Chin. Ann. of Math.
6B (2) 1985

A NEW PROOF FOR THE CONVEXITY OF THE BERNSTEIN-BÉZIER SURFACES OVER TRIANGLES

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Abstract

A necessary and sufficient condition for the convexity of the Bézier net is presented 'The convex Bézier net implies the convexity of the corresponding Bernstein-Bézier surface over triangles', a theorem established by Chang Geng-zhe and Philip J. Davis, is reproved by a new approach.

§ 1. Definitions and Notations

Let T be a triangle arbitrarily given and let P be an arbitrary point in the plane on which triangle T lies. It is well-known that P can be expressed uniquely by the barycentric coordinates (u, v, w) with respect to T and that $u+v+w=1$ is satisfied. We write $P=(u, v, w)$ to identify the point and its barycentric coordinates. If $P\in T$, then we have further restrictions $0\leqslant u, v, w\leqslant 1$. For some fixed yet arbitrary positive integer n, set $P_{i,j,k}=(i/n, j/n, k/n)$ where i, j, k are nonnegative integers such that $i+j+k=n$. Assigning a real number $f_{i,j,k}$ to the point $P_{i,j,k}$, a point $F_{i,j,k}=(P_{i,j,k}, f_{i,j,k})$ is obtained. There are altogether $(n+1)(n+2)/2$ such points in the space. If line segments between any two of the following three

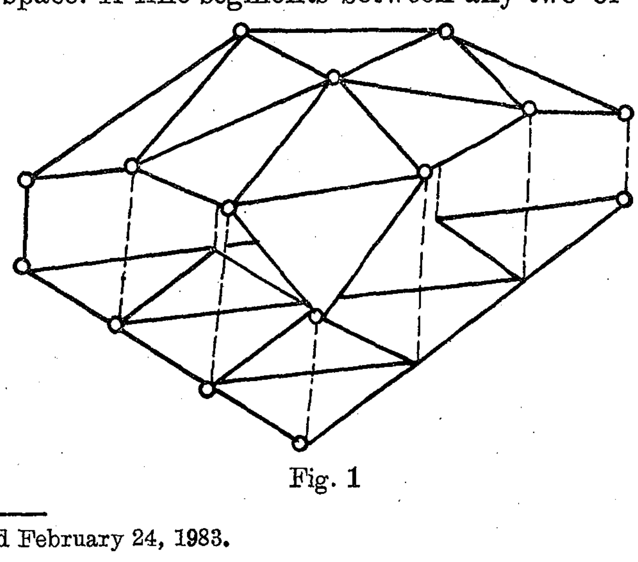

Fig. 1

Manuscript received February 24, 1983.

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points $F_{i+1,\ j,\ k}$, $F_{i,\ j+1,k}$, $F_{i,\ j,\ k+1}$ $(i+j+k=n-1)$ are connected, a piecewise linear function $\hat{f}_n$, which is called an nth Bézier net over T, is obtained. The projection of $\hat{f}_n$ onto triangle T produces a subdivision of T denoted by $S_n(T)$. Points $P_{i,\ j,\ k}$ are said to be the nodes of $S_n(T)$. $S_3(T)$ and Bézier net are illustrated in Fig. 1. $\{f_{i,\ j\ k}: i+j+k=n\}$ is called the set of Bézier ordinates of $\hat{f}_n$.

To each Bézier net f_n over T, there is a polynomial

$$B^n(f;P)=\sum_{i+j+k=n} f_{i,\ j,\ k}\, J^n_{i,\ j,\ k}(P) \tag{1}$$

associated with it, where

$$J^n_{i,\ j,\ k}(P)=\frac{n!}{i!j!k!}\, u^i v^j w^k, \tag{2}$$

$i+j+k=n$, are called the Bernstein basis polynomials. $B^n(f;\ P)$ is called the Bernstein-Bézier polynomial (or the B-B polynomial for brevity) over triangle T. The B-B surfaces have been studied extensively by Farin[1-4], Barnhill and Farin[5], and Goldman[6]. Stimulated by interests both in theory and application, Geng-zhe Chang and Philip J. Davis investigated the convexity of these surfaces in [7]. They call the Bézier net $\hat{f}_n$ convex in u-direction if inequalities

$$f_{i+1,\ j,\ k}+f_{i-1,j+1,\ k+1}\geqslant f_{i,\ j+1,\ k}+f_{i,\ j,\ k+1} \tag{3}$$

hold for all $i,\ j,\ k$ such that $i\geqslant 1$ and $i+j+k=n-1$. Similar definitions may be applied to the v-direction and the w-direction. If $\hat{f}_n$ is convex in u-direction, v-direction and w-direction, then it is said to be convex in 3-direction. They have shown that a sufficient condition for convexity of $B^n(f;P)$ is that $\hat{f}_n$ is convex in 3-direction. Since the convexity of $\hat{f}_n$ over T implies that of $\hat{f}_n$ in 3-direction, they conclude

Theoem 1. *The convexity of $\hat{f}_n$ implies that of $B^n(f;P)$ over triangle T.*

In this paper, we first establish the equivalence of the convexity of $\hat{f}_n$ and the convexity of $\hat{f}_n$ in 3-direction. Then based on a theorem of Farin, Theorem 1 is reproved by a new approach.

§ 2. Characterization for Convexity of f_n

The following lemma will be useful in the sequel.

Lemma. *Let $y=f(x)$ be a continuous curve which can be divided into two convex pieces having not only a point but a segment in common. Then $y=f(x)$ itself is a convex curve in its domain.*

Proof Assume that the common segment shared by the two convex pieces is represented by $y=f(x)$ restricted on interval $(a,\ b)$ where $a<b$. We have to prove that the inequality

$$f((x+y)/2)\leqslant(f(x)+f(y))/2 \tag{4}$$

holds for any $x<y$ in the domain of f.

Let $m=(x+y)/2$, i. e., m is the midpoint of the interval (x, y). We first consider the case: $m\in(a, b)$. There exist x_1 and y_1 close enough to m such that $m=(x_1+y_1)/2$ and that

$$x<x_1<m<y_1<y,$$
$$a<x_1<y_1<b.$$

From the first inequality we see that there exist two positive numbers α and β with $\alpha+\beta=1$ such that

$$x_1=\alpha x+\beta m,$$
$$y_1=\beta m+\alpha y.$$

Since $f(x)$ is convex for $x>a$ and for $x<b$, we have

$$2f(m)=2f((x_1+y_1)/2)\leqslant f(x_1)+f(y_1)=f(\alpha x+\beta m)+f(\beta m+\alpha y)$$
$$\leqslant \alpha f(x)+\beta f(m)+\beta f(m)+\alpha f(y).$$

It follows that

$$2\alpha f(m)\leqslant\alpha(f(x)+f(y))$$

and then (4) is obtained. Now we turn to the other case: $m\bar{\in}(a, b)$. Without loss of generality, we can assume $b\leqslant m<y$. If at least one of x and y is in (a, b), then (4) is valid by the hypothesis. Hence it suffices to consider the case in which $x\leqslant a<b\leqslant y$. Fix arbitrarily a point z in (a, b) and then find a point x' such that

$$z=(x+x')/2.$$

It is obvious that $z<x'<y$. Hence there exists a real number λ satisfying $0<\lambda<1$ such that $x'=\lambda z+(1-\lambda)y$. Then we get

$$m=(2-\lambda)z/2+\lambda y/2.$$

Since the midpoint z of interval (x, x') is in (a, b), by the fact we have just shown before

$$2f(z)=2f((x+x')/2)\leqslant f(x)+f(x')$$
$$\leqslant f(x)+\lambda f(z)+(1-\lambda)f(y).$$

It follows that

$$f(z)\leqslant f(x)/(2-\lambda)+f(y)(1-\lambda)/(2-\lambda). \tag{5}$$

Since $a<z<m<y$, we have

$$f(m)\leqslant f(z)(2-\lambda)/2+f(y)\lambda/2\leqslant(f(x)+f(y))/2,$$

(5) has been used in the last step. This completes the proof of the lemma.

The main theorem of this section is stated as follows.

Theorem 2. *A necessary and sufficient condition for Bézier net being convex is that it is convex in 3-direction.*

Proof The proof of the necessity of the condition is easier and could be found in [7]. It remains to prove the sufficiency of the condition. Proceed induction on degree of Bézier net. For $n=2$, convexity in 3-direction says that

$$\left.\begin{array}{l} f_{2,0,0}+f_{0,1,1}\geqslant f_{1,0,1}+f_{1,1,0},\\ f_{0,2,0}+f_{1,0,1}\geqslant f_{1,1,0}+f_{0,1,1},\\ f_{0,0,2}+f_{1,1,0}\geqslant f_{0,1,1}+f_{1,0,1}. \end{array}\right\} \tag{6}$$

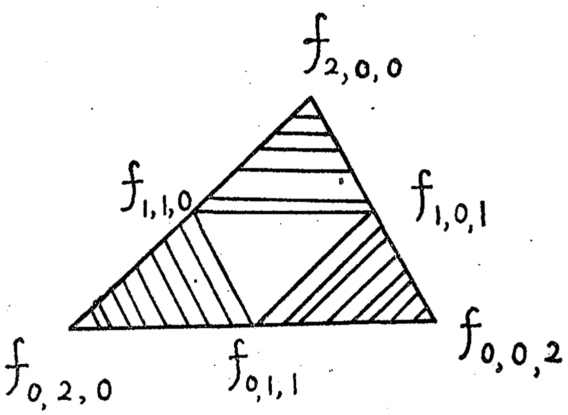

Fig. 2

It is intuitively easy to see that each inequality of (6) ensures the convexity of $\hat{f}_2$ restricted to the corresponding parallelogram in Fig. 2. (This can be made rigorous of course.) Now suppose P and Q are two arbitrary points in T. There exists at least one shaded triangle shown in Fig. 2, which does contain neither P nor Q. This means that line segment $\overline{PQ}$ is contained completely by the union of two parallelograms. The Bézier net $\hat{f}_2$ restricted to $\overline{PQ}$ is denoted by $\overline{P'Q'}$. If $\overline{PQ}$ is not a part of the boundary of T, then $\overline{P'Q'}$ itself is either a convex curve or a union of two convex curves having a line segment in common, thus $\widehat{P'Q'}$ is a convex curve by the lemma.

It follows immediately from (6) that

$$\left.\begin{array}{l} f_{2,0,0}+f_{0,2,0}\geqslant 2f_{1,1,0},\\ f_{0,2,0}+f_{0,0,2}\geqslant 2f_{0,1,1},\\ f_{0,0,2}+f_{2,0,0}\geqslant 2f_{1,0,1}. \end{array}\right\} \tag{7}$$

This means that each of the boundary curves of $\hat{f}_2$ is convex. Hence we have shown $\hat{f}_2$ is convex over T.

Assume that each $(n-1)$th Bézier net which is convex in 3-direction is convex. Now we consider a nth Bézier net which is convex in 3-direction. If any side of T and all subtriangles of $S_n(T)$ having at least one point in common with the side are deleted, then the$(n-1)$th Bézier net left, which is still convex in 3-direction, is convex by the induction hypothesis. This means that $\hat{f}_n$ is the union of three convex surfaces. Applying the same argument used in the case of $n=2$, we can prove that $\hat{f}_n$ is convex over T.

§ 3. Proof of Theorem 1

A simple calculation shows

$$J^n_{i,j,k}=[(i+1)J^{n+1}_{i+1,j,k}+(j+1)J^{n+1}_{i,j+1,k}+(k+1)J^{n+1}_{i,j,k+1}]/(n+1)$$

where $i+j+k=n$. This formula enables us to express the B-B polynomial of degree n in terms of the Bernstein basis polynomials of degree $n+1$ as the following

$$B^n(f; P) = \sum_{i+j+k=n+1} f^*_{i,j,k} J^{n+1}_{i,j,k}(P), \tag{8}$$

where

$$f^*_{i,j,k} = (if_{i-1,j,k} + jf_{i,j-1,k} + kf_{i,j,k-1})/(n+1). \tag{9}$$

If $f^*_{i,j,k}$ defined in (9) is assigned to the node $(i/(n+1),\ j/(n+1),\ k/(n+1))$ of $S_{n+1}(T)$, then a $(n+1)$th Bézier net $E\hat{f}_n$ is obtained. (8) says that Bézier nets $\hat{f}_n$ and $E\hat{f}_n$ define the same B-B surface over triangle T. This procedure is called degree elevation by Farin in [1, 2]

We have the following

Theorem 3. *If $\hat{f}_n$ is convex over T, then so is $E\hat{f}_n$.*

Proof Since $\hat{f}_n$ is convex over T, by Theorem 2 it is convex in 3-direction. From (9) we have

$$\begin{aligned}
f^*_{i+1,j,k} &= [(i+1)f_{i,j,k} + jf_{i+1,j-1,k} + kf_{i+1,j,k-1}]/(n+1),\\
f^*_{i-1,j+1,k+1} &= [(i-1)f_{i-2,j+1,k+1} + (j+1)f_{i-1,j,k+1} + (k+1)f_{i-1,j+1,k}]/(n+1),\\
f^*_{i,j+1,k} &= [if_{i-1,j+1,k} + (j+1)f_{i,j,k} + kf_{i,j+1,k-1}]/(n+1),\\
f^*_{i,j,k+1} &= [if_{i-1,j,k+1} + jf_{i,j-1,k+1} + (k+1)f_{i,j,k}]/(n+1),
\end{aligned}$$

thus

$$\begin{aligned}
&(n+1)(f^*_{i+1,j,k} + f^*_{i-1,j+1,k+1} - f^*_{i,j+1,k} - f^*_{i,j,k+1})\\
&\quad = (i-1)(f_{i,j,k} + f_{i-2,j+1,k+1} - f_{i-1,j+1,k} - f_{i-1,j,k+1})\\
&\qquad + j(f_{i+1,j-1,k} + f_{i-1,j,k+1} - f_{i,j,k} - f_{i,j-1,k+1})\\
&\qquad + k(f_{i+1,j,k-1} + f_{i-1,j+1,k} - f_{i,j+1,k-1} - f_{i,j,k}) \geqslant 0
\end{aligned}$$

by the fact of $\hat{f}_n$ being convex in u-direction. The result we have just shown is that $E\hat{f}_n$ is convex in u-direction. Its convexity in 3-direction can be proved by symmetry. Hence $E\hat{f}_n$ is convex over T by Theorem 2.

If the process of degree elevation is continued, the sequence of Bézier nets

$$E\hat{f}_n,\ E^2\hat{f}_n,\ E^3\hat{f}_n \cdots\cdots$$

is obtained. Each term in the sequence is convex over T if $\hat{f}_n$ is so. In 1979, Farin proved that

$$\lim_{m\to\infty} Em\hat{f}(P) = B^n(f; P),$$

Theorem 1 follows immediately from this fact.

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