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ANOTHER PROOF FOR THE SOLVABILITY OF FINITE GROUPS WITH AT MOST TWO CONJUGACY CLASSES OF MAXIMAL SUBGROUPS

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Abstract

In this paper we give a very short and elementary proof to the following theorem due to S. Adnan.

Theorem 2. Finite groups with at most two conjugacy classes of maximal subgroups are solvable.

Let i be a positive integer. A finite group G is called an M_i-group, if the number of conjugacy classes of maximal subgroups of G is $\leqslant i$.

It is nearly obvious that an M_1-group must be a cyclic p-group. (See Proposition 1). And for M_2-groups, several authors conjectured that they are solvable (See [1, 3]), and S. Adnan[4] has proved this conjecture. The purpose of this paper is to give a short and elementary proof to this conjecture.

The notations and terminology are standard. Readers may refer to [2] when necessary.

Proposition 1. *M_1-groups are cyclic p-groups.*

Proof Suppose that G is an M_1-group and M is a maximal subgroup of G. Then $\bigcup\limits_{x\in G} M^x \subsetneqq G$. Let $a\in G-\bigcup\limits_{x\in G} M^x$, then we have $\langle a\rangle=G$. (If not, $\langle a\rangle$ must be contained in a maximal subgroup M^x. This contradicts the choice of a.) Therefore G is cyclic. Further, if G is not a p-group, then G has at least two maximal subgroups, which are of course non-conjugate, this contradicts the fact that G is an M_1-group.

Conversely, it is clear that a cyclic p-group is an M_1-group.

Theorem 2. *M_2-groups are solvable.*

Proof Let G be an M_2-group. By Proposition 1 we may assume that G has exactly two conjugacy classes of maximal subgroups. Let L and K be the represen-

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tatives of the two classes respectively. We discuss the following three cases.

Case 1. $L\lhd G$ and $K\lhd G$. In this case, all maximal subgroups of G are normal in G. This implies that G is nilpotent. The conclusion is true.

Case 2 $L\lhd G$ but $K\ntriangleleft G$.

Let $p\mid |G:K|$, $P\in \mathrm{Syl}_p(G)$, $N=N_G(P)$. If $N<G$, then $N\leqslant L$ as $p\nmid |G:N|$. Because P is a Sylow subgroup and $N=N_G(P)$, we have $N_G(L)=L$. This contradicts the fact that $L\lhd G$. Thus we have $N=G$, i. e., $P\lhd G$.

Set $\overline{G}=G/P$. If $\overline{M}=M/P$ is one of the maximal subgroup of $\overline{G}$, then M is maximal in G. Since G is an M_2-group and $M\geqslant P$, we get $M=L$. This implies that $\overline{G}$ is an M_1-group. By Proposition 1 $\overline{G}$ is cyclic, the conclusion holds.

Case 3. Neither L nor K is normal in G.

We shall show that this is impossible. Let $|G:L|=r$, $|G:K|=s$. We have $(r, s)=1$. (If not, suppose $p\mid(r, s)$ and $P\in\mathrm{Syl}_p(G)$, we have $P\nleqslant L$, $P\nleqslant K$. Hence there exists a maximal subgroup $M\geqslant P$ which is conjugate to neither L nor K, contradicting the fact that G is an M_2-group.)Hence we have

$$|G:L^x\cap K^y|=|G:L^x|\,|G:K^y|=rs,\ \forall x,\ y\in G.$$

It follows that

$$|L^x\cap K^y|=|G|/rs=|L\cap K|,\ \forall x,\ y\in G. \tag{1}$$

Without loss of generality we may assume that $r>s$. Then we have

$$|G|\geqslant|K^zK^w|=\frac{|K^z|\,|K^w|}{|K^z\cap K^w|}=\frac{|G|^2}{s^2|K^z\cap K^w|},\ \forall z,\ w\in G.$$

It follows that

$$|K^z\cap K^w|\geqslant|G|/s^2>|G|/rs=|L\cap K|,\ \forall z,\ w\in G. \tag{2}$$

If we can prove

$$|K^z\cap K^w|_p\leqslant|L\cap K|_p, \tag{3}$$

for any prime p and any $K^z\cap K^w$, which will imply that $|K^z\cap K^w|\leqslant|L\cap K|$, contradicting(2), the proof will be completed.

In order to prove(3), we choose two conjugate subgroups $K^{z_0}\neq K^{w_0}$ of K such that $|K^{z_0}\cap K^{x_0}|_p$ reaches the maximum value. Without loss of generality we may assume that $z_0=1$ and $|K\cap K^{w_0}|_p>1$. Suppose that $P\in\mathrm{Syl}_p(K\cap K^{w_0})$, $P_1\in\mathrm{Syl}_p(K)$. Then $P_1\geqslant P$. Set $N=N_G(P)$. We may assume $N\neq G$. (If not, we get $P\lhd G$. Set $\overline{G}=G/P$, we can complete the proof by induction on$|G|$.) Let M be a maximal subgroup of G containing N. We discuss the following three possibilities.

(i) $P_1>P$: In this case, we have $M\cap K\geqslant N\cap K=N_K(P)$. Then $|M\cap K|_p>|P|$ as $P_1>P$. Similarly, $|M\cap K^{w_0}|_p>|P|$. By the choice of K^{w_0}, we get M is not conjugate to K, i. e., $M=L^x$ for some $x\in G$. Without loss of generality we may assume $M=L$. Hence

$$|L\cap K|_p=|M\cap K|_p>|P|=|K\cap K^{w_0}|_p,$$

(3) holds.

(ii) $P_1=P\notin Syl_p(G)$: In this case, we have $p\mid|G:K|$, hence $p\nmid|G:L|$. Therefore $|L|_p=|G|_p>|P|$. By (1) we get $|L\cap K|_p=|P|=|K\cap K^{w_0}|_p$, (3) holds.

(iii) $P_1=P\in Syl_p(G)$: Suppose that M is conjugate to K. Without loss of generality we may assume that $M=K^{w_0}$. Since $P\leqslant K^{w_0}$ and $P^{w_0}\leqslant K^{w_0}$, by Sylow theorem there exists an element $k\in K^{w_0}$ such that $P=P^{w_0k}$. Then $w_0k\in N_G(P)\leqslant K^{w_0}$. This implies $w_0\in K^{w_0}$ and $K=K^{w_0}$, a contradiction. Thus, we have $M=L^x$ for some $x\in G$. Therefore $|L|_p=|K|_p=|G|_p$. By (1) we get $|L\cap K|_p=|G|_p=|K\cap K^{w_0}|_p$, (3) holds. The proof is completed.

Corollary 3 *The number of conjugacy classes of maximal subgroups of a finite nonabelian simple group is* $\geqslant 3$.

Because A_5 and PSL (2, 7) has exactly 3 conjugacy classes of maximal subgroups, the bound in Corollary 3 is best possible.

Remark. Applying Theorem 2, we can slightly simplify the group-theoretic proof of Burnside's p^aq^b-theorem for the case of odd order. In analyzing the minimal counterexample G of this case, we can deduce that G has exactly two conjugacy classes of maximal subgroups (See [5, Lemma 4. 135]), then Theorem 2 can be applied to deduce the final contradiction.

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