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COMPLETE HYPERSURFACES WITH CONSTANT SCALAR CURVATURE AND CONSTANT MEAN CURVATURE IN R^4

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Abstract

Let M be a 3-dimersional complete and connected hypersurface immersed in R^4. If the scalar curvature R and the mean curvature $|H|$ of M are constants, where $|H|\neq 0$, $R\geqslant 0$, then there are only three cases: $R=6|H|^2$, $\frac{9}{2}$ $|H|^2$ and 0. Moreoven we can find some hypersurfaces appropriate to these cases.

Introduction

In 1974, Mosafumi Okumura proved the following theorem: Let M be an n-dimensional complete and connected hypersurface in an $(n+1)$-dimensional Riemannian manifold N with non-negative constant curvature C. Let S be the square norm of second fundamental tensor $\widetilde{H}$ of M. If both S and trace $\widetilde{H}$ are constants, where trace $\widetilde{H}>0$, and in addition, $S<2C+\frac{1}{n-1}$ (trace $\widetilde{H})^2$, then M is a sphere (see [1]).

Suppose that $|H|=\frac{1}{n}$ trace $\widetilde{H}$, and R is the scalar curvature of M. At first, we have $R+S=n^2|H|^2$. In particular when $n=3$, $N=R^4$, the above theorem indicates that if $S<\frac{9}{2}|H|^2$, we must have $M=S^3$. At that time, $S=3|H|^2$.

Assume $R\geqslant 0$. It is obvious that $S\leqslant 9|H|^2$. In this paper, we obtain the following result: Let M be a 3-dimensional complete and connected hypersurface in R^4. If $|H|$ and R are constants, and $|H|\neq 0$, $R\geqslant 0$, then there are only three cases: $R=6|H|^2$, $\frac{9}{2}|H|^2$ and 0, when $R=6|H|^2$, $M=S^3\left(\frac{1}{|H|}\right)$; when $R=\frac{9}{2}$ $|H|^2$, $M=S^2\left(\frac{2}{3|H|}\right)\times R^1$; When $R=0$, for example, $M=S^1\left(\frac{1}{3|H|}\right)\times R^2$.

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§ 1. Preliminaries

Let M be an n-dimensional hypersurface immersed in $(n+1)$-dimensional Euclidean space R^{n+1}. We choose a local vector field of orthonormal frames $e_1, \cdots, e_{n+1}$ in R^{n+1} such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. And consequently, the remaining vector e_{n+1} is normal to M. We shall agree with the following convention on the ranges of indices:

$$1\leqslant A, B, C, \cdots\leqslant n+1,\ 1\leqslant i, j, k, l, s, t, \cdots\leqslant n.$$

Let $\omega^1, \cdots, \omega^{n+1}$ be a local field of dual frames in R^{n+1}. It's well known that the structure equation and fundamental formulas are given by

$$d\omega^A=\sum_B \omega^B\wedge\omega_B^A,\ \omega_B^A+\omega_A^B=0,$$
$$d\omega_B^A=\sum_C \omega_B^C\wedge\omega_C^A. \tag{1.1}$$

We restrict these forms to M. Then we can see

$$\omega^{n+1}=0,\ \omega_j^{n+1}=\sum_i h_{ji}\omega^i,\ h_{ji}=h_{ij},$$
$$d\omega^i=\sum_j \omega^j\wedge\omega_j^i,\ \omega_j^i+\omega_i^j=0,$$
$$d\omega_j^i=\sum_l \omega_j^l\wedge\omega_l^i+\frac{1}{2}\sum_{l,s} R_{jls}^i\omega^l\wedge\omega^s,$$

where

$$R_{jls}^i=h_{js}h_{il}-h_{jl}h_{is}. \tag{1.2}$$

We call h_{ij} the components of the second fundamental form. Let

$$S=\sum_{i,j}(h_{ij})^2.$$

Mean curvature vector $H=\frac{1}{n}\cdot\sum_i h_{ii}e_{n+1}$. When $|H|\neq0$, we can choose e_{n+1} as the unit vector of H, and make use of $|\ \ |$ to express the length of a vector. We can see $\sum_i h_{ii}=n|H|$.

From [6] or [7], we know

$$\sum_k h_{ijk}\omega^k=dh_{ij}-\sum_l h_{il}\omega_j^l-\sum_l h_{lj}\omega_i^l,\ h_{ijk}=h_{ikj},$$
$$\sum_l h_{ijkl}\omega^l=dh_{ijk}-\sum_l h_{ljk}\omega_i^l-\sum_l h_{ilk}\omega_j^l-\sum_l h_{ijl}\omega_k^l$$
$$h_{ijkl}-h_{ijlk}=\sum_s h_{is}R_{jkl}^s+\sum_s h_{sj}R_{ikl}^s,$$
$$\sum_s h_{ijkls}\omega^s=dh_{ijkl}-\sum_s h_{sjkl}\omega_i^s-\sum_s h_{iskl}\omega_j^s-\sum_s h_{ijsl}\omega_k^s-\sum_s h_{ijks}\omega_l^s,$$
$$h_{ijkls}-h_{ijksl}=\sum_t h_{tjk}R_{ils}^t+\sum_t h_{itk}R_{jls}^t+\sum_t h_{ijt}R_{kls}^t,$$
$$\Delta h_{ij}=\sum_k h_{ijkk},\ \Delta h_{ijk}=\sum_l h_{ijkll}. \tag{1.3}$$

The Ricci curvature R_{ij} of M is determined by $R_{ij}=\sum_l R_{ilj}^l$. The scalar curvature

$R=\sum_i R_{ii}$. From (1.2), we know

$$R_{ij}=n|H|h_{ij}-\sum_l h_{il}h_{lj},\ R=n^2|H|^2-S. \tag{1.4}$$

In this paper, we require that $|H|$ and R are constants ($|H|\neq 0$). We make use of $\widetilde{H}$ to express matrix (h_{ij}). $\sum$ indicates the sum of all the same indices. By a long calculation and by virtue of (1.2)—(1.4), we obtain

$$\Delta h_{ij}=n|H|\sum_l h_{il}h_{lj}-Sh_{ij}, \tag{1.5}$$

$$\begin{aligned}\sum h_{ijk}\Delta h_{ijk}=&6\sum h_{ijk}h_{lsk}h_{il}h_{sj}-3\sum h_{ijk}h_{sik}h_{ls}h_{jl}-6\sum h_{ijk}h_{lsk}h_{ij}h_{ls}\\&+3n|H|\sum h_{ijk}h_{sik}h_{sj}-S\sum h_{ijk}^2,\end{aligned} \tag{1.6}$$

$$\begin{aligned}\Delta(\text{trace }\widetilde{H}^3)&=3\sum\Delta h_{ij}h_{jl}h_{li}+6\sum h_{ijk}h_{jlk}h_{li}\\&=3n|H|\ \text{trace }\widetilde{H}^4-3S\ \text{trace }\widetilde{H}^3+6\sum h_{ijk}h_{jlk}h_{li}.\end{aligned} \tag{1.7}$$

$$\begin{aligned}\Delta(\text{trace }\widetilde{H}^4)=&4n|H|\ \text{trace }\widetilde{H}^5-4S\ \text{trace }\widetilde{H}^4+8\sum h_{ijs}h_{jls}h_{lk}h_{ki}\\&+4\sum h_{ijs}h_{jl}h_{lks}h_{ki}.\end{aligned} \tag{1.8}$$

We also have

$$\frac{1}{2}\Delta S=\sum h_{ij}\Delta h_{ij}+\sum h_{ijk}^2=n|H|\ \text{trace }\widetilde{H}^3-S^2+\sum h_{ijk}^2, \tag{1.9}$$

$$\frac{1}{4}\Delta^2 S=\frac{1}{2}n|H|\Delta(\text{trace }\widetilde{H}^3)-\frac{1}{2}\Delta S^2+\sum h_{ijk}\Delta h_{ijk}+\sum(h_{ijkl})^2. \tag{1.10}$$

Now because S is a constant too, using (1.6) and (1.7), we have

$$\begin{aligned}\sum h_{ijkl}^2=&-\frac{3}{2}n^2|H|^2\ \text{trace }\widetilde{H}^4+\frac{3}{2}Sn|H|\ \text{trace }\widetilde{H}^3-6n|H|\sum h_{ijk}h_{jlk}h_{li}\\&-6\sum h_{ijk}h_{lsk}h_{il}h_{sj}+3\sum h_{ijk}h_{sik}h_{ls}h_{jl}+S\sum h_{ijk}^2.\end{aligned} \tag{1.11}$$

Let $b_{ij}=|H|\delta_{ij}-h_{ij}$, then $b_{ij}=b_{ji}$. The following formulas are easily seen:

$$\sum_i b_{ii}=0,\ \sum b_{ij}^2=S-n|H|^2,$$

$$\text{trace }\widetilde{H}^3=3|H|S-2n|H|^3-\text{trace }B^3,$$

$$\text{trace }\widetilde{H}^4=6|H|^2S-5n|H|^4-4|H|\ \text{trace }B^3+\text{trace }B^4,$$

where B denotes the matrix (b_{ij}). (1.12)

§ 2. 3-dimensional complete hypersurfaces in R^4

In this section, $n=3$. On the other word, M is a complete and connected hypersurface with constant mean curvature and constant scalar curvature in R^4.

At the beginning of this section, we know all formulas in § 1 are valid. Because $\widetilde{H}=(h_{ij})$ is a symmetric matrix, we choose e_1, e_2, e_3 at a point P of M such that $h_{ij}=\lambda_i\delta_{ij}$. From the definition of b_{ij}, at the same time, we have $b_{ij}=\mu_i\delta_{ij}$ and $\lambda_i+\mu_i=|H|$.

It is easy to know that

$$\mu_1+\mu_2+\mu_3=0,\ \mu_1^2+\mu_2^2+\mu_3^2=S-3|H|^2,$$

$$\mu_1^4+\mu_2^4+\mu_3^4=(\mu_1^2+\mu_2^2+\mu_3^2)^2-\frac{1}{2}\left[(\mu_1+\mu_2+\mu_3)^2-(\mu_1^2+\mu_2^2+\mu_3^2)\right]^2$$

$$=\frac{1}{2}(S-3|H|^2)^2. \tag{2.1}$$

From (2.1)、(1.9) and (1.12), we get

$$\Delta(\text{trace }\widetilde{H}^4)=4|H|\ \Delta(\text{trace }\widetilde{H}^3)=-\frac{4}{3}\Delta(\sum h_{ijk}^2). \tag{2.2}$$

Because of (1.8) and (2.2), at P, we have

$$2\sum {}_{ijk}^2\lambda_i^2+\sum h_{ijk}^2\lambda_i\lambda_j=-\frac{1}{3}\Delta(\sum h_{ijk}^2)-3|H|\sum_i\lambda_i^5+S\sum_i\lambda_i^4. \tag{2.3}$$

Using the relation between λ_i and μ_i, we know

$$\sum h_{ijk}^2\lambda_i^2+2\sum h_{ijk}^2\lambda_i\lambda_j=3|H|^2\sum h_{ijk}^2-6|H|\sum h_{ijk}^2\mu_i+\frac{1}{3}\sum h_{ijk}^2(\mu_i+\mu_j+\mu_k)^2. \tag{2.4}$$

From (1.7), we have

$$6|H|\sum h_{ijk}^2\mu_i=9|H|^2\text{ trace }\widetilde{H}^4-3S|H|\text{ trace }\widetilde{H}^3$$
$$+6|H|^2\sum h_{ijk}^2+\frac{1}{3}\Delta(\sum h_{ijk}^2), \tag{2.5}$$

and from

$$6\sum h_{ijk}^2\lambda_i\lambda_j-3\sum h_{ijk}^2\lambda_j^2=5(\sum h_{ijk}^2\lambda_i^2+2\sum h_{ijk}^2\cdot\lambda_i\lambda_j)-4(2\sum h_{ijk}^2\lambda_i^2+\sum h_{ijk}^2\lambda_i\lambda_j), \tag{2.6}$$

we can obtain

$$6\sum h_{ijk}^2\lambda_i\lambda_j-3\sum h_{ijk}^2\lambda_j^2=-\frac{1}{3}\Delta(\sum h_{ijk}^2)+12|H|\text{ trace }\widetilde{H}^5$$
$$-(4S+45|H|^2)\text{ trace }\widetilde{H}^4$$
$$+15|H|(S+3|H|^2)\text{trace }\widetilde{H}^3-15S^2|H|^2$$
$$+\frac{5}{3}\sum h_{ijk}^2(\mu_i+\mu_j+\mu_k)^2. \tag{2.7}$$

We substitute (1.7)、(1.9)、(2.2) and (2.7) into (1.11), then

$$\sum h_{ijkl}^2=\frac{4}{3}\Delta(\sum h_{ijk}^2)+S^3+15S^2|H|^2+4S\text{ trace }\widetilde{H}^4$$
$$-\frac{45}{2}S|H|\text{ trace }\widetilde{H}^3-12|H|\text{ trace }\widetilde{H}^5+\frac{117}{2}|H|^2\text{ trace }\widetilde{H}^4$$
$$-45|H|^3\text{ trace }\widetilde{H}^3-\frac{5}{3}\sum h_{ijk}^2(\mu_i+\mu_j+\mu_k)^2. \tag{2.8}$$

Using (2.1), we can see

$$\text{trace}\widetilde{H}^5=-\frac{9}{2}|H|^5-5|H|^3S+\frac{5}{2}|H|S^2-10|H|^2\sum_i\mu_i^3-\sum_i\mu_i^5 \tag{2.9}$$

and

$$\sum_i\mu_i^5=-5\mu_1\mu_2(\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)$$
$$=\left(\frac{5}{3}\sum_i\mu_i^3\right)\cdot\left(\frac{1}{2}\sum_i\mu_i^2\right)=\frac{5}{6}(S-3|H|^2)\sum_i\mu_i^3. \tag{2.10}$$

From (1.12) and (2.10), evidently

$$\text{trace}\ \widetilde{H}^5 = \frac{5}{6}(S+9|H|^2)\ \text{trace}\ \widetilde{H}^3 + \frac{9}{2}|H|^3(9|H|^2-5S). \tag{2.11}$$

We rely on (1.12)、(1.9) and (2.11). Let $\sum h_{ijk}^2 = f$. f is a differentiable function of M. Then (2.8) becomes

$$\sum h_{ijkl}^2 = \frac{4}{3}\Delta f + \frac{5}{2}(S-3|H|^2)\left(S-\frac{9}{2}|H|^2\right)(9|H|^2-S) + \frac{11}{2}(S-6|H|^2)f - \frac{5}{3}\sum h_{ijk}^2(\mu_i+\mu_j+\mu_k)^2. \tag{2.12}$$

Combining(2.5) with (1.9)、(1.12), we have

$$\frac{1}{3}\Delta f = 6|H|\sum h_{ijk}^2\mu_i + (6|H|^2-S)f + (S-3|H|^2)\left(S-\frac{9}{2}|H|^2\right)(S-9|H|^2). \tag{2.13}$$

Therefore

$$\sum h_{ijkl}^2 + \frac{5}{3}\sum h_{ijk}^2(\mu_i+\mu_j+\mu_k)^2 - 24|H|\sum h_{ijk}^2\mu_i = \frac{3}{2}(S-3|H|^2)\left(S-\frac{9}{2}|H|^2\right)(S-9|H|^2) + \frac{3}{2}(S-6|H|^2)f. \tag{2.14}$$

The above equality is a very important formula in this paper.

If, in the whole of M, each λ_i $(1\leqslant i\leqslant 3)$ is constant, we say M owns constant principal curvatures. Secondly, we introduce some res $u|t$ which we shall use later on

(1) In R^{n+1}, a complete and connected hypersurface with constant principal curvatures must be $S^k\times R^{n-k}$ $(0\leqslant k\leqslant n)$ (see [2]).

Hence, in R^4, non-totally geodesic complete and connected hypersurfaces with constant principal curvatures are certainly S^3, $S^2\times R^1$, $S^1\times R^2$, where S are $3|H|^2$, $\frac{9}{2}|H|^2$ or $9|H|^2$, respectively.

(2) $-\frac{1}{\sqrt{6}}(S-3|H|^2)^{3/2}\leqslant\sum_i \mu_i^3\leqslant\frac{1}{\sqrt{6}}(S-3|H|^2)^{3/2}$, the equality holds if and only if there are at least two equal μ_i (cf. lemma in [1]).

From (1.4). we know

$$R_{ii} = 3|H|\lambda_i - \lambda_i^2 > -3|H|\sqrt{S} - S. \tag{2.15}$$

In this paper, the Ricci curvature of M is bounded below.

Now we shall set up a theorem.

Theorem. *In R^4, let M be complete and connected hypersurface with non-zero constant mean curvature vector H and non-negative constant scalar curvature R. Then there are only three cases:* $R=6|H|^2$, $\frac{9}{2}|H|^2$ and 0. *When* $R=6|H|^2$, $M=S^3\left(\frac{1}{|H|}\right)$; *when* $R=\frac{9}{2}|H|^2$, $M=S^2\left(\frac{2}{3|H|}\right)\times R^1$; *when* $R=0$, *for example* $M=S^1\left(\frac{1}{3|H|}\right)\times R^2$.

Proof Because $|H|$ and S are constants, from (1.9) and (1), M owns constant principal curvatures if and only if f is constant. Hence it is considerable that f is not constant.

We assume f is not constamt. Masafumi Okumura proved that when

$$S<\frac{9}{2}|H|^2,\ M=S^3\left(\frac{1}{|H|}\right).$$

When $S=\frac{9}{2}|H|^2$, by virtue of the work of Chen and Okumura in 1973 (see [4]), we can see M has non-negative sectional curvature.

From the work of Nomizu and Smyth in 1969 (cf. [5]), we can find

$$M=S^2\left(\frac{2}{3|H|}\right)\times R^1.$$

Now we consider $S>\frac{9}{2}|H|^2$. In 1975, S. T. Yau discovered the following theorem[3]:

Let $\widetilde{M}$ be a complete Riemannian manifold. The Ricci curvature of $\widetilde{M}$ is bounded below. In addition. F is a c^2 function and sup $F<\infty$. *Then there is a sequence of points $\{P_\nu\}$ in $\widetilde{M}$ such that*

$$\lim_{\nu\to\infty}|dF(P_\nu)|=0,\quad \lim_{\nu\to\infty}\Delta F(P_\nu)\leqslant 0,$$
$$\lim_{\nu\to\infty}F(P_\nu)=\sup F. \tag{2.16}$$

Let $F=-f$. Obviously, M and F satisfy the condition in S. T. Yau's theorem. Hence, $\lim\limits_{\nu\to\infty}f(P_\nu)=\inf f$. Because of (1.9) and (1.12), evidently

$$\lim_{\nu\to\infty}\sum_i\mu_i^3(P_\nu)=\inf\sum_i\mu_i^3\geqslant-\frac{1}{\sqrt{6}}(S-3|H|^2)^{3/2}, \tag{2.17}$$

while we know $\mu_i(P_\nu)\in[-(S-3|H|^2)^{1/2},\ (S-3|H|^2)^{1/2}]$ $(1\leqslant i\leqslant 3)$. It is well known that in a closed interval a set of infinite points has at least an accumulation point. Therefore we can select a subsequence $\{P_{\nu'}\}$ of $\{P_\nu\}$ such that $u_1(P_{\nu'})$ has a limit point. Next, we can choose a subsequence $\{P_{\nu''}\}$ of $\{P_{\nu'}\}$ such that $\mu_2(P_{\nu''})$ has a limit point, etc. At last, there exists a subsequence of $\{P_\nu\}$, which, for simplicity, we still write as $\{P_\nu\}$. Moreover, at the same time, $\mu_i(P_\nu)(1\leqslant i\leqslant 3)$ has limit points respectively.

Now we conclude

$$\lim_{\nu\to\infty}\sum_i\mu_i^3(P_\nu)=-\frac{1}{\sqrt{6}}(S-3|H|^2)^{3/2},\ \inf f=0. \tag{2.18}$$

At the beginning, we prove the first equality. If it were not true, i. e.,

$$\inf\sum_i\mu_i^3>-\frac{1}{\sqrt{6}}(S-3|H|^2)^{3/2},$$

from (2), we should know that any two numbers among $\lim_{\nu\to\infty}\mu_1(P_\nu)$, $\lim_{\nu\to\infty}\mu_2(P_\nu)$ and $\lim_{\nu\to\infty}\mu_3(P_\nu)$ are different. Because there are two equal numbers, by calculation, $\inf\sum_i\mu_i^3$ must be $\frac{1}{\sqrt{6}}(S-3|H|^2)^{3/3}=\sup\sum_i\mu_i^3$. Hence $\sum_i\mu_i^3$ is a constant. Consequently f is a constant too. It is contrary to the hypothesis. Therefore we can see that when ν is sufficiently large, any two numbers among $\lambda_1(P_\nu)$, $\lambda_2(P_\nu)$ and $\lambda_3(P_\nu)$ are different.

It's easy to know that

$$A(P_\nu)=\begin{vmatrix}1 & 1 & 1\\ \lambda_1, & \lambda_2, & \lambda_3\\ \lambda_1^2, & \lambda_2^2, & \lambda_3^2\end{vmatrix}(P_\nu)\neq 0 \text{ and } \lim_{\nu\to\infty}A(P_\nu)\neq 0. \tag{2.19}$$

In view of (1.9), we know $df=-3|H|\,d(\text{trace }\widetilde{H}^3)$. Then $\lim_{\nu\to\infty}|d(\text{trace }\widetilde{H}^3(P_\nu))|=0$. And because $|H|$, S are constants, at P_ν wehave

$$\sum_i h_{iik}=0.\quad \sum_i\lambda_i h_{iik}=0.\quad \sum_i\lambda_i^2 h_{iik}=a_k, \tag{2.20}$$

where a_k depends on P_ν, $1\leqslant k\leqslant 3$, and $\lim_{\nu\to\infty}a_k(P_\nu)=0$. Because of (2.20), at point P_ν we get

$$h_{11k}=A^{-1}a_k(\lambda_3-\lambda_2),\ h_{22k}=A^{-1}a_k(\lambda_1-\lambda_3),\ h_{33k}=A^{-1}a_k(\lambda_2-\lambda_1). \tag{2.21}$$

By virtue of (2.19), we find $\lim_{\nu\to\infty}h_{iik}(P_\nu)=0$. from (2.1), we can see

$$\sum h_{ijk}^2\mu_i=\sum_i h_{iii}^2\mu_i+\sum_{i\neq k}h_{iik}^2(2\mu_i+\mu_k). \tag{2.22}$$

Evidently

$$\lim_{\nu\to\infty}\sum h_{ijk}^2\mu_i(P_\nu)=0. \tag{2.23}$$

Similarly, we have

$$\lim_{\nu\to\infty}\sum h_{ijk}^2(\mu_i+\mu_j+\mu_k)^2(P_\nu)=0. \tag{2.24}$$

Evaluate formula (2.13) at P_ν and let $\nu\to\infty$. Then using $\lim_{\nu\to\infty}\Delta f(P_\nu)\geqslant 0$ and (2.23), we can see

$$(S-6|H|^2)\inf f\leqslant(S-3|H|^2)\left(S-\frac{9}{2}|H|^2\right)(S-9|H|^2). \tag{2.25}$$

As in the work of Peng and Terng[6] we obtain

$$\sum h_{ijkl}^2\geqslant 3\sum_{i\neq j}h_{ijij}^2. \tag{2.26}$$

Define $t_{ij}=h_{ijij}-h_{jiji}$. From (1.3), we have

$$t_{ij}=\lambda_i\lambda_j(\lambda_i-\lambda_j). \tag{2.27}$$

When $i\neq j$,

$$h_{ijij}^2+h_{jiji}^2=\frac{1}{2}(h_{ijij}-h_{jiji})^2+\frac{1}{2}(h_{ijij}+h_{jiji})^2. \tag{2.28}$$

Hence

$$\sum h_{ijkl}^2\geqslant\frac{3}{4}\sum\lambda_i^2\lambda_j^2(\lambda_i-\lambda_j)^2. \tag{2.29}$$

From (2.14), we know $\lim_{\nu\to\infty}\sum h^2_{ijkl}(P_\nu)$ exists. From (2.29) and the fact that any two numbers among $\lim_{\nu\to\infty}\lambda_1(P_\nu)$, $\lim_{\nu\to\infty}\lambda_2(P_\nu)$ and $\lim_{\nu\to\infty}\lambda_3(P_\nu)$ are different, we have

$$\lim_{\nu\to\infty}\sum h^2_{ijkl}(P_\nu)>0. \tag{2.30}$$

Making use of (2.23)、(2.24)、(2.25) and (2.30), from (2.14), we can see

$$0<3(S-3|H|^2)\Big(S-\frac{9}{2}|H|^2\Big)(S-9|H|^2). \tag{2.31}$$

Because $\frac{9}{2}|H|^2<S\leqslant 9|H|^2$, it is impossible. Therefore, we obtain the first formula in (2.18).

Using (1.9) and (1.12), at P_ν, we have

$$f=(S-3|H|^2)(S-6|H|^2)+3|H|\sum_i \mu_i^3. \tag{2.32}$$

Because $f\geqslant 0$, we have $\inf f\geqslant 0$. In view of the first formula in (2.18), obviously

$$(S-3|H|^2)(S-6|H|^2)-\frac{3}{\sqrt{6}}|H|(S-3|H|^2)^{3/2}=\inf f\geqslant 0. \tag{2.33}$$

(2.33) is equivalent to

$$-3(S-3|H|^2)\Big[|H|-\frac{1}{\sqrt{6}}(S-3|H|^2)^{1/2}\Big]\cdot\Big[|H|+\frac{2}{\sqrt{6}}(S-3|H|^2)^{1/2}\Big]$$
$$=\inf f\geqslant 0.$$

By virtue of $\frac{9}{2}|H|^2<S\leqslant 9|H|^2$, the left hand side of (2.34)$\leqslant 0$. Hence $\inf f=0$. At the moment

$$|H|-\frac{1}{\sqrt{6}}(S-3|H|^2)^{1/2}=0,\ \text{i. e.},\ S=9|H|^2. \tag{2.35}$$

In sum, when $0<S\leqslant 9|H|^2$, S gets only three values: $3|H|^2$, $\frac{9}{2}|H|^2$ and $9|H|^2$, i. e., when $R\geqslant 0$, R has only three values: $6|H|^2$, $\frac{9}{2}|H|^2$ and 0.

When $M=S^1\Big(\frac{1}{3|H|}\Big)\times R^2$, M owns constant principal curvatures. We know $S=9|H|^2$ and $R=0$ (cf. [2]).

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