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ALMOST PERIODIC SOLUTIONS OF THE EQUATION $\dot{x}=x^3+\lambda g(t)x+\mu f(t)$ AND THEIR STABILITY

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Abstract

By using the Liapunov function and the contraction mapping principle, the author investigates the existence and stability of almost periodic solutions of the first order nonlinear equations

$$\frac{dx}{dt}=-h_1(x)+h_2(x)g(t)+f(t)$$

and

$$\frac{dx}{dt}=r(t)x^n+\lambda g(t)x+\mu f(t),$$

where $r(t)$, $g(t)$, $f(t)$ are given almost periodic functions, $n(\geqslant 2)$ integer, and λ, μ real parameters.

As a special case, for the equation

$$\frac{dx}{dt}=-x^3+\lambda g(t)x+\mu f(t),$$

under the conditions $1\leqslant|g(t)|<3$, $|f(t)|\leqslant 1$, the author constructs regions in the (λ, μ)-plane such that for (λ, μ) in these regions there are either one or three almost periodic solutions. Similar conditions and regions are also obtained such that the equation

$$\frac{dx}{dt}=-x^2+\lambda g(t)x+\mu f(t)$$

has either two or no almost periodic solution. Moreover, by using the successive approximation method, sufficient condition is obtained for the existence of almost periodic solution of a quasilinear system.

We consider the first order nonlinear differential equation

$$\frac{dx}{dt}=x^3+\lambda g(t)x+\mu f(t), \tag{1}$$

where $g(t)$, $f(t)$ are almost periodic functions (written as a. p. for simplicity) and λ, μ are real parameters. Using a fixed point theorem and Liapunov function, we discuss the existence and stability of a. p. solutions of (1). For given f, g satisfying some conditions, we construct regions in the (λ, μ)-plane such that for (λ, μ) in these regions there are either one or three a. p. solutions. Although we cannot

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determine the number of a. p. solutions for all values of (λ, μ), the exact regions are constructed for the autonomous equation $f(t)=g(t)=1$ (or $f(t)=g(t)=-1$) for all t.

If t is replaced by $-t$, equation (1) has the form

$$\frac{dx}{dt}=-x^3+\lambda g(t)x+\mu f(t). \tag{2}$$

Clearly, (1) has a. p. solution for (λ, μ) if and only if (2) does for $(-\lambda, -\mu)$, and in all conclusions on stability we must change the direction of the time-axis. In the following, we give a sufficient condition for the existence of an a. p. solution of some equations which are more general than (2). In § 1 we suppose $g(t)\geqslant 0$ (or $\leqslant 0$), however, in § 2 we will remove this restriction.

§ 1.

Theorem 0. *Let $F(x, t)$ be a scalar function, almost periodic in t uniformly for x in compact sets, that suppose F is continuous and monotone decreasing with respect to $x\in[a,b]$, such that all equation in the hull of*

$$\frac{dx}{dt}=F(x,t) \tag{0}$$

has a unique solution to the initial value problem. If (0) has a bounded solution $\varphi(t)$ such that $\{\varphi(t)\,|\,0\leqslant t<\infty\}\subset[a, b]$, then it has an almost periodic solution $x(t)$, Range $(x)\subset[a, b]$. *Moreover, if F decreases strictly with respect to $x\in[a, b]$, then $x(t)$ is the unique a. p. solution with* Range $(x)\subset[a, b]$.

Proof The first part is just Theorem 12.8 in [1]. By Theorem 6.2 of the same reference, we have Range $(x)\subset[a, b]$.

If $x_1(t)$, $x_2(t)$ are two distinct a. p. solutions with Range $(x_i)\subset[a, b]$ $(i=1, 2)$, by the uniqueness of solution of the initial value problem, we may assume $x_1(t)>x_2(t)$. We have

$$\frac{d(x_2-x_1)}{dt}=G(t),$$

where $G(t)=F(x_2(t), t)-F(x_1(t), t)>0$ is an a. p. function. Therefore

$$x_2(t)-x_1(t)=x_2(0)-x_1(0)+\int_0^t G(t)dt.$$

Let $t\to+\infty$, from Theorem 3.8 of [1], the mean value G_0 of $G(t)$ is positive. Noticing

$$G_0=\lim_{T\to\infty}\frac{1}{T}\int_0^T G(t)dt,$$

we get $\int_0^{+\infty} G(t)dt=+\infty$, which contradicts the fact that $x_2(t)-x_1(t)$ is bounded. So the second part is proved.

We now consider the equation

$$\frac{dx}{dt}=-h_1(x)+h_2(x)g(t)+f(t), \tag{3}$$

where $h_1(x),\ h_2(x)\in C^1,\ g(t),\ f(t)\in$ A. P. (the set of a. p. functions). If

$$F(x,t)=-h_1(x)+h_2(x)g(t)+f(t),$$

then $F(x, t)$ is almost periodic in t uniformly, for x in every closed interval. Obviously, each equation in the hull of (3) has a unique solution to the initial value problem.

Theorem 1. *Suppose that $g(t)\leqslant 0$, $h_1(x)$, $h_2(x)$ are continuous and monotone increasing functions, $h_1(+\infty)=+\infty, h_1(-\infty)=-\infty$, and at least one of the following two conditions is satisfied:*

i) *$h_1(x)$ increases strictly,*

ii) *$h_2(x)$ increases strictly and $g(t)<0$.*

Then (3) has a unique a. p. solution which is uniformly asymptotically stable. Moreover, if one of the following two conditions is satisfied:

iii) $h_1'(x)\geqslant\alpha>0$,

iv) $h_2'(x)\geqslant\gamma>0,\ g(t)\leqslant\beta<0$,

where α, β, γ are constants, then this a. p. solution is uniformly asymptotically stable in the large. All the other solutions are asymptotically almost periodic.

Proof First, we prove that (3) has bounded solutions on $[0, +\infty)$. Let $x(t; 0, x_0)$ be a solution with $x(0)=x_0$. Since $g(t)$, $f(t)$ are bounded, and

$$h_1(+\infty)=+\infty,\quad h_1(-\infty)=-\infty,$$

there exists $K>0$ such that $F(K, t)<0$, $F(-K, t)>0$ for $t\geqslant 0$. Therefore, if we take $|x_0|<K$, $x(t; 0, x_0)$ is well defined for $t\geqslant 0$ and $|x(t; 0, x_0)|\leqslant K$. Under condition i) or ii), $F(x, t)$ is decreasing strictly with respect to $x\in[-K, K]$, where K may be taken arbitrarily large. By Theorem 0, there exists a unique a. p. solution.

If $x-u(t)=y$, where $u(t)$ is the a. p. solution of (3), then

$$\frac{dy}{dt}=-[h_1(y+u)-h_1(u)]+g(t)[h_2(y+u)-h_2(u)]. \tag{4}$$

We consider a Liapunov function $V(t, y)=\frac{y^2}{2}$. Since

$$[h_1(y+u)-h_1(u)|y\geqslant 0,\ [h_2(y+u)-h_2(u)|y\geqslant 0,\ g(t)\leqslant 0,$$

it follows that

$$\left.\frac{dv}{dt}\right|_{(4)}=-[h_1(y+u)-h_1(u)]y+g(t)[h_2(y+u)-h_2(u)y\leqslant 0,$$

and hence, the solution $y\equiv 0$ of (4) and the solution $u(t)$ of equation (3) are uniformly stable [2].

We consider now the nontrivial solution $y(t; t_0, y_0)$ of (4). By the uniqueness of the solution of the initial value problem, $y(t)\neq 0$ for $t\geqslant t_0$. We rewrite (4) as follows:

$$\frac{dy}{dt}=-\frac{h_1(y+u)-h_1(u)}{y}y+\frac{g(t)[h_2(y+u)-h_2(u)]}{y}y. \tag{5}$$

From i) or ii) we have $y\dfrac{dy}{dt}<0$, and hence $|y(t)|$ is decreasing. Now we seek to prove that $\lim\limits_{t\to+\infty}y(t)=0$. If this is not true, then $|y(t)|\geqslant d>0$, where d is a constant. Since $\dfrac{h_1(y+u)-h_1(u)}{y}$ is continuous on the compact set

$$\{(u, y)\mid |u|\leqslant\sup_t|u(t)|,\ d\leqslant y\leqslant y_0\ (\text{or } y_0\leqslant y\leqslant -d)\}$$

and positive because of i), there exists a constant $c>0$ such that $\dfrac{h_1(y+u)-h_1(u)}{y}\geqslant c$. Thus

$$\frac{1}{y}\frac{dy}{dt}\leqslant -c \quad (t\geqslant t_0)$$

from (5), and hence

$$|y|\leqslant|y_0|e^{-c(t-t_0)}\to 0, \quad (\text{as } t\to+\infty),$$

which contradicts the fact that $|y(t)|\geqslant d>0$. Thus we have $\lim\limits_{t\to+\infty}y(t)=0$. Under condition ii) we have

$$\frac{1}{y}\frac{dy}{dt}\leqslant cg(t),$$

where $cg(t)<0$ and $cg(t)\in$A. P., and hence $\int_{t_0}^{+\infty}cg(t)dt=-\infty$ as proved in Theorem 0. From this fact we can derive the same conclusion. By [2], Theorem 7.8 and Definition 7.7, the trivial solution of (4) and hence $u(t)$ are uniformly asymptotically stable.

Finally, if iii) or iv) holds, by the mean value theorem we have

$$\left.\frac{dv}{dt}\right|_{(4)}=[-h_1'(\theta_1(t))+g(t)h_2'(\theta_2(t))]y^2\leqslant -by^2,$$

where $b>0$ is a constant. Thus the solution $y\equiv0$ and hence the solution $u(t)$ of (3) are uniformly asymptotically stable in the large [2], and all other solutions are asymptotically almost periodic. This completes the proof.

In equation (2), $h_1(x)=x^3$, $h_2(x)=x$. we have

Corollary 1. *If $\lambda g(t)\leqslant0$, then (2) has a unique a. p. solution which is uniformly asymptotically stable. Moreover, if $\lambda g(t)\leqslant-\beta<0$, then the a. p. solution is uniformlly asymptotically stable in the large. All other solutions are asymptotically almost periodic.*

Now we consider the case when $h_1(x)$ is not monotonic. For simplicity we discuss only the type of equation

$$\frac{dx}{dt}=-x^n+\lambda g(t)x+\mu f(t), \tag{6}$$

where $n\geqslant2$ is positive integer. Without loss of generality we can assume that

$$\inf|g(t)|=\sup|f(t)|=1.$$

Using a fixed point theorem we will prove the following

Theorem 2. *Suppose* $\inf|g(t)|=\sup|f(t)|=1$ *and let* $M=\left(\frac{|\mu|}{n-1}\right)^{\frac{1}{n}}$. *If*

$$|\lambda|>nM^{n-1}, \tag{7}$$

then (6) *has an a. p. solution* $u(t)$, $|u(t)|\leqslant M$. *It is uniformly asymptotically stable if* $\lambda g(t)<0$, *unstable if* $\lambda g(t)>0$.

Proof If $\mu=0$, obviously, (6) has the trivial a. p. solution $x\equiv 0$. If $\mu\neq 0$, then $M>0$. By (7), we have

$$M^{n-1}+\frac{|\mu|}{M}-|\lambda|=M^{n-1}[1+|\mu|M^{-n}]-|\lambda|=nM^{n-1}-|\lambda|<0,$$

i. e.

$$\frac{M^n+|\mu|}{|\lambda|}<M. \tag{8}$$

Since $\inf|\lambda g(t)|=|\lambda|\neq 0$, either $\inf \lambda g(t)=|\lambda|>0$ or $\sup \lambda g(t)=-|\lambda|<0$. Since these two cases are analogous, we only consider the first case. Therefore

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \lambda g(t)dt\geqslant|\lambda|>0.$$

By [1], Theorem 6.6, for any a. p. function $\varphi(t)$, the equation

$$\frac{dx}{dt}=\lambda g(t)x+\mu f(t)-\varphi^n(t) \tag{9}$$

has a unique a. p. solution*, say $T\varphi$,

$$T\varphi=\int_t^\infty\left(\exp\int_s^t \lambda g(u)du\right)(\varphi^n(s)-\mu f(s))ds.$$

Let

$$B=\{\varphi(t)\in \text{A. P.}\,|\,\|\varphi\|\leqslant M\},\ \|\varphi\|=\sup_t|\varphi(t)|.$$

Clearly B is a completemetric space. From (8) we have

$$|T\varphi|=\left|\int_t^\infty\left(\exp\int_s^t \lambda g(u)du\right)(\varphi^n(s)-\mu f(s))ds\right|\leqslant\int_t^\infty e^{-|\lambda|(s-t)}(M^n+|\mu|)ds$$
$$\leqslant\frac{M^n+|\mu|}{|\lambda|}<M,$$

therefore $\|T\varphi\|\leqslant M$.

$$|T\varphi_1-T\varphi_2|=\left|\int_t^\infty\left(\exp\int_s^t \lambda g(u)du\right)(\varphi_1^n(s)-\varphi_2^n(s))\,ds\right|$$
$$\leqslant\int_t^\infty e^{-|\lambda|(s-t)}\cdot nM^{n-1}|\varphi_1(s)-\varphi_2(s)|ds\leqslant\frac{nM^{n-1}}{|\lambda|}\|\varphi_1-\varphi_2\|=\rho\|\varphi_1-\varphi_2\|,$$

where $\rho=\frac{nM^{n-1}}{|\lambda|}<1$ from (7). Therefore T is a contraction mapping, and hence there exists a unique fixed point u. Relation (9) shows that $u(t)$ is an a. p. solution of (6), and

*) In fact, the uniqueness is proved in Theorem 6.6.

$$\|u\| \leqslant M = \left(\frac{|\mu|}{n-1}\right)^{\frac{1}{n}} \tag{10}$$

In order to discuss the stability, let $y = x - u(t)$. We obtain

$$\frac{dy}{dt} = (\lambda g - nu^{n-1})y + O(y^2). \tag{11}$$

In case $\lambda g(t) < 0$, from (10) and (7), we have

$$\lambda g(t) - nu^{n-1}(t) \leqslant \lambda g(t) + |nu^{n-1}(t)| \leqslant -|\lambda| + nM^{n-1} < 0.$$

By the uniqueness of the solution of the initial value problem, the nontrivial solution of (11) satisfies $y(t) \neq 0$ for $t \geqslant t_0$. Thus we can rewrite (11) as follows

$$\frac{1}{y}\frac{dy}{dt} = \lambda g - nu^{n-1} + O(y).$$

Therefore

$$\frac{1}{y}\frac{dy}{dt} < -a \quad (|y| \ll 1),$$

where $a > 0$ is a constant. Consequently

$$|y(t| < |y_0| e^{-a(t-t_0)} \to 0 \quad (|y_0| \ll 1,\ t \to +\infty).$$

So the solution $y \equiv 0$ of (11) and hence the solution $\mathbf{u}(t)$ of (6) are uniformly asymptotically stable. In case $\lambda g(t) > 0$, we have

$$\lambda g(t) - nu^{n-1}(t) > 0.$$

So the solution $y \equiv 0$ of (11) and hence the solution $u(t)$ of (6) are unstable.

The proof is complete. Being different from Theorem 1, Theorem 2 does not show the uniqueness of the a. p. solution in the large, but it gives us a bound for the norm of a. p. solution. This estimate provides some informations on the a. p. solution as the parameter varies. In fact, for given $g(t)$, $f(t)$ and fixed $\lambda \neq 0$ (7) is satisfied for $|\mu|$ small enough. Let $u_\mu(t)$ be an a. p. solution satisfying the bound given in Theorem 2. From (10) we have

Corollary 2. *Suppose $g(t)$, $f(t)$ are given,* $\inf |g(t)| = \sup |f(t)| = 1$, and λ is fixed. If (7) holds, then $u_\mu(t)$ approaches zero uniformly as $\mu \to 0$.

Remark 1. Condition (7) is related to the bifurcation curve [3] for the equilibrium solutions of the autonomous equation obtained by setting $g(t) = f(t) \equiv 1$ in (6), that is, the equation

$$\frac{dx}{dt} = -x^n + \lambda x + \mu.$$

In the (λ, μ)-plane, the bifurcation curve for equilibrium solutions is given by

$$\begin{cases} -x^n + \lambda x + \mu = 0. \\ -nx^{n-1} + \lambda = 0. \end{cases}$$

Eliminating the parameter x, we obtain

$$\lambda = n\left(\frac{-\mu}{n-1}\right)^{\frac{n-1}{n}},$$

Therefore, we find that the bifurcation curve is a branch of the curve defined by

$$|\lambda|=n\left(\frac{|\mu|}{n-1}\right)^{\frac{n-1}{n}}.$$

As we will see later, for the almost periodic equation, the situation is much more complicated than the autonomous equation.

Remark 2. If the term $-x^n$ is replaced by x^n in equation (6), then the theorem is obviously true. Moreover, the theorem may be generalized. In fact, consider the equation

$$\frac{dx}{dt}=r(t)x^n+\lambda g(t)x+\mu f(t), \tag{6'}$$

where $r(t)\in$ A. P. and $n\geq 2$ is a positive integer. If $|r(t)|\leq 1$, the statement and the proof are still valid without any change. If $\sup|r(t)|=R>1$, instead of (7) we require

$$|\lambda|>nRM^{n-1}. \tag{7'}$$

Noticing that $R>1$ implies $n+R<nR+1$, we can prove

$$\frac{RM^n+|\mu|}{|\lambda|}<M. \tag{8'}$$

In fact, we derive that

$$RM^{n-1}+\frac{|\mu|}{M}-|\lambda|=M^{n-1}(R+|\mu|M^{-n})-|\lambda|$$
$$=M^{n-1}(R+n-1)-|\lambda|<nRM^{n-1}-|\lambda|<0.$$

Thus, (8′) is true. In this case

$$T\varphi=\int_t^\infty\left(\exp\int_s^t\lambda g(u)du\right)(r(s)\varphi^n(s)-\mu f(s))ds$$

is also a contraction mapping and (11) has the form

$$\frac{dy}{dt}=(nr(t)u^{n-1}+\lambda g(t))y+O(y^2). \tag{11'}$$

The stability may be discussed in the same way. Thus Theorem 2 may be generalized to the following

Theorem 2′. *Suppose*

$$\inf|g(t)|=\sup|f(t)|=1,\ \sup|r(t)|=R.$$

Let

$$M=\left(\frac{|\mu|}{n-1}\right)^{\frac{1}{n}}.\ \textit{If}\ |\lambda|>\max(1,R)\cdot nM^{n-1},$$

then (6′) *has an a. p. solution* $u(t)$, $|u(t)|\leq M$. *It is uniformly asymptotically stable if* $\lambda g(t)<0$, *unstable if* $\lambda g(t)>0$.

Remark 3. We can also apply another method [6] to prove the existence of a. p. solution in Theorems 2 and 2′. The advantage of this method is that we can apply it to a higher-dimesional case. Consider the system

$$\frac{dx}{dt}=A(t)x+\mu f(t)+q(x,t), \tag{12}$$

for which the following conditions hold:

i) Square matrix $A(t)$ is almost periodic and the homogeneous equation

$$\frac{dx}{dt}=A(t)x \tag{13}$$

satisfies an exponential dichotomy [5];

ii) Vector $f(t)$ is almost periodic;

iii) Vector $q(x, t)$ is almost periodic in t uniformly with respect to x in a neighbourhood of $x=0$, and there is a $\delta>0$ such that

$$|q(x, t)|O(|x|^{1+\delta}), \quad \left|\frac{\partial q}{\partial x}\right|=O(|x|^{\delta}) \quad (\text{as } x\to 0),$$

uniformly for $t\in(-\infty, +\infty)$.

We now prove the following

Theorem 3. *If conditions* i), ii), iii) *are satisfied and* $|\mu|$ *is small enough, then* (12) *has a unique a. p. solution in the neighbourhood of* $x=0$.

Proof We only prove the existence of a bounded solution; for almost periodicity, see [6]. From i), the fundamental matrix $X(t)$ of (13) can be decomposed as follows

$$X(t)=X_1(t)+X_2(t), \quad X^{-1}(s)=Z_1(s)+Z_2(s),$$
$$X(t)X^{-1}(s)=X_1(t)Z_1(s)+X_2(t)Z_2(s),$$

and there are two positive constants α and β such that

$$|X_1(t)Z_1(s)|\leqslant\beta\exp(-\alpha(t-s)), t\geqslant s;$$
$$|X_2(t)Z_2(s)|\leqslant\beta\exp(\alpha(t-s)), t\leqslant s.$$

From iii) there is a constant $K>0$ such that

$$|q(x, t)|\leqslant Kh^{1+\delta}, \quad \left|\frac{\partial q}{\partial x}\right|\leqslant Kh^{\delta} \text{ for } |x|\leqslant h, \ -\infty<t<+\infty,$$

where h may be taken arbitrarily small. Let

$$x_0(t)\equiv 0,$$
$$x_m(t)=\int_{-\infty}^{t} X_1(t)Z_1(s)\{\mu f(s)+q(x_{m-1}(s), s)\}ds$$
$$-\int_{t}^{\infty} X_2(t)Z_2(s)\{\mu f(s)+q(x_{m-1}(s), s)\}ds, m=1,2, \cdots.$$

Take $h\ll 1$ such that

$$\frac{4\beta Kh^{\delta}}{\alpha}<1,$$

and then take $|\mu|\ll 1$ such that

$$\frac{4\beta|\mu|\,\|f\|}{\alpha}<h, \quad \|f\|=\sup_t |f(t)|.$$

we have

$$\|x_m(t)\|\leqslant\beta\{|\mu|\,\|f\|+Kh^{1+\delta}\}\left\{\int_{-\infty}^{t} e^{-\alpha(t-s)}ds+\int_{t}^{\infty} e^{\alpha(t-s)}ds\right\}$$
$$=\frac{2\beta}{\alpha}\left\{|\mu|\,\|f\|+Kh^{1+\delta}\right\}<h,$$

$$\|x_{m+1}-x_m\| \leqslant \beta K h^\delta \|x_m-x_{m-1}\| \left\{\int_{-\infty}^{t} e^{-\alpha(t-s)}\,ds+\int_{t}^{\infty} e^{\alpha(t-s)}\,ds\right\}$$
$$=\frac{2\beta K h^\delta}{\alpha}\|x_m-x_{m-1}\|<\frac{1}{2}\|x_m-x_{m-1}\|.$$

Therefore, $\{x_m(t)\}$ converges uniformly to $x(t)$, and $|x(t)|\leqslant h$. Differentiating the equality

$$x(t)=\int_{-\infty}^{t} X_1(t)Z_1(s)\{\mu f(s)+q(x(s),\ s)\}-\int_{t}^{\infty} X_2(t)Z_2(s)\{\mu f(s)+q(x(s),\ s)\}ds,$$

we see that $x(t)$ satisfies (12). The uniqueness can be easily proved.

We also mention that every nontrivial solution of (13) is unbounded if i) is satisfied, so (13) has no nontrivial a. p. solution in this case. Let $X(t)\eta$ be a nontrivial solution of (13). Corresponding to i), we decompose η as $\eta=\eta_1+\eta_2$, where

$$\eta_1=\begin{pmatrix}\eta_{11}\\ 0\end{pmatrix},\quad \eta_2=\begin{pmatrix}0\\ \eta_{22}\end{pmatrix}.$$

For any vector ξ, we have

$$|X_1(t)Z_1(s)\xi|\leqslant|\xi|\beta e^{-\alpha(t-s)},\ t\geqslant s;$$
$$|X_2(t)Z_2(s)\xi|\leqslant|\xi|\beta e^{\alpha(t-s)},\ t\leqslant s.$$

Taking $\xi=X(s)\eta$ we get

$$|X(s)\eta|\geqslant\frac{1}{\beta}e^{\alpha(t-s)}|X_1(t)Z_1(s)X(s)\eta|=\frac{1}{\beta}e^{\alpha(t-s)}|X_1(t)\eta_1|$$
$$=\frac{1}{\beta}e^{\alpha(t-s)}|X(t)\eta_1|,\ t\geqslant s;$$
$$|X(s)\eta|\geqslant\frac{1}{\beta}e^{-\alpha(t-s)}|X_2(t)Z_2(s)X(s)\eta|=\frac{1}{\beta}e^{-\alpha(t-s)}|X_2(t)\eta_2|$$
$$=\frac{1}{\beta}e^{-\alpha(t-s)}|X(t)\eta_2|,\ t\leqslant s.$$

Since $X(t)\eta\neq0$, we have $X(t)\eta_1\neq0$ or $X(t)\eta_2\neq0$. In case $X(t)\eta_1\neq0$ we have $|X(s)\eta|\to+\infty$ as $s\to-\infty$, and in case $X(t)\eta_2\neq0$ we have $|X(s)\eta|\to+\infty$ as $s\to+\infty$; so in both cases, $X(s)\eta$ is unbounded.

Applying Theorem 2 to equation (2), we have

Corollary 3. *Suppose* $\inf|g(t)|=\sup|f(t)|=1$. *If* $|\lambda|>3\left(\frac{|\mu|}{2}\right)^{\frac{2}{3}}$, *then* (2) *has an a. p. solution* $u(t)$, $|u(t)|\leqslant\left(\frac{|\mu|}{2}\right)^{\frac{1}{3}}$. *It is uniformly asymptotically stable if* $\lambda g(t)<0$, *unstable if* $\lambda g(t)>0$.

Corollary 1 shows that an a. p. solution of (2) is unique when $\lambda g(t)\leqslant0$. Now we consider $\lambda g(t)>0$ and prove the following

Theorem 4. *Suppose* $\inf|g(t)|=\sup|f(t)|=1$, $\sup|g(t)|<3$. *If* $\lambda g(t)>0$, *then there exists* $k>1$ *such that, for all* (λ,μ) *satisfying the condition*

$$|\lambda|>3k\cdot\left(\frac{|\mu|}{2}\right)^{\frac{2}{3}}, \tag{14}$$

(2) *has exactly three a. p. solutions.*

Proof Clearly (14) implies $|\lambda|>3\left(\frac{|\mu|}{2}\right)^{\frac{2}{3}}$. We suppose $\lambda>3\left(\frac{|\mu|}{2}\right)^{\frac{2}{3}}$ and $1\leqslant g(t)\leqslant G<3$. The other case is similar. From Corollary 3, (2) has an a. p. solution $u(t)$, $|u(t)|\leqslant\left(\frac{|\mu|}{2}\right)^{\frac{1}{3}}$. Put $x-u(t)=y$, then

$$\frac{dy}{dt}=-y\,[y^2+3uy+3u^2-\lambda g]. \tag{15}$$

If $F(y,t)=-y\,[y^2+3uy+3u^2-\lambda g]$, then we have $F'_y(y,t)=-(3y^2+6uy+3u^2-\lambda g)$. From $F(y,t)=0$ we have $y_0=0$,

$$\tilde{y}_1(t)=-\frac{3u}{2}+\sqrt{\lambda g-\frac{3u^2}{4}}>0,\quad \tilde{y}_2(t)=-\frac{3u}{2}-\sqrt{\lambda g-\frac{3u^2}{4}}<0.$$

From $F'_y(y,t)=0$ we have

$$\bar{y}_1(t)=-u+\sqrt{\frac{\lambda g}{3}}>0,\quad \bar{y}_2(t)=-u-\sqrt{\frac{\lambda g}{3}}<0.$$

We claim $\sup\bar{y}_1(t)<\inf\tilde{y}_1(t)$. If

$$\lambda=3k\left(\frac{|\mu|}{2}\right)^{\frac{2}{3}},\ k>1,$$

then
$$\left(\frac{|\mu|}{2}\right)^{\frac{1}{3}}=\sqrt{\frac{\lambda}{3k}}=k_1\sqrt{\frac{\lambda}{3}},$$

and we have
$$|u(t)\leqslant k_1\sqrt{\frac{\lambda}{3}},$$

where
$$0<k_1=\sqrt{\frac{1}{k}}<1.$$

Since
$$y_1(t)=-u+\sqrt{\frac{\lambda g}{3}}\leqslant k_1\sqrt{\frac{\lambda}{3}}+\sqrt{\frac{\lambda G}{3}}=\sqrt{\frac{\lambda}{3}}(k_1+\sqrt{G}),$$

it follows that

$$\sup\bar{y}_1(t)\leqslant\sqrt{\frac{\lambda}{3}}\,(k_1+\sqrt{G}).$$

Since

$$\tilde{y}_1(t)=-\frac{3u}{2}+\sqrt{\lambda g-\frac{3u^2}{4}}\geqslant-\frac{3k_1}{2}\sqrt{\frac{\lambda}{3}}+\sqrt{\lambda-\frac{\lambda k_1^2}{4}}=\sqrt{\lambda}\left(\sqrt{1-\frac{k_1^2}{4}}-\frac{\sqrt{3}\,k_1}{2}\right),$$

we have
$$\inf\tilde{y}_1(t)\geqslant\sqrt{\lambda}\left(\sqrt{1-\frac{k_1^2}{4}}-\frac{k_1\sqrt{3}}{2}\right).$$

By a simple computation, we conclude that

$$\sqrt{\frac{\lambda}{3}}(k_1+\sqrt{G})<\sqrt{\lambda}\left(\sqrt{1-\frac{k_1^2}{4}}-\frac{\sqrt{3}\,k_1}{2}\right)\Leftrightarrow\sqrt{G}<-\frac{5k_1}{2}+\sqrt{3\left(1-\frac{k_1^2}{4}\right)}.$$

Therefore, if $G<3$ is given, the last inequality holds true for all $k_1>0$ small enough, and hence we have $\sup\bar{y}_1(t)<\inf\tilde{y}_1(t)$ for all $(\lambda,\ \mu)$ which lie on the curve $\lambda=3k\cdot\left(\frac{|\mu|}{2}\right)^{\frac{2}{3}}$, where $k>1$ is large enough. If $\sup \bar{y}_1(t)=A_1$, then $F(A_1, t)>0$. Since $u(t)$, $g(t)$ are bounded, we can take $B_1>A_1$ such that $F(B_1, t)<0$. Then (15) has a bounded solution $y(t)$, $A_1\leqslant y(t)\leqslant B_1$ for $0\leqslant t<\infty$. Since $F'_y(y, t)<0$ for $A_1\leqslant y\leqslant B_1$, by Theorem 0, we conclude that there exists a unique a. p. solution of (15) in $[A_1,B_1]$. In the same way, let $B_2=\inf \bar{y}_2(t)$, $B_2<0$ and take $A_2<B_2$, $|A_2|$ large enough. Then (15) has another a. p. solution in $[A_2, B_2]$. Using the fact that an a. p. function $\varphi(t)$ for which $\lim\limits_{t\to\infty} \varphi(t)=l$ must satisfy $\varphi(t)\equiv l$ for all t, we conclude that there is no a. p. solution in every horizontal strip in the (y, t)-plane where $F(y, t)\geqslant 0$ (or $\leqslant 0$). Thus we have proved that (15) has exactly three a. p. solutions, i. e. (2) has exactly three a. p. solutions. This completes the proof.

If $g=f\equiv 1$, then (2) has the form

$$\frac{dx}{dt}=-x^3+\lambda x+\mu.$$

In the (λ, μ)-plane, the bifurcation curve for equilibrium solutions of this equation is given by $\lambda=3\left(\frac{-\mu}{2}\right)^{\frac{2}{3}}$. The number of equilibrium solutions for a given (λ, μ) is shown in Figs 1 and 3. For the case f, $g\in$ A. P., by Corollaries 1 and 3 and Theorem 4, for some (λ, μ), we can show the number of a. p. solutions in Figs. 2 and 4.

If $n=2$ in (6), we have

$$\frac{dx}{dt}=-x^2+\lambda g(t)x+\mu f(t). \tag{16}$$

Similarly we can prove

Theorem 5. *Suppose* $\inf|g(t)|=\sup|f(t)|=1$, $\sup|g(t)|<2$. *Then there exists* $k>1$ *such that, for all* (λ, μ) *satisfying the coudition* $|\lambda|>2k\sqrt{|\mu|}$, (16) *has exactly two a. p. solutions.*

If we do not change the variable and we use the same method as in the proof of Theorem 4 directly to equation (16), then the following theorem can be proved:

Theorem 6. *Suppose* $\inf|g(t)|=\sup|f(t)|=1$, $\sup|g(t)|<2$, $\inf|f(t)|=\sigma\geqslant 0$. *If* $\mu f(t)>0$ *or* $\mu f(t)<0$ *and* $\lambda^2>4|\mu|$, $|\mu|\ll 1$, *then* (16) *has exactly two a. p. solutions. In the first case, one of them is positive, and the other is negative. In the second case, these two solutions are both positive (or negative) if* $\lambda g(t)>0$ (*or* $\lambda g(t)<0$). *If* $\mu f<0$ *and* $\lambda^2\leqslant|\mu|\sigma$, *then* (16) *has no a. p. solution.*

Theorem 6 is a generalization in the almost periodic case of [7], Chapter 11, Theorem 3.1 and Theorem 3.2 for special Reccati equation. Being different from Theorem 4, Theorms 5 and 6 are true for both $\lambda g>0$ and $\lambda g<0$. For some (λ, μ), the number of a. p. solutions are shown in Figs. 6 and 8. The boundary of the region in which there is no a. p. solution depends on inf $|f(t)|$. For the autonomous equations, the numbe r of equilibria for a given (λ, μ) is shown in Figs. 5 and 7. Figs. 1—8 show that our results in almost periodic case are far from a complete one. In the periodic case, the bifurcation diagrams for (λ, μ) small are similar to that for the autonomous case (see [3], Chapter 9). It is still an unknown but interesting problem to determine the complete bifurcation diagram for (λ, μ) small.

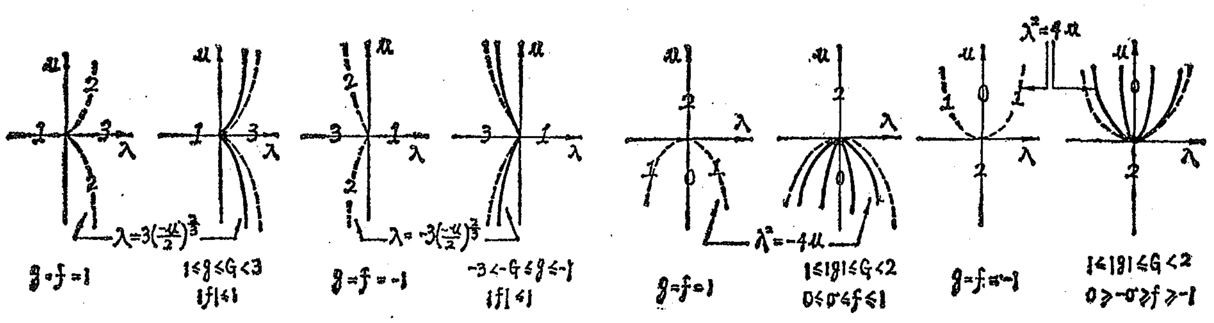

Fig.1 Fig.2 Fig.3 Fig.4 Fig.5 Fig.6 Fig.7 Fig.8

At the end of this section, we apply Theorem 2′ to the periodic equation. It is well known that the a. p. solution must be periodic and has period T provided $F(x, t)$ has period T and the scalar equation $\dot{x}=F(x, t)$ has a unique solution to the initial value problem [4]. So we can make

Remark 4. If $r(t)$, $g(t)$, $f(t)$ have period T, and the conditions in Theorem 2′ are satisfied, then (6′) has a periodic solution with period T.

For example, we cosider the equation

$$\frac{dx}{dt}=r(t)x^n+\lambda(2+\sin t)x+\mu\cos t, \tag{17}$$

where $n\geqslant 2$ is an integer, $r(t+2\pi)=r(t)$, $|r(t)|\leqslant 1$. If $|\mu|\geqslant 1$, we can easily prove $n\left(\frac{|\mu|}{n-1}\right)^{\frac{n-1}{n}}<3|\mu|$, and hence, (7′) holds for $n\geqslant 2$ provided $\frac{|\lambda|}{3}\geqslant|\mu|\geqslant 1$. Remark 4 shows that (15) has a periodic solution with period 2π for $n\geqslant 2$. It is uniformly asymptotically stable if $\lambda<0$, unstable if $\lambda>0$. When n is odd and $r(t)\equiv -1$, since there exists a bounded positive trajectory, the existence of periodic solution may be shown by Massera theorem [4]. But in the general case, the existence of a bounded positive trajectory is not clear.

§ 2.

In this section we shall consider the equation

$$\frac{dx}{dt}=-h(x)+g(t)x+f(t) \tag{18}$$

in the case that the sign of $g(t)$ may change.

Theorem 7. *Suppose* i) $h'(x)\geqslant 0$, $\lim_{x\to\pm\infty}\frac{x}{h(x)}=0$; ii) *there exists an* $\varepsilon>0$ *such that* $\int_0^t (2g(s)+\varepsilon)ds$ *is bounded for* $t\geqslant 0$. *Then* (16) *has a unique a. p. solution which is uniformly asymptotically stable in the large, and all of the other solutions are asymptotically almost periodic.*

Proof If $F(x, t)=-h(x)+g(t)x+f(t)$, then $F(x, t)$ is almost periodic in t uniformly for $x\in(-\infty, +\infty)$. From i), we have $h(-\infty)=-\infty$, $h(+\infty)=+\infty$. Noticing $h(x)\neq 0$ for $|x|$ large, we may write (18) as follows

$$\frac{dx}{dt}=h(x)\left[-1+\frac{xg(t)}{h(x)}+\frac{f(t)}{h(x)}\right].$$

Therefore, there exist $K>0$, $L>0$ such that

$$\frac{dx}{dt}<-L \text{ for } x\geqslant K; \quad \frac{dx}{dt}>L \text{ for } x\leqslant -K.$$

Thus
$$x(t)<-L(t-t_0)+x_0 \text{ for } x_0>K,\ x(t)\geqslant K,\ t\geqslant t_0;$$
$$x(t)>L(t-t_0)+x_0 \text{ for } x_0<-K,\ x(t)\leqslant -K,\ t\geqslant t_0.$$

Hence, we can see that the solutions of (18) are ultimately bounded for bound K.

We consider the system

$$\begin{cases}\dfrac{dx}{dt}=-h(x)+g(t)x+f(t)\\[2mm] \dfrac{dy}{dt}=-h(y)+g(t)y+f(t)\end{cases} \tag{19}$$

and take the Liapunov function

$$V(t, x, y)=(x-y)^2\exp\left(-\int_0^t (2g(s)+\varepsilon)ds\right),\ 0\leqslant t<\infty,\ |x|\leqslant K^*,\ |y|\leqslant K^*,$$

where $K^*>K$ is a constant. Since $\int_0^t (2g(s)+\varepsilon)ds$ is bounded, there are constants α, β such that $\alpha\leqslant -\int_0^t (2g(s)+\varepsilon)ds\leqslant\beta$. Then we can derive:

i) $e^{\alpha}(x-y)^2\leqslant V(t, x, y)\leqslant e^{\beta}(x-y)^2$,

ii) $|V(t, x_1, y_1)-V(t, x_2, y_2)|=\left|\exp\left(-\int_0^t (2g(s)+\varepsilon)ds\right)[(x_1-y_1)^2-(x_2-y_2)^2]\right|$
$\leqslant 4K^*e^{\beta}\{|x_1-x_2|+|y_1-y_2|\}$,

iii) $\dot{V}(t, x, y)|_{(19)}=2(x-y)\exp\left(-\int_0^t (2g(s)+\varepsilon)ds\right)\left[h(y)-h(x)+\frac{\varepsilon}{2}(y-x)\right]$
$=-2(x-y)^2\exp\left(-\int_0^t (2g(s)+\varepsilon)ds\right)\left[h'(\xi)+\frac{\varepsilon}{2}\right]\leqslant -\varepsilon e^{\alpha}(x-y)^2$,

using the mean value theorem in the last equality. By [2], Theorem 19.2, (16) has a unique a. p. solution which is uniformly asymptotically stable in the large.

And hence all of the other solutions are asymptotically almost periodic.

Applying Theorem 7 to equation (2), we have

Corollary 4. *If there exists an $\varepsilon>0$ such that $\int_0^t (2\lambda g(s)+\varepsilon)ds$ is bounded for $t\geqslant 0$, then (2) has a unique a. p. solution which is uniformly asymptotically stable in the large, and all of the other solutions are asymptotically almost periodic.*

If $\mu=0$, then (2) has the form

$$\frac{dx}{dt}=-x^3+\lambda g(t)x, \tag{20}$$

and Corollary 1 and Corollary 4 hold for (20). Since (20) has the trivial a. p. solution $x\equiv 0$, we have

Corollary 5. *If either i) there exists $\varepsilon>0$ such that $\int_0^t (2\lambda g(s)+\varepsilon)ds$ is bounded for $t\geqslant 0$ or ii) $\lambda g(t)\leqslant\beta<0$, then the trivial solution of (20) is uniformly asymptotically stable in the large, and (20) does not have a nontrivial a. p. solution.* In Corollary 1, it is possible that $g(t)\equiv 0$, so that neither Corollary 2 nor Corollary (4) implies Corollary 1.

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