\\ E_3 & 0 & E_4 & g_1F_3 & 0 & g_1F_4\\ g_1^{-1}G_1 & 0 & g_1^{-1}G_2 & H_1 & 0 & H_2\\ 0 & 1 & 0 & 0 & 0 & 0\\ g_1^{-1}G_3 & 0 & g_1^{-1}G_4 & H_3 & 0 & H_4\end{pmatrix}P_1^{-1},$$

$$\Lambda Q_{i,\nu+1}T_{i,\nu+1}(1)=P_2\begin{pmatrix} E_5 & 0 & E_6 & g_2F_5 & 0 & g_2F_6\\ 0 & 0 & 0 & 0 & -1 & 0\\ E_7 & 0 & E_8 & g_2F_7 & 0 & g_2F_8\\ g_2^{-1}G_5 & 0 & g_2^{-1}G_6 & H_5 & 0 & H_6\\ 0 & 1 & 0 & 0 & 1 & 0\\ g_2^{-1}G_7 & 0 & g_2^{-1}G_8 & H_7 & 0 & H_8\end{pmatrix}P_2^{-1}.$$

Use C to denote the contralizer of $Q_{i,\nu+i}$ and $Q_{i,\nu+i}$, $T_{i,\nu+i}(1)$ in C_{T_i}. It is easy to see that

$$C(R)=\left\{\begin{pmatrix} A_1 & 0 & A_2 & B_1 & 0 & B_2 \\ 0 & a & 0 & 0 & 0 & 0 \\ A_3 & 0 & A_4 & B_3 & 0 & B_4 \\ C_1 & 0 & C_2 & D_1 & 0 & D_2 \\ 0 & 0 & 0 & 0 & a & 0 \\ C_3 & 0 & C_4 & D_3 & 0 & D_4 \end{pmatrix} \middle| \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{pmatrix} \in S_{p_{n-2}}(R),\ a^2=1\right\}.$$

Suppose

$$\pi_i(R)=\left\{\begin{pmatrix} A_1 & 0 & A_2 & B_1 & 0 & B_2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ A_3 & 0 & A_4 & B_3 & 0 & B_4 \\ C_1 & 0 & C_2 & D_1 & 0 & D_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ C_3 & 0 & C_4 & D_3 & 0 & D_4 \end{pmatrix} \middle| \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{pmatrix} \in, S_{p_{n-2}}(R)\right\}.$$

Then $\pi_i(R)\subseteq C(R)$. So $\Lambda\pi_i(R)\subseteq\Lambda C(R)$ and $\Lambda C(R)$ are the centralizer of $\Lambda Q_{i,\nu+i}$ and $\Lambda(Q_{i,\nu+i}T_{i,\nu+i}(1))$ in $C_{\Lambda T_i'}$.

Let

$$X=\begin{pmatrix} X_1 & 0 & X_2 & Y_1 & 0 & Y_2 \\ 0 & x & 0 & 0 & y & 0 \\ X_3 & 0 & X_4 & Y_3 & 0 & Y_4 \\ Z_1 & 0 & Z_2 & U_1 & 0 & U_2 \\ 0 & z & 0 & 0 & u & 0 \\ Z_3 & 0 & Z_4 & U_3 & 0 & U_4 \end{pmatrix}$$

be any element in $\Lambda C(R)$. Then $\begin{pmatrix} x & y \\ z & u \end{pmatrix}\in SL_2(R_1)$ which commute with the following matrices

$$C_1=\begin{pmatrix} 1 & e_1 \\ & g_1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & e_1 \\ & g_1 \end{pmatrix}^{-1},\quad C_2=\begin{pmatrix} 1 & e_2 \\ & g_2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & e_2 \\ & g_2 \end{pmatrix}^{-1}.$$

So we have

$$\begin{pmatrix} 1 & e_1 \\ & g_1 \end{pmatrix}^{-1}\begin{pmatrix} x & y \\ z & u \end{pmatrix}\begin{pmatrix} 1 & e_1 \\ & g_1 \end{pmatrix}=\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & e_2 \\ & g_2 \end{pmatrix}^{-1}\begin{pmatrix} x & y \\ z & u \end{pmatrix}\begin{pmatrix} 1 & e_2 \\ & g_2 \end{pmatrix}=\begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2-y_2 \end{pmatrix},$$

with $x_i, y_i\in R_1$. Hence $y_1=y_2=0$, $x_1=x_2$ by $e_k\in J(R_1)$. Therefore we have $x=u$, $y=z=0$, $\Lambda C(R)\subseteq C(R_1)$, and $\Lambda\pi_i(R)\subseteq C(R_1)$. It is clear that $\pi_i^2(R)\simeq S^2_{p_{n-2}}(R)$ by $\pi_i(R)\simeq S_{p_{n-2}}(R)$. So since $2\in R^*$ and $S_{p_n}(R)$ is generated by symplectic trasvection, we have $S^2_{p_{n-2}}(R)=S_{p_{n-2}}(R)$, $\pi_i^2(R)=\pi_i(R)$. Thus

$$\Lambda\pi_i(R)=\Lambda\pi_i^2(R)=(\Lambda\pi_i(R))^2\subseteq C^2(R_1)=\pi_i(R_1).$$

The condition $\lambda_{S_t}(\Lambda X)=\lambda_{S_t}X$, $\forall t\in T$, $X\in S_{0,1}$ impleis $\lambda_t(\Lambda^{-1}X)=\lambda_t X$, $\forall t\in T$ and $X\in S_{0,1}$. So we have $\Lambda^{-1}\pi_i(R_1)\subseteq\pi_i(R)$ by the same reason. Therefore

$$\Lambda\pi_i(R)=\pi_i(R_1).$$

Let C_{π_i} denote the centralizer of π_i in C_{T_i}. We have $\boldsymbol{a}_i\subseteq C_{\pi_i(R)}$. So

$$\Lambda\boldsymbol{a}_i\subseteq\Lambda C_{\pi_i(R)}=C_{\Lambda\pi_i(R)}=C_{\pi_i(R_1)}.$$

It is easy to see that

$$C_{\pi_i(R_1)}=\left\{\begin{pmatrix} xI & & & & & 0 & \\ & x_1 & & & & y_1 & \\ & & xI & & & & 0 \\ 0 & & & xI & & & \\ & z_1 & & & & u_1 & \\ & & 0 & & & & xI \end{pmatrix}\middle|\; x^2=1,\ \begin{pmatrix} x_1 & y_1\\ z_1 & u_1\end{pmatrix}\in SL_2(R_1)\right\}.$$

Recall that $\boldsymbol{a}_i\simeq SL_2(R)$, so $\boldsymbol{a}_i^2=\boldsymbol{a}_i$. Hence

$$\Lambda\boldsymbol{a}_i=\Lambda\boldsymbol{a}_i^2=(\Lambda\boldsymbol{a}_i)^2\subseteq C^2_{\pi_i(R_1)}\subseteq\boldsymbol{b}_i.$$

Similarly, we have $\Lambda^{-1}\boldsymbol{b}_i\subseteq\boldsymbol{a}_i$, that is, $\Lambda\boldsymbol{a}_i=\boldsymbol{b}_i$.

Lemma 9. *Suppose that* $\lambda_{S_t}(\Lambda X)=\lambda_{S_t}X$ *for all* $t\in T$ *and* $X\in S_{0,1}$, *then there exists* $P\in S_{p_n}(R_1)$ *such that*

$$\Lambda_p\Lambda T_i=T_i,\ i=1,\cdots,\nu.$$

Proof Clearly, $(\Lambda T_i)^2=I$ and $\lambda_{S_t}(\Lambda T_i)=\lambda_{S_t}T_i$, $i=1,\cdots,\nu$. By lemma 6, we have $Q\in S_{p_n}(R_1)$such that $\Lambda_Q\Lambda T_1=T_1$, $\lambda_{S_t}Q=I$, $\forall t\in T$. So we also have

$$\lambda_{S_t}(\Lambda_Q\Lambda X)=\lambda_{S_t}X$$

for all $t\in T$, $X\in S_{0,1}$. Now assume that there exists P_1 in $S_{p_n}(R_1)$ such that

$$\Lambda_{p_1}\Lambda T_i=T_i,\ i=1,\cdots,k-1,\ k\geqslant 2,$$

and $\lambda_{S_t}(\Lambda_{p_1}\Lambda X)=\lambda_{S_t}X$ for all $t\in T$, $X\in S_{0,1}$. Consider T_k. Since

$$T_kT_i=T_iT_k,\ i=1,\cdots,k-1,$$

by Lemma 8, we have

$$\Lambda_{p_1}\Lambda T_k=\begin{pmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_{k-1} & & & b_{k-1} \\ & & & A & & & B \\ c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_{k-1} & & & d_{k-1} \\ & & & C & & & D \end{pmatrix},$$

where $\lambda_{S_t}a_i=\lambda_{S_t}d_i=1$, $\lambda_{S_t}b_i=\lambda_{S_t}c_i=0$ for all $t\in T$ and $i=1,\cdots,k-1$. But $(\Lambda_{p_1}\Lambda T_k)^2=I$, so we have $a_i=d_i=1$, $b_i=c_i$ by lemma 6.

* For any group G, we mean that $G^2=\langle gh\,|\,g,h\in G\rangle$.

Hence

$$\Lambda_{p_1}\Lambda T_k=\begin{pmatrix} I^{(k-1)} & & O^{(k-1)} & \\ & A & & B \\ O^{(k-1)} & & I^{(k-1)} & \\ & C & & D \end{pmatrix},$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in S_{p_{n-2k+2}}(R_1),\quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}^2=I$$

and

$$\lambda_{S_t}\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} -1 & & & \\ & I & & \\ & & -1 & \\ & & & I \end{pmatrix}$$

for all $t\in T$. Therefore, applying Lemma 6 again, we have

$$Q_1=\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}\in S_{p_{n-2k+2}}(R_1)$$

such that

$$Q_1\begin{pmatrix} A & B \\ C & D \end{pmatrix}Q_1^{-1}=\begin{pmatrix} -1 & & & \\ & I & & \\ & & -1 & \\ & & & I \end{pmatrix}.$$

Put

$$P_2=\begin{pmatrix} I & & 0 & \\ & Q_{11} & & Q_{12} \\ 0 & & I & \\ & Q_{21} & & Q_{22} \end{pmatrix}.$$

Then $P_2\in S_{p_n}(R_1)$ and $\Lambda_{p_2}\Lambda_{p_1}\Lambda T_i=T_i$, $i=1, \cdots, k$. Our lemma therefore follows from the above discussion.

Lemma 10. *Suppose $\Lambda T_i=T_i$, $i=1, \cdots, \nu$, $\lambda_{S_t}(\Lambda X)=\lambda_{S_t}X$ for all $t\in T$ and $X\in S_{0,1}$. Then there exists $P\in GS_{p_n}(R_1)$ such that*

$$\Lambda_p\Lambda Y=Y,$$

where Y may be any one of the following symplectic matrices

$$T_i,\ T_{i,\nu+i}(1),\ T_{\nu+i,i}(1),\ J_{1j},\ J_{j1},\ i=1, \cdots, \nu;\ j=2, \cdots, \nu.$$

Proof It is clear that $J_{12}T_i=T_iJ_{12}$ for $i>2$ and $T_{1,\nu+1}(1)\in\boldsymbol{a}_1$. So by Lemma 8, we have

$$\Lambda J_{12}=\begin{pmatrix} x_1 & x_{12} & & & y_1 & y_{12} & & \\ x_{21} & x_2 & & & y_{21} & y_2 & & \\ & & \ddots & & & & \ddots & \\ & & & x_\nu & & & & y_\nu \\ z_1 & z_{12} & & & u_1 & u_{12} & & \\ z_{21} & z_2 & & & u_{21} & u_2 & & \\ & & \ddots & & & & \ddots & \\ & & & z_\nu & & & & u_\nu \end{pmatrix},\quad \Lambda T_{1,\nu+1}(1)=\begin{pmatrix} a & & b & \\ & I & & 0 \\ c & & d & \\ & 0 & & I \end{pmatrix}.$$

By $(\Lambda J_{12})^2=I$ and (1) in Lemma 6, we have $x_i=u_i=1$, $z_i=y_i=0$ for

$$i>2,\ y_1=y_2=z_1=z_2=0$$

and

$$\begin{pmatrix} u_1 & u_{12} \\ u_{21} & u_2 \end{pmatrix}' = \begin{pmatrix} x_1 & x_{12} \\ x_{21} & x_2 \end{pmatrix}, \quad y_{21}=-z_{12}, \quad y_{21}=-z_{12},$$

that is

$$AJ_{12}=\begin{pmatrix} x_1 & x_{12} & & 0 & y_{12} & \\ x_{21} & x_2 & & -y_{21} & 0 & \\ & & I & & & 0 \\ 0 & z_{12} & & x_1 & x_{21} & \\ -z_{21} & 0 & & x_{12} & x_2 & \\ & & 0 & & & I \end{pmatrix}.$$

By using $J_{12}\in S_{0,1}$, it is easy to see that x_1+1, $x_{12}+1$, x_2, y_{12}, z_{12} are all in $J(R_1)$. In particular x_{12}, $x_1\in R_1^*$.

Obviously, $T_{1,\nu+1}(1)J_{12}=J_{12}T_{1,\nu+1}(1)$. So

$$AT_{1,\nu+1}(1)AJ_{12}=AJ_{12}AT_{1,\nu+1}(1), \tag{1}$$

therefore, we have

$$\begin{cases} a=1-z_{12}x_{12}^{-1}b, \\ c=-z_{12}^2x_{12}^{-2}b, \\ d=1+z_{12}x_{12}^{-1}b, \end{cases} \tag{2}$$

where $\lambda_{S_t}b=1$, $\forall t\in T$. Hence $b\in R_1^*$. Put

$$P_1=\begin{pmatrix} I & \\ & bI \end{pmatrix}\begin{pmatrix} I & & \\ -z_{12}x_{12}^{-1} & & \\ & 0 & I \end{pmatrix}=\begin{pmatrix} I & & \\ -bz_{12}x_{12}^{-1} & & \\ & 0\ b & I \end{pmatrix}\in GS_{p_n}(R_1).$$

Clearly $\lambda_{S_t}P_1=I$, $\forall t\in T$. So, using the equalities (2), we have

$$A_{p_1}AT_{1,\nu+1}(1)=T_{1,\nu+1}(1),\ A_{p_1}AT_i=T_i,\ i=1,\ \cdots,\ \nu.$$

Furthermore, we have

$$A_{p_1}AJ_{12}=\begin{pmatrix} -1 & x_{12} & & 0 & b^{-1}y_{12} & \\ & 1 & & -b^{-1}y_{12} & 0 & \\ & & I & & & 0 \\ & & & -1 & & \\ & & & x_{12} & 1 & \\ & & & & & I \end{pmatrix}$$

by equality (1). Let

$$P_2=T_{2,\nu+2}(b^{-1}y_{12}x_{12}^{-1})\begin{pmatrix} 1 & & & & & \\ & -x_{12} & & & & \\ & & I & & & \\ & & & 1 & & \\ & & & & -x_{12}^{-1} & \\ & & & & & I \end{pmatrix}.$$

Then $P_2\in S_{p_n}(R_1)$, $\lambda_{S_t}P_2=I$ for all $t\in T$. It is easy to see that

$$A_{p_2}A_{p_1}AT_i=T_i,\ i=1,\ \cdots,\ \nu,$$

$$\Lambda_{p_2}\Lambda_{p_1}\Lambda T_{1,\nu+1}(1)=T_{1,\nu+1}(1),$$
$$\Lambda_{p_2}\Lambda_{p_1}\Lambda J_{12}=J_{12},$$

and $\lambda_{S_t}(\Lambda_{p_2}\Lambda_{p_1}\Lambda X)=\lambda_{S_t}X$ for all $t\in T$ and $X\in S_{0,1}$. By lemma 8, $\Lambda_{p_2p_1}\Lambda\boldsymbol{a}_1=\boldsymbol{b}_1$. So there is a $P_3=T_{1,\nu+1}(x)$, $\lambda_{S_t}x=0$ for all $t\in T$ such that

$$\Lambda_{p_3}\Lambda_{p_2}\Lambda_{p_1}\Lambda T_{1,\nu+1}(1)=T_{1,\nu+1}(1),\quad \Lambda_{p_3}\Lambda_{p_2}\Lambda_{p_1}\Lambda T_{\nu+1,1}(1)=T_{\nu+1,1}(1)$$

and $\Lambda_{p_3}\Lambda_{p_2}\Lambda_{p_1}\Lambda$ still preserves J_{12}, $T_i(i=1,\cdots,\nu)$.

Using the equalities $J_{21}T_i=T_iJ_{21}$, $i>2$, $J_{21}T_{\nu+1,1}(1)=T_{\nu+1,1}(1)J_{21}$, $J_{21}^2=I$, $\lambda_{S_t}\Lambda J_{21}=\lambda_{S_t}J_{21}$, $\forall t\in T$ and $T_{\nu+1,1}(1)\in\boldsymbol{a}_i$, we know that there is matrix

$$P_4=\begin{pmatrix} I & -y_{12}x_{21}^{-1} & \\ & & 0 \\ & & c^{-1}I\end{pmatrix},\ \lambda_{S_t}c=1,\ \lambda_{S_t}x_{21}=-1,\ \lambda_{S_t}y_{12}=0,\ \forall t\in T$$

such that

$$\Lambda_{p_4p_3p_2p_1}\Lambda J_{21}=\begin{pmatrix} -1 & & & & & \\ y & 1 & & & & \\ & & I & & & \\ & z & & -1 & y & \\ -z & & \cdot & & 1 & \\ & & 0 & & & I\end{pmatrix},$$

where $\lambda_{S_t}y=-1$, $\lambda_{S_t}z=0$ for all $t\in T$ Let $P_5=T_{\nu+2,2}(y^{-1}z)$. we have

$$\Lambda_{p_5p_4p_3p_2p_1}\Lambda J_{21}=\begin{pmatrix} -1 & & & & & \\ y & 1 & & & & \\ & & I & & & \\ & & & -1 & y & \\ & & & & 1 & \\ & & & & & I\end{pmatrix}$$

and $\Lambda_{p_5p_4p_3p_2p_1}\Lambda$ still preserves T_i, J_{12}, $T_{1,\nu+1}(1)$, $T_{\nu+1,1}(1)$, $i=1,\cdots,\nu$, and $\lambda_{S_t}\Lambda_{p_5p_4p_3p_2p_1}\Lambda X=\lambda_{S_t}X$ for all $t\in T$, $X\in S_{0,1}$.

But $(T_1J_{12}T_1J_{21})^3=T_1T_2$, so $(T_1J_{12}T_1\Lambda_{p_5p_4p_3p_2p_1}\Lambda J_{21})^3=T_1T_2$, from which we know that $y=-1$, i. e.

$$\Lambda_{p_6}\Lambda J_{21}=J_{21},\ p_6=p_5p_4p_3p_2p_1.$$

Applying

$$F_{12}=J_{12}T_1J_{21}T_1J_{12}T_1,\ T_{2,\nu+2}(1)=F_{12}T_{1,\nu+1}(1)F_{12}^{-1},$$
$$T_{\nu+2,2}(1)=F_{12}^{-1}T_{\nu+1,1}(1)F_{12},$$

we have

$$\Lambda_{p_6}\Lambda T_{2,\nu+2}(1)=T_{2,\nu+2}(1),\ \Lambda_{p_6}\Lambda T_{\nu+2,2}(1)=T_{\nu+2,2}(1).$$

For any integers $k(\geqslant 3)$, we assume that there is

$$Q\in S_{p_n}(R_1),\ \lambda_{S_t}Q=I,\ \forall t\in T$$

such that $\Lambda_Q\Lambda Y=Y$, where Y may be any one of T_1, J_{1i}, J_{j1}, $T_{1,\nu+1}(1)$, $T_{\nu+1,1}(1)$, $T_{j,\nu+j}(1)$, $T_{\nu+j,j}(1)$, $j=2,\cdots,k-1$.

Using the argument as above, we can see that $\Lambda_Q\Lambda$ preserves J_{1k}.

Choose another matrix $Q_1\in GS_{p_n}(R_1)$, $\lambda_{S_t}Q_1=I$ for all $t\in T$. Then Λ_{Q_1Q} preserves $J_{k1}T_{k,\nu+k}(1)$ and $T_{\nu+k,k}(1)$ besides the above matrices.

Our lemma therefore follows from the above discussion.

§ 2. The proof of our main theorem

Recall that the isomorphism Λ of $S_{p_n}(R)$ onto $S_{p_{n_1}}(R_1)$ induces a set of isomorphisms

$$\Lambda_{S_t}: S_{p_n}(R/M_t) \to S_{p_{n_1}}(R_1/N_{S_t}),\ \forall t \in T$$

with $\Lambda_{S_t}(\lambda_t X) = \lambda_{S_t}(\Lambda X)$ for all X in $S_{p_n}(R)$, therefore, according to the result on symplectic groups over fields, $n = n_1$ and for each $t \in T$ there is $P_{S_t} \in GS_{p_n}(R_1)$ such that

$$\Lambda_{S_t}(\lambda_t X) = P_{S_t}(\lambda_t X)^{\sigma_{S_t}} P_{S_t}^{-1},$$

that is,

$$\lambda_{S_t}(\Lambda X) = P_{S_t}(\lambda_t X)^{\sigma_{S_t}} P_{S_t}^{-1},\ \forall t \in T,\ X \in S_{p_n}(R),$$

where σ_{S_t} is the isomorphism of R/M_t onto R_1/N_{S_t}.

Since R_1 is a Φ–surjective ring, there exists $P_1 \in S_{p_n}(R_1)$ such that $\lambda_{S_t} P_1 = P_{S_t}$ for all $t \in T$. Hence

$$\lambda_{S_t}(\Lambda_{P_1^{-1}} \Lambda X) = (\lambda_t X)^{\sigma_{S_t}},\ \forall t \in T.$$

If $X \in S_{0,1}$, it is clear that $(\lambda_t X)^{\sigma_{S_t}} = \lambda_{S_t} X$. So we have

$$\lambda_{S_t}(\Lambda_{P_1^{-1}} \Lambda X) = \lambda_{S_t} X,\ \forall t \in T \text{ and } X \in S_{0,1}.$$

Now $\Lambda_{P_1^{-1}}\Lambda$ is also an isomorphism of $S_{p_n}(R)$ onto $S_{p_n}(R_1)$. By lemma 9 and lemma 10, there is $P_2 \in GS_{p_n}(R)$ such that $\Lambda_{P^{-1}}\Lambda Y = Y$, $P = P_1 P_2$, where Y may be any one of the following matrices

$$T_i,\ T_{i,\nu+i}(1),\ T_{\nu+i,i}(1),\ J_{1j},\ J_{j1},\ i=1, \cdots, \nu,\ j=2, \cdots, \nu.$$

By Lemma 8, we have $\Lambda_{P^{-1}}\Lambda \boldsymbol{a}_i = \boldsymbol{b}_i$. But $\boldsymbol{a}_i \simeq SL_2(R)$, $\boldsymbol{b}_i \simeq SL_2(R_1)$. So, there is an isomorphism σ of R onto R_1 such that $\Lambda_{P^{-1}}\Lambda X = X^\sigma$, $\forall X \in \boldsymbol{a}_i$.

Using lemma, and the fact that $\begin{pmatrix} -I & \\ & I \end{pmatrix}$ can be expressed by

$$T_{i,\nu+i}(1),\ T_{\nu+i,i}(1),\ i=1, \cdots, \nu,$$

we can show

$$\Lambda_{P^{-1}}\Lambda X = X^\sigma,\ \forall X \in S_{p_n}(R),$$

i. e.

$$\Lambda X = P X^\sigma P^{-1},\ \forall X \in S_{p_n}(R),$$

where $P \in GS_{p_n}(R_1)$.

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