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ARTINIAN RADICAL AND ITS APPLICATION*)

XU YONGHUA(许永华)**

abstract

Let R be a left and right Neotherian ring with identity. Let A be the Artinian radical. Lenagan [3] pointed out that R has Artinian quotient ring if $A=0$ and the Krull dimension of R is one. In this paper first the structure of Artinian radical is investigated. Then for R with Krull dimension one the author gives a necessary and sufficient condition under which R has Artinian quotient ring. The main results are as follows: (i) $A=eR$, where e is a central idempotent element of R, if and only if $r(A)^\lambda=l(A)^\lambda=(\bigcap_{\substack{i=1,\cdots,n_k\\ k=1,\cdots,m}} \mathfrak{p}(a_i^{(k)}))^\lambda$, where λ is a positive integer, $\mathfrak{p}(a_i^{(k)})$ are prime ideals of R and $r(A)(lA))$ is the notation of right(:eft) annihilatorof A(see Theorem 7). (ii) In the case(i)$R=A\oplus r(A)^\lambda$. (iii) If R has Krull dimension one, then R has Artinian quotient ring if and only if there exists a positive integer λ such that $r(A)^\lambda=l(A)^\lambda=(\bigcap_{\substack{i=1,\cdots,n_k\\ k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)}))^\lambda$.

§ 1.

In this paper "Ring" always means associative ring with identity. Unless otherwise stated all concepts and terms used here are cited from[1]. As we know, a right (left) ideal of a ring R is said to be Artinian if it is Artinian as a right (left) R-module. As usual we define the Artinian radical A of R to be the sum of all the Artinian right ideals of R. When R is left and right Noetherian, it is clear that A is also the sum of all Artinian left ideals of R(see[1]).

Let R be a left and right Noetherian ring and $\hat{E}$ be the right socle of R, i. e. $\hat{E}$ is the sum of all minimal right ideals of R. If $a'\in\hat{E}$, then it is clear that there exists a minimal ideal (a) such that $(a)\subset(a')$, where (a) denotes the principal ideal of R. Without loss of genarality we can assume that Ra and aR are minimal left and right ideals respectively. It can be easily shown that

$$\hat{E}=\sum_{i=1}^{n}\oplus Ra_iR\oplus\sum_{j=1}^{s}\oplus Rb_jR\oplus L, \tag{*}$$

where $(a_i)=Ra_iR$ and $(b_j)=Rb_jR$ are minimal ideals of R, $(a_i)^2=(a_i)$, $(b_j)^2=0$, and L is a right ideal of R which doesn't contain any non-zero ideal of R. For the

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*) Projects Supported by the Science Fund of the Chinese Academy of Sciences.
** Department of Mathematics, Fudan University, Shanghai, China.

convenience we call (*) a right standard formula of $\hat{E}$, and $n+s$ the length of the right standard formula (*) of $\hat{E}$, n the lenth of idempotent part of $\hat{E}$ and s the length of nilpotent part of $\hat{E}$. Finally we say that a right standard formula of $\hat{E}$ has most length, if every length of right standard formula of $\hat{E}$ is not greater than this length. Let the formula (*) be a right standard formula of $\hat{E}$ having the most length, and

$$\hat{E}=\sum_{i=1}^{n'}\oplus(a'_i)\oplus\sum_{j=1}^{s'}\oplus(b'_j)\oplus L'$$

be another right standard formula having the most length, then it is clear that $n=n'$, $(a_i)=(a'_i)$ $i=1, \cdots, n$ and $s=s'$,

$$\sum_{j=1}^{s}\oplus(b_j)=\sum_{j=1}^{s'}\oplus(b'_j).\tilde{L}$$

In fact, if $(b'_j)\cap((b_1)\oplus\cdots\oplus(b_s))=0$, then $((a_1)\oplus\cdots\oplus(a_n))\cap((b_1)\oplus\cdots\oplus(b_s)\oplus(b'_j))=0$. This shows that

$$\hat{E}=\sum_{i=1}^{n}\oplus(a_i)\oplus\sum_{k=1}^{s}\oplus(b_k)\oplus(b'_j)\oplus\tilde{L},$$

where $\tilde{L}$ is a right ideal of R. But this contradicts the fact that $\hat{E}$ has the most length $n+s$. Therefore $(b'_j)\subset(b_1)\oplus\cdots\oplus(b_s)$ for any (b'_j), i. e.

$$\sum_{j=1}^{s'}\oplus(b'_j)\subseteq\sum_{j=1}^{s}\oplus(b_j).$$

Similarly, we can show that $\sum_{j=1}^{s}\oplus(b_j)\subseteq\sum_{j=1}^{s'}\oplus(b'_j)$. We have already shown that $\sum_{j=1}^{s}\oplus(b_j)=\sum_{j=1}^{s'}\oplus(b'_j)$. Using the same method we can show that $\sum_{i=1}^{n}\oplus(a_i)=\sum_{i=1}^{n'}\oplus(a'_i)$. From this it follows that $(a_i)^2=\sum_{j=1}^{n'}(a_i)(a'_j)$. Therefore there exists an element, for example (a'_l), in the set $\{(a'_1), (a'_2), \cdots, (a'_{s'})\}$ such that $(a_1)^2=(a_1)(a'_1)$ and $(a_1)(a'_i)=0$, $i=2, \cdots, s'$. Hence $(a_1)=(a'_1)$. We can go on in this way and obtain $(a_i)=(a'_i)$, $i=1, \cdots, n$, and $n\leqslant n'$. Analogously we can show that $n'\leqslant n$ and $(a'_i)=(a_i)$ for $i=1, \cdots, n$. Since $n+s=n'+s'$, it follows that $s=s'$.

Denote by $\hat{E}=\sum_{j=1}^{n}\oplus(a_i)\oplus\sum_{j=1}^{s}\oplus(b_j)\oplus$L a right standard formula having the most length. Then we call $\hat{E}=\sum_{i=1}^{n}\oplus(a_i)\oplus\sum_{j=1}^{s}\oplus(b_j)$ the normal socle of R and call $E_1=\sum_{i=1}^{n}\oplus(a_i)$ the non-nilpotent part of $\hat{E}$ and $E_2=\sum_{j=1}^{s}\oplus(b_j)$ the nilpotent part of $\hat{E}$. It is clear that $\hat{E}$, E_1 and E_2 are independent of the choice of the forms of right standard formulas with the most length.

Similarly, denote by $\tilde{E}$ the left socle of R and

$$\tilde{E}=\sum_{i=1}^{\tilde{n}}\oplus R\tilde{a}_iR\oplus\sum_{j=1}^{\tilde{n}}\oplus R\tilde{b}_jR\oplus\tilde{L}, \tag{**}$$

where $(\tilde{a}_i)=R\tilde{a}_iR$, $(\tilde{b}_j)=R\tilde{b}_jR$ are minimal ideals of R with $(\tilde{a}_i)^2\neq0$, $(\tilde{b}_j)^2=0$ and $R\tilde{a}_i$, $R\tilde{b}_j$ and $\tilde{a}_iR$, $\tilde{b}_jR$ are minimal left and right ideals respectively, $\tilde{L}$ is a left ideal of R. Then we have the notion of left standard formula having the most length as the right one we stated above If (**) is such a left standard formula having the most length, then $\sum_{i=1}^{\tilde{n}}\oplus(\tilde{a}_i)\oplus\sum_{j=1}^{\tilde{s}}\oplus(\tilde{b}_j)$ is called the left normal socle of R. Denote $\tilde{E}_1=\sum_{i=1}^{\tilde{n}}\oplus(\tilde{a}_i)$, $\tilde{E}_2=\sum_{j=1}^{\tilde{s}}\oplus(\tilde{b}_i)$. Then $\tilde{E}_1$ is called the non-nilpotent part of $\tilde{E}$, $\tilde{E}_2$ the nilpotent part of $\tilde{E}$. Similarly, we can show that $\tilde{E}$, $\tilde{E}_1$ and $\tilde{E}_2$ are independent of the choice of the forms of standard formulas with the most length. It is easy to see that the normal socle of R is equal to the left normal socle of R. Infact, let $E=\sum_{i=1}^{n}\oplus(a_i)\oplus\sum_{j=1}^{s}\oplus(b_j)$ be the right normal socle, where $(a_i)=Ra_iR$, $(b_j)=Rb_jR$ and Ra_i, Rb_j and a_iR, b_jR are left and right minimal ideals respectively. Hence $E\subset\tilde{E}$, $n+s\leqslant\tilde{n}+\tilde{s}$, i. e. the most, length of right normal socle of R is not less than the length of left normal socle of R. Symmetrically we can show that $\tilde{n}+\tilde{s}\leqslant n+s$. Hence $E=\tilde{E}$, i. e. the right normal socle is equal to the left normal socle. It is clear that we can introduce the notion of the normal socle of R. Using the normal socle E of R we set $\bar{R}=R/E$. Clearly $\bar{R}$ is left and right Noetherian. Then we obtain again the normal socle $\bar{E}^{(1)}$ of $\bar{R}$ and

$$\bar{E}_1^{(1)}=\sum_{i=1}^{n_1}\oplus(\bar{a}_i^{(1)})\oplus\sum_{j=1}^{s_1}\oplus(\bar{b}_j^{(1)}),$$

where $(\bar{a}_i^{(1)})^2\neq0$, $(\bar{b}_j^{(1)})^2=0$, and $(\bar{a}_i)$, $(\bar{b}_j)$ are all minimal ideals of $\bar{R}$. We can go on in this way and obtain inductively the following formula

$$E^{(m)}=E^{(m-1)}+\sum_{i=1}^{n_{m-1}}(a_i^{(m-1)})+\sum_{j=1}^{s_{m-1}}(b_j^{(m-1)}), \tag{$\triangle$}$$

where $E^{(0)}=E$, $E^{(-1)}=0$ and

$$\bar{E}^{(m)}=\sum_{i=1}^{n_{m-1}}\oplus(\bar{a}_i^{(m-1)})\oplus\sum_{j=1}^{s_{m-1}}\oplus(\bar{b}_j^{(m-1)})$$

is the normal socle of $R/E^{(m-1)}=\bar{R}$,

$$\bar{E}_1^{(m)}=\sum_{j=1}^{n_{m-1}}\oplus(\bar{a}_i^{(m-1)})$$

is the non-nilpotent part of $\bar{E}^{(m)}$,

$$\bar{E}_2^{(m)}=\sum_{j=1}^{s_{m-1}}\oplus(\bar{b}_j^{(m-1)})$$

is the nilpotent part of $\bar{E}^{(m)}$.

We call $E^{(m)}$ of ($\triangle$) the m-graded socle of R, where $m=0, 1, \cdots$. Denote by A the Artinian radical of R. Then because R is left and right Noetherian, there exists a non-negative integer m such that $A=E^{(m)}$.

Definition 1. *Let A be an Artinian radical of R, and E be the normal socle of R, i. e. 0-gradational socle of R. Denote $E=E^{(0)}$. Then the following chain of ideals*

$$E=E^{(0)}\subset E^{(1)}\subset\cdots\subset E^{(l-1)}\subset E^{(l)}\subset\cdots\subset E^{(m)}=A$$

is called the acsending chain of normal socles of Artinian radical, where $E^{(l)}$ *is the normal socle of* $\bar{R}=R/E^{(l-1)}$, i. e.

$$E^{(l)}=E^{(l-1)}+(a_1^{(l)})+\cdots+(a_{n_l}^{(l)})+(b_1^{(l)})+\cdots+(b_{s_l}^{(l)})$$

for every 1 *and* $(\bar{a}^{(l)})$, $(\bar{b}^{(l)})$ *are minimal ideals of* $\bar{R}$ *and* $(\bar{a}_i^{(l)})^2=(\bar{a}_i^{(l)})$, $(\bar{b}_j^{(l)})^2=\bar{0}$. *Moreover, we call*

$$E_1^{(l)}=E^{(l-1)}+(a_1^{(l)})+\cdots+(a_{n_l}^{(l)})$$

and

$$E_2^{(l)}=E^{(l-1)}+(b_1^{(l)})+\cdots+(b_{s_l}^{(l)})$$

the non-nilpotent part and the nilpotent part of $E^{(l)}$ *respectively.*

Now we consider the following

$$E^{(k)}=E_1^{(k)}+E_2^{(k)},\quad E_1^{(k)}=E^{(k-1)}+\sum_{i=1}^{n_{k-1}}(a_i^{(k)}),$$

$$E_2^{(k)}=E^{(k-1)}+\sum_{j=1}^{s_1}(b_j^{(k)}).$$

Write $\bar{R}=R/E^{(k-1)}$. Then

$$\bar{E}_1^{(k)}=\sum_{i=1}^{n_{k-1}}\oplus(\bar{a}_i^{(k)}),\quad \bar{E}_2^{(k)}=\sum_{j=1}^{s_{k-1}}\oplus(\bar{b}_j^{(k)})$$

are non-nilpotant part and nilpotent part of R respectively. Denote

$$\bar{\mathfrak{p}}_{(\bar{a}_i^{(k)})}=\{\bar{r}\in\bar{R}\mid(\bar{a}_i^{(k)})\bar{r}=\bar{0}\}.$$

Since $(\bar{a}_i^{(k)})$ is minimal ideal of $\bar{R}$, it is clear that $\bar{\mathfrak{p}}_{(\bar{a}_i^{(k)})}$is prime ideal of $\bar{R}$. Because $(\bar{a}_i^{(k)})$ is Artinian right ideal, $\bar{R}/\bar{\mathfrak{p}}_{(\bar{a}_i^{(k)})}$is Artinian ring by Lemma4.5c)in[1]. From Theorem 1.24 in [1] it follows that $\bar{\mathfrak{p}}_{(\bar{a}_i^{(k)})}$is maximal prime ideal of $\bar{R}$. Write

$$\mathfrak{p}_{(a_i^{(k)})}=\{r\in R\mid(a_i^{(k)})r\subset E^{(k-1)}\}.$$

Then from $\bar{R}/\bar{\mathfrak{p}}_{(\bar{a}_i^{(k)})}\cong R/\mathfrak{p}_{(a_i^{(k)})}$ it follows that $\mathfrak{p}_{(a_i^{(k)})}$ is maximal prime ideal of R. Denote

$$r^{(k)}(E_1^{(k)})=\{r\in R\mid E_1^{(k)}r\subset E^{(k-1)}\},\quad l^{(k)}(E_1^{(k)})=\{r\in R\mid rE_1^{(k)}\subset E^{(k-1)}\},$$

We can show that

$$r^{(k)}(E_1^{(k)})=\bigcap_{i=1}^{n_k}\mathfrak{p}_{(a_i^{(k)})}=l^{(k)}(E_1^{(k)}).\tag{1}$$

In fact, from $(a_i^{(k)})r(E_1^{(k)})\subset E^{(k-1)}$ and $(a_i^{(k)})\not\subset\mathfrak{p}_{(a_i^{(k)})}$ it follows that

$$r^{(k)}(E_1^{(k)})\subseteq\bigcap_{i=1}^{n_k}\mathfrak{p}_{(a_i^{(k)})}.$$

Since$(a_i^{(k)})\cap\mathfrak{p}_{(a_i^{(k)})}\subseteq E^{(k-1)}$ for every $(a_i^{(k)})$, $i=1,\cdots,n_k$ and $\mathfrak{p}_{(a_i^{(k)})}$ is atmaximal prime ideal, it is easy to show that (1) is true.

Now we want to show that the foregoing ideals $\mathfrak{p}_{(a_1^{(k)})},\cdots,\mathfrak{p}_{(a_{n_k}^{(k)})}$ are all minimal prime ideals of R. In fact,let $\bar{N}$ be the nilpotent radical of $\bar{R}=R/E^{(k-1)}$, then

$$\bar{N}=\bigcap_{i=1}^{t}\bar{P}_i,$$

where $\bar{P}_i$ are minimal prime ideals of $\bar{R}$. From $(\bar{a}_i^{(k)})\bar{\mathfrak{p}}_{(\bar{a}_i^{(k)})}=\bar{0}$ it follows that there exists an element $\bar{P}_i$ of $\bar{P}_1,\cdots,\bar{P}_t$ such that $\bar{\mathfrak{p}}_{(\bar{a}_i^{(k)})}\subseteq\bar{P}_i$. Because of the maximum of

$\bar{\mathfrak{p}}^{(\bar{a}_i^{(k)})}$ we have $\bar{\mathfrak{p}}^{(\bar{a}_i^{(k)})}=\bar{P}_i$. Hence $\mathfrak{p}^{(a_i^{(k)})}$ is a minimal prime ideal of R.

We are going to show that

$$\bar{E}_1^{(k)}\oplus r^{(k)}(\bar{E}_1^{(k)})=\bar{R}. \tag{2}$$

In fact, since $\bar{R}=(\bar{a}_i^{(k)})\oplus\bar{\mathfrak{p}}^{(\bar{a}_i^{(k)})}$ for $i=1,\cdots,n_k$, we have $\bar{\mathfrak{p}}^{(\bar{a}_2^{(k)})}=(\bar{a}_1^{(k)})\oplus\bar{\mathfrak{p}}^{(\bar{a}_1^{(k)})}\cap\bar{\mathfrak{p}}^{(\bar{a}_2^{(k)})}$, $\bar{R}=(\bar{a}_1^{(k)})\oplus(\bar{a}_2^{(k)})\oplus\bar{\mathfrak{p}}^{(\bar{a}_1^{(k)})}\cap\bar{\mathfrak{p}}^{(\bar{a}_2^{(k)})}$. We can go on in such way and obtain

$$\bar{R}=\bar{E}_1^{(k)}\oplus r^{(k)}(\bar{E}_1^{(k)}).$$

Now we consider

$$\bar{E}_2^{(k)}=\sum_{j=1}^{s_k}\oplus(\bar{b}_j^{(k)}),\quad(\bar{b}_j^{(k)})^2=\bar{0},$$

and denote $\mathfrak{p}'^{(b_j^{(k)})}=\{r\in R\mid(b_j^{(k)})r\subset E^{(k-1)}\}$. As before we can show that $\mathfrak{p}'^{(b_j^{(k)})}$ is a maximal prime ideal of R and $(b_j^{(k)})\subset\mathfrak{p}'^{(b_j^{(k)})}$ for $j=1,\cdots,s_k$. Similarly we can prove that

$$r^{(k)}(\bar{E}_2^{(k)})=\bigcap_{j=1}^{s_k}\bar{\mathfrak{p}}'^{(\bar{b}_j^{(k)})}, \tag{3}$$

$$l^{(k)}(\bar{E}_1^{(k)})=\bigcap_{j=1}^{s_k}\bar{\mathfrak{p}}''^{(\bar{b}_j^{(k)})}, \tag{4}$$

where $\bar{\mathfrak{p}}''^{(\bar{b}_j^{(k)})}$ are maximal prime ideals of R. In general $\bar{\mathfrak{p}}'^{(\bar{b}_j^{(k)})}\neq\bar{\mathfrak{p}}''^{(\bar{b}_j^{(k)})}$, $\bar{\mathfrak{p}}^{(k)}(\bar{E}_2^{(k)})\neq l^{(k)}(\bar{E}_2^{(k)})$.

From (1), (2) and $\bar{N}^{(k)}=\bigcap_{i=1}^{t}\bar{P}_i^{(k)}$, where $\bar{N}^{(k)}$ is the nilpotent radical of $\bar{R}=R/E^{(k-1)}$, we have

$$\bar{P}_{n+1}^{(k)}\cap\cdots\cap\bar{P}_t^{(k)}=\bar{E}_1^{(k)}\oplus\bar{N},\quad r^{(k)}(E_1^{(k)})=\bigcap_{i=1}^{n_k}\bar{P}_i. \tag{5}$$

We sum up the obtained results in the following

Proposition 1. *Let R be a left and right Noetherian ring, A be the Artinian Radical of R. Then there exists an ascending chain of different graded normal socles of R (see Definition 1). Let $E^{(k)}$ be the normal socle of $\bar{R}=R/E^{(k-1)}$ and let $N^{(k)}$ be the original imagee of nilpotent radical $\bar{N}^{(k)}$ of $\bar{R}$, $\bar{\mathfrak{p}}^{(a_i^{(k)})}$ and $r^{(k)}(E_1^{(k)})$ having the same meaning asbefore. Then we have the following relations:*

$$r^{(k)}(E_1^{(k)})=\bigcap_{i=1}^{n_k}\mathfrak{p}^{(a_i^{(k)})}=l^{(k)}(E_1^{(k)}), \tag{6}$$

$$r^{(k)}(E_2^{(k)})=\bigcap_{j=1}^{s_k}\mathfrak{p}'^{(b_j^{(k)})}, \tag{7}$$

$$l^{(k)}(E_2^{(k)})=\bigcap_{j=1}^{s_k}\mathfrak{p}''^{(b_j^{(k)})}, \tag{8}$$

$$R=E_1^{(k)}+r^{(k)}(E_1^{(k)}),\ E_1^{(k)}\cap r^{(k)}(E_1^{(k)})\subset E^{(k-1)}, \tag{9}$$

$$E^{(k)}=E_1^{(k)}+E_2^{(k)}, \tag{10}$$

where $\mathfrak{p}^{(a_i^{(k)})}$, $\mathfrak{p}'^{(b_j^{(k)})}$ and $\mathfrak{p}''^{(b_j^{(k)})}$ are maximal prime ideals of R. Moreover, if we set

$$N^{(k)}=\bigcap_{i=1}^{t_k}P_i^{(k)},$$

the intersection of prime ideals $P_i^{(k)}$ of R, then

$$r^{(k)}(E_1^{(k)})=\bigcap_{i=1}^{n_k}P_i^{(k)},\ P_i^{(k)}=\mathfrak{p}(a_i^{(k)}),\ i=1,\cdots,n_k\leqslant t_k, \tag{11}$$

$$E_1^{(k)}+N^{(k)}=P_{n_k+1}^{(k)}\cap\cdots\cap P_{t_k}^{(k)}. \tag{12}$$

Denote by m the least positive integer which satisfies $A=E^{(m)}$, and by $E_1^{(m)}$ the non-nilpotent part of $E^{(m)}$. Then

$$R=E_1^{(m)}+\bigcap_{k=1}^{n_m}\mathfrak{p}_k,\ E_1^{(m)}\cap\Big(\bigcap_{k=1}^{n_m}\mathfrak{p}_k\Big)\subset E^{(m-1)}$$

and $\mathfrak{p}_1,\cdots,\mathfrak{p}_{n_m}$ are all maximal prime ideals of R.

Using the foregoing notations and results we can prove the following

Theorem 1. *Let R be a left and right Noetherian ring, A be the Artinian radical of R. Denote by $r(A)$ and $l(A)$ the right and left annihilators of A respectively. Then*

$$\Big(\bigcap_{k=0}^{m}r^{(k)}(E^{(k)})\Big)^{m+1}\subset r(A)\subset\bigcap_{k=0}^{m}r^{(k)}(E^{(k)}), \tag{13}$$

$$\Big(\bigcap_{k=0}^{m}l^{(k)}(E^{(k)})\Big)^{m+1}\subset l(A)\subset\bigcap_{k=0}^{m}l^{(k)}(E^{(k)}), \tag{14}$$

where m, $r^{(k)}$ and $l^{(k)}$ have the same meaning as before.

Proof We only prove the form (13), the form (14) can be proved similarly. Let $r\in r(b)$. Then $Ar=0$. Hence $k\in r^{(k)}(E^{(k)})$ for $k=0,1,\cdots,m$, and $r^{(0)}(E^{(0)})=r(E)$. On the other hand, if $x\in\bigcap_{k=0}^{m}r^{(k)}(E^{(k)})$, then $E^{(k)}x\subset E^{(k-1)}$ for $k=0,1,\cdots,m$, where $E^{(-1)}=0$. Hence $A(x)^{m+1}=0$, $(x)^{m+1}\subset r(A)$. This proves the theorem.

As in [1] denote $C(I)=\{c\in R\,|\,c+I$ is regular element of $R/I\}$, where I is an arbitrary ideal of R.

Theorem 2. *Let R be a left and right Noetherian ring, A be the Artinian radical of R. Then $\Big(A+\Big(\bigcap_{k=0}^{m}r^{(k)}(E^{(k)})\Big)\Big)^h$ contains elements of $C(N)$, where N is the nilpotent radical of R and h is arbitrary positive number.*

Proof. If $\Big(A+\bigcup_{k=0}^{m}r^{(k)}(E^{(k)}(E^{(k)})\Big)^h\cap xR\subset N$, then we can show that $xR\subset N$. In fact, we have $xRA^h\subset N$ and $xR\Big(\bigcap_{k=0}^{m}r^{(k)}(E^{(k)})\Big)^h\subset N$. Since $N=P_l\cap\cdots\cap P_t$ is the intersection of prime ideals P_i, we have $\Big(\bigcap_{k=0}^{m}r^{(k)}(E^{(k)})\Big)^h\subset P$ if $xR\not\subset P$, where $P\in\{P_1,\cdots,P_t\}$. By Proposition 1 we have

$$\bigcap_{k=0}^{m}\Big(\bigcap_{i=1}^{n_k}\mathfrak{p}(a_i^{(k)})\cap\Big(\bigcap_{j=1}^{s_k}\mathfrak{p}'(b_j^{(k)})\Big)\Big)\subset P. \tag{15}$$

Therefore P is one of $\{\mathfrak{p}(a_1^{(k)}),\cdots,\mathfrak{p}(a_{n_k}^{(k)}),\mathfrak{p}'(b_1^{(k)}),\cdots,\mathfrak{p}'(b_{s_k}^{(k)})\}$, for example $P=\mathfrak{p}$. But since $R/\mathfrak{p}$ is Artinian and $P=\mathfrak{p}$ is aminimal prime ideal, we have $A\not\subset P$ by Lemma 4.10 in [1]. Hence from $xRA^h\subset P$ it follows that $xR\subset P$. This contradicts $xR\not\subset P$. Since P is any element of $\{P_1,\cdots,P_t\}$, we have $xR\subset N$. By Goldie theorem

$$\left(A+\left(\bigcap_{k=0}^{m} r^{(k)}(E^{(k)})\right)^{h}\right.$$

contains elements of $C(N)$.

Therefore the following theorem (see [1]) follows immediately from Theorems 1 and 2.

Corollarly. *Let R be a left and right Noetherian ring, A be the Artinian radical of R. Then $A+r(A)$ contains elements of $C(N)$.*

Theorem 3. *Let R be a left and right Noetherian ring, N be the nilpotent radical and $N=\bigcap_{i=1}^{t} P_i$, the intersection of minimal prime ideals P_i. Write $P\in\{P_1, \cdots, P_t\}$. Then either* (i) *$A\subset P$, $r(A)\not\subset P$ or* (ii) *$A\not\subset P$, $r(A)\subset P$.*

Proof If $r(A)\subset P$, then $\bigcap_{k=0}^{m} r^{(k)}(E^{(k)})\subset P$ by Theorem 1. Hence R/P is Artinian for P. But P is a minimal prime ideal. Hence $P\not\supset A$ by the Lemma 4, 10 in [1].

Corollary. *Let R be a left and right Noetherian ring, P a minimal prime ideal of R. Then R/P is Artinian if and only if $A\not\subset P$.*

Proof The necessity of the condition has been proved by Lemma 4.10 in [1]. The sufficiency of the condition is now to prove. In fact, if $A\not\subset P$, then $r(A)\subset P$. Hence R/P is Artinian by the proof of Theorem 3.

Theorem 4. *Let R be a left and right Noetherian ring and A be the Artinian radical, N be the nilpotent radical of R. Then $R=A+\bigcap_{\substack{i=1,\cdots,n_k\\ k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})$ and $A\cap\Big(\bigcap_{\substack{i=1,\cdots,n_k\\ k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})\Big)\subset N$, where $\mathfrak{p}(a_i^{(k)})$ is the maximal prime ideal stated in form* (6).

Proof By (6) $\mathfrak{p}(a_i^{(k)})$ is a maximal prime ideal. Hence $R=(a_i^{(k)})+\mathfrak{p}(a_i^{(k)})$. If $k=0$, then

$$R=E_1+\bigcap_{i=1}^{n_0}\mathfrak{p}(a_i^{(0)})$$

from (2). If $k=1$, then it is easy to see that

$$\mathfrak{p}(a_i^{(1)})=E_1+\left(\bigcap_{i=1}^{n_0}\mathfrak{p}(a_i^{(0)})\cap\mathfrak{p}(a_i^{(1)})\right).$$

Hence

$$R=E_1+(a_1^{(1)})+\left(\bigcap_{i=1}^{n_0}\mathfrak{p}(a_i^{(0)})\cap\mathfrak{p}(a_i^{(1)})\right).$$

By induction we obtain

$$R=E_1+E_1^{(1)}+\bigcap_{i=1}^{n_0}\mathfrak{p}(a_i^{(0)})\cap\left(\bigcap_{j=1}^{n_1}\mathfrak{p}(a_j^{(1)})\right)$$

and

$$R=\sum_{k=0}^{m}E_1^{(k)}+\bigcap_{\substack{i=1,\cdots,n_k\\ k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}).$$

Since $\sum_{k=0}^{m}E_1^{(k)}\subset A$,

$$R=A+\bigcap_{\substack{i=1,\cdots,n_k\\ k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}).$$

Now we want to show that $A\cap(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))\subset N$. Let $x\in A\cap(\bigcap_{i,k}\mathfrak{p}(a_i^{(k)}))$. Then for any k we have $(a_i^{(k)})(x)\subset E^{(k-1)}$. Since $x\in A$, there exists a positive integer k such that $x\in E^{(k-1)}+(a_1^{(k)})+\cdots+(a_{n_k}^{(k)})+(b_1^{(k)})+\cdots+(b_{s_k}^{(k)})$, $x\notin E^{(k-1)}$. Therefore

$$x=e^{(k-1)}+a_1'+\cdots+a_{n_k}'+b_1'+\cdots+b_{s_k}',\ e^{(k-1)}\in E^{(k-1)},\ a_i'\in(a_i^{(k)}),\ b_j'\in(b_j^{(k)}).$$

Suppose that there exists an element $a_i'\in\{a_1',\cdots,a_{n_k}'\}$ such that $a_i'\notin E^{(k-1)}$, then

$$(a_i')+E^{(k-1)}=(a_i^{(k)})+E^{(k-1)},\ (a_i^{(k)})x+E^{(k-1)}=(a_i^{(k)})a_i'+E^{(k-1)}.$$

This contradicts $(a_i^{(k)})(x)\subset E^{(k-1)}$. Hence $a_i'\in E^{(k-1)}$ for $i=1,\cdots,n_k$. Therefore $(x)^2\subset E^{(k-1)}$, since $(b_j')^2\subset E^{(k-1)}$. Also we have

$$(x)^2\subset E^{(k-2)}+\sum_{i=1}^{n_{k-1}}(a_i^{(k-1)})+\sum_{j=1}^{s_{k-1}}(b_j^{(k-1)}).$$

Let $y\in(x)^2$. Then $(a_i^{(k)})(y)\subset E^{(k-1)}$ for all k. Analogously we can prove that

$$(y)^2\subset E^{(k-2)}+\sum_{j=1}^{s_{k-1}}(b_j^{(k-1)}).$$

Since y is an arbitrary element of (x) and R is a left and right Noetherian ring, there exists a positive integer λ such that

$$(x)^\lambda\subset E^{(k-2)}+\sum_{j=1}^{s_{k-1}}(b_j^{(k-1)}).$$

With this procedure we can find a positive integer n^* such that $(x)^{n^*}=0$. This proves our theorem.

Theorem 5. *Let R be a left and right Noetherian ring, and A be the Artinian radical of R. If A contains no nilpotent ideals of R, then*

$$R=A+r(A),\quad A=\sum_{k=1}^{m}E_1^{(k)}$$

and

$$r(A)=\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}).$$

Proof By the hypothesis and Theorem 4 we have $R=A\oplus\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})$. Hence

$$\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\subseteq r(A).$$

From this it follows that

$$r(A)=\bigcup_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\oplus(A\cap r(A)).$$

But A contains no nilpotent ideals of R. So $r(A)\cap A=0$. On the other hand, from the proof of Theorem 4 it follows that

$$R=\sum_{k=1}^{m}E_1^{(k)}+\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}).$$

Therefore $A=\sum_{k=1}^{m}E_1^{(k)}$. This proves our theorem.

Theorem 6. *Let R be a left and right Noetherian ring which has an Artinian* *quotient ring. Then for evevy nilpotent ideal I of R which is contained in A $I^{2m}=0$ holds, where m is the least integer satisfying $E^{(m)}=A$. Moreover*

$$R=A\oplus(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^{\lambda},$$

where λ is an integer$>2^m$.

Proof By Theorem 5.1 of [1] $A=eR$, e is a central idempotent element. As before we set

$$E^{(k)}=E^{(k-1)}+\sum_{i=1}^{n_k}(a_i^{(k)})+\sum_{j=1}^{s_k}(b_j^{(k)})$$

for $k=1, \cdots, m$. Since $e\in A=E^{(m)}$,

$$e=e^{(m-1)}+a_1'^{(m)}+\cdots+a_{n_m}'^{(m)}+b_1'^{(m)}+\cdots+b_{s_m}'^{(m)}=\tilde{e}^{(m-1)}+\sum_{i=1}^{n_m}a_i'^{(m)*},$$

where $\tilde{e}^{(m-1)}\in E^{(m-1)}$. Therefore we have

$$A=Re=E^{(m-1)}+\sum_{i=1}^{n_m}(a_i^{(m)})=E^{(m-2)}+\sum_{i=1}^{n_{m-1}}(a_i^{(m-1)})+\sum_{j=1}^{s_{m-1}}(b_j^{(m-1)})+\sum_{i=1}^{n_m}(a_i^{(m)}),$$

$$e=e^{(m-2)}+\sum_{i=1}^{n_{m-1}}a_i'^{(m-1)}+\sum_{j=1}^{s_{m-1}}b_j'^{(m-1)}+\sum_{i=1}^{n_m}a_i''^{(m)}=\tilde{e}^{(m-2)}+\sum_{i=1}^{n_{m-1}}a_i'^{2(m-1)}+\sum_{i=1}^{n_m}\tilde{a}_i^{(m)}.$$

Hence

$$A=eR=E^{(m-2)}+\sum_{i=1}^{n_{m-1}}(a_i^{(m-1)})+\sum_{i=1}^{n_m}(a_i^{(m)}).$$

Going on in such way we have

$$A=\sum_{i=1}^{n_0}(a_i)+\sum_{j=1}^{s_0}(b_j)+\sum_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}(a_i^{(k)}).$$

But $A=eR=A^2$, Hence

$$A=\sum_{i=1}^{n_0}(a_i)+\sum_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}(a_i^{(k)}).$$

Now we want to prove that

$$A\cap(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})^{\lambda}=0$$

for any integer $\lambda>2^m$, where $(a_i^{(0)})=(a_i)$. In fact, if $x\in A\cap(\bigcap\mathfrak{p}(a_i^{(k)}))^{\lambda}$, then $x=c^{\lambda}$, $c\in\bigcap\mathfrak{p}(a_i^{(k)})$. Since $A=eR$ and e is a centarl idempotent element, it follows that $x=ex=ec^{\lambda}=\underbrace{ec\cdots ec}_{\lambda\text{ time}}$. But $ec\in A\cap(\bigcap\mathfrak{p}(a_i^{(k)}))\subset N$ and N is nilpotent ideal. Applying the first assertion which shall be proved below we have $(ec)^{2m}=0$. But $0\neq x=\underbrace{ec\cdots ec}_{\lambda\text{ time}}$, $\lambda>2^m$. This is impossible. Hence $x=0$. On the other hand, since $R=eR\oplus(1-e)R$, we have $A(1-e)R=0$. Thereofore $(1-e)R\subset\bigcap\mathfrak{p}(a_i^{(k)})$, $(1-e)R\subset(\bigcap\mathfrak{p}(a_i^{(k)}))^{\lambda}$. This proves that $R=A\oplus(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^{\lambda}$, $(1-e)R=(\bigcap\mathfrak{p}(a_i^{(k)}))^{\lambda}$ for $\lambda>2^m$.

Now we want to prove the first assertion. Let I be a nilpotent ideal of R and $I\subset A$. We prove first that $I\subset E^{(m-1)}$. Suppose that $I\not\subset E^{(m-1)}$, Then from

$$E^{(m-1)}+(a_1^{(m)})+\cdots+(a_{n_m}^{(m)})=E^{(m)}$$

it follows that $E^{(m-1)}+I$ contains, for example, $(a_1^{(m)})$. And from $E^{(m-1)}+I\subset\mathfrak{p}(a_1^{(m)})$ it follows that $(a_1^{(m)})\subset\mathfrak{p}(a_1^{(m)})$. This is impossible. Now suppose that $I\subset E^{(k)}$, $I\not\subset E^{(k-1)}$. Since $E^{(k)}=E^{(k-1)}+(a_1^{(k)})+\cdots+(a_{n_k}^{(k)})+(b_1^{(k)})+\cdots+(b_{s_k}^{(k)})$, then if $E^{(k-1)}+I$ contains one ideal $(a_t^{(k)})$, $i\leqslant t\leqslant n_k$, then as above we obtain a contradiction that $(\bar{a}_t^{(k)})\subset\mathfrak{p}(a_t^{(k)})$. Hence $E^{(k-1)}+I\subseteq E^{(k-1)}+(b_1^{(k)})+\cdots+(b_{s_k}^{(k)})$, since $(\bar{a}_t^{(k)})$ is a minimal ideal in $\bar{R}=R/E^{(k-1)}$ and $(\bar{a}_t^{(k)})^2=(\bar{a}_t^{(k)})$. But $(\bar{b}_i^{(k)})(\bar{b}_j^{(k)})=\bar{0}$ for $i,\ j=1,\ \cdots,\ s_k$. Hence $\bar{I}^2=\bar{0}$. Write $I_1=I^2$. Then as before we can prove that there exists an $E^{(k'-1)}$ such that the ideal $\tilde{I}_1$ of $\tilde{R}=R/E^{(k'-1)}$ satisfies $\tilde{I}_1^2=\tilde{O}$. We go on in such way and finally obtain $I^{2^m}=0$. This proves our theorem.

Corollary. *Let R be a left and right Noetherian ring which has an Artinian quotient ring. Then the Artinian radical*

$$A=\sum_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}(a_i^{(k)}),\ r(A)^\lambda=l(A)^\lambda=\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\Big)^\lambda=(1-e)R,\ R=A\oplus r(A)^\lambda,$$

where $\lambda>2^m$ and e is a central idempotent element of R such that $A=eR$.

Proof We *only* show that

$$r(A)^\lambda=l(A)^\lambda=\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\Big)^\lambda.$$

By Theorem 6 $\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\Big)^\lambda=(1-e)R$. On the other hand, from $eR(1-e)R=A(1-e)R=0$ it follows that $(1-e)R\subseteq r(A)$. Clearly $r(A)=\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})$. Hence

$$r(A)^\lambda=\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\Big)^\lambda.$$

Similarly, from $(1-e)RA=0$ if follows that $(1-e)R\subset l(A)\subset\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})$. Hence

$$l(A)^\lambda=\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\Big)^\lambda.$$

Theorem 7. *Let R be a left and right Noetherian ring, A be the Artinian radical. Then A has a central idempotent element e such that $A=eR$ if and only if there exists a positive integer $\lambda>2^m$ such that*

$$r(A)^\lambda=l(A)^\lambda=\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\Big)^\lambda.$$

Proof The necessity of the condition has been already proved by Theorem 6 and its corollary· Now we want to prove the sufficiency of the condition. From Theorem 4 it follows that

$$R=A+\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\Big)^\lambda,$$

where λ is an arbitrary positive integer. Then

$$1=e+e',\ e\in A,\ e'\in\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})\Big)^\lambda.$$

By the hypothesis of our theorem it is easy to see that $e=e^2$, $e'=e'^2$, $ee'=e'e=0$. Hence $R=eR\oplus(1-e)R$. Since $eR\subset A, (1-e)R\subset r(A)^\lambda=l(A)^\lambda$, $\lambda>2^m$, we get

$$eR(1-e)R=0, (1-e)ReR=0,$$

i. e. for any $r\in R$ we have $er=re$, e is a central idempoteot element of R. Since

$$(1-e)R\subset(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^\lambda, R=eR+(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^\lambda.$$

If $x=er=c^\lambda$, $c\in\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)})$, then $x=ec^\lambda=\underbrace{ec\cdots ec}_{\lambda\text{ time}}$. From Theoran 4 it follows that

$$A\cap(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))\subset N.$$

Because R is left and right Noetherian, there exists a positive integer t such that $N^t=0$. Choosing $\lambda>t+2^m$ we obtain $x=\underbrace{ec\cdots ec}_{\lambda\text{ time}}=0$ from $ec\in A\cap(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))$. This proves that

$$eR\cap(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^\lambda=0$$

for a suitable integer λ. Hence $R=eR\oplus(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^\lambda$. From this it follows that

$$(1-e)R=(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^\lambda=r(A)^\lambda.$$

It is clear that $A=eR\oplus A\cap(1-e)R$. Write $x\in A\cap(1-e)R$. Then from $Ar(A)^\lambda=0$ it follows that $x(1-e)=0$, i. e. $x\in eR$, $x=0$. This proves $A=eR$.

Theorem 8. *Let R be a left and right Noetherian ring and A be the Artinian radical of R. Then A has a central idempotent element e such that $A=eR$ if and only if the $\mathfrak{p}'(b_j^{(k)})=\{r\in R\,|\,(b_j^{(k)})r\subset E^{(k-1)}\}$ and the $\mathfrak{p}''(b_j^{(k)})=\{r\in R\,|\,r(b_j^{(k)})\subset E^{(k-1)}\}$ all belong to the set $\{\mathfrak{p}(a_i^{(k)})\}_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}$, where $j=1,\cdots,s_k$; $k=0,\cdots,m$.*

Proof Necessity: By Theorem 7

$$r(A)^\lambda=l(A)^\lambda=(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^\lambda.$$

It is easy to see from the above definition that

$$r^{(k)}(E^{(k)})=\Big(\bigcap_{i=1}^{n_k}\mathfrak{p}(a_i^{(k)})\Big)\cap\Big(\bigcap_{j=1}^{s_k}\mathfrak{p}'(b_j^{(k)})\Big),$$

$$l^{(k)}(E^{(k)})=\Big(\bigcap_{i=1}^{n_k}\mathfrak{p}(a_i^{(k)})\Big)\cap\Big(\bigcap_{j=1}^{s_k}\mathfrak{p}''(b_j^{(k)})\Big).$$

Hence from (13) and (14) it follows that any $\mathfrak{p}'(b_j^{(k)})$ and $\mathfrak{p}''(b_j^{(k)})$ contain $(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}))^\lambda$, Therefore from the property of maximal prime ideal $\mathfrak{p}(a_i^{(k)})$, follows. immediately the assertion of the theorem, where $k=0,\cdots,m$, $i=1,\cdots,n_k$.

Sufficiency: By the hypothesis

$$\bigcap_{k=0}^{m}r^{(k)}(E^{(k)})=\bigcap_{k=0}^{m}r^{(k)}(E_1^{(k)})=\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}}\mathfrak{p}(a_i^{(k)}).$$

From (1), (13) and (14) it follows that

$$\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)}) \supset r(A) \supset \Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})\Big)^{m+1}, \quad \bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)}) \supset l(A) \supset \Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})\Big)^{m+1}.$$

From the proof of Theorem 7 we have

$$\Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})\Big)^{\lambda} + A = R,$$

where λ can be an arbitraay positive integer. Therefore $R = A + (r(A) \cap l(A))^{\lambda}$. Write $1 = e + e'$, $e \in A$, $e' \in r(A) \cap l(A)$. We have $e^2 = e$, $e'^2 = e'$, $ee' = e'e = 0$, $R = eR \oplus (1-e)R$. Hence $eR(1-e) = 0$, $(1-e)Re = 0$. This means that for any $r \in R$ we have $r = er$, where e is a central idempotent element of R, Because of

$$(1-e)R \subset (r(A) \cap l(A))^{\lambda} \subset \Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})\Big)^{\lambda}, \quad R = eR + \Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})\Big)^{\lambda}.$$

Similarly, as in the proof of Theorem 7 we have

$$eR \cap \Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})\Big)^{\lambda} = 0.$$

Hence

$$(1-e)R = \Big(\bigcap_{\substack{i=1,\cdots,n_k\\k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)})\Big)^{\lambda} = (r(A) \cap l(A))^{\lambda}.$$

It is easy to see that $A = eR \oplus A \cap (1-e)R$. If $x \in A \cap (1-e)R$, then from $x(r(A) \cap l(A))^{\lambda} \subset Ar(A) = 0$ it follows that $x(1-e) = 0$, $x = ex \in eR \cap (1-e)R = 0$. Hence $A = eR$.

Theorem 9. *Let R be a left and right Noetherian ring which has Artinian quotient ring and A be the Artinian radical. Then $\bigcap_c r(A)^{\lambda} c = \bigcap_c c r(A)^{\lambda} = 0$, $\bigcap_c l(A)^{\lambda} c = \bigcap_c c l(A)^{\lambda} = 0$, where c denotes all regular elements of R, λ is a positive integer.*

Proof By *Theorem* 5.1 $A = \bigcap_c Rc = \bigcap_c cR$. By Theorem 6 and its corollary

$$\bigcap_c Rc = A \oplus \bigcap_c (Rc \cap r(A)^{\lambda}) = A \oplus \bigcup_c r(A)^{\lambda} c.$$

Hence $\bigcap_c r(A)^{\lambda} c = 0$. Analogously we can show that $\bigcap_c c r(A)^{\lambda} = 0$. On the other hand, since R is left and right Noetherian, the right Artinian radical is also lelt one. Faom the A assumption that R has Artinian quotient ring it follows symmetrically that $\bigcap_c l(A)^{\lambda} c = \bigcap_c c(A)^{\lambda} = 0$.

§ 2.

In this section we shall discuss mainly the following problem: When does a left and right Noetherian ring with Krull dimension 1 have quotient ring? In the preceding section we knew from Theorems 7 and 8 that if R is a left and right Noetherian ring which has Artinian quotient ring, then the Artinian radical A of R must have

$$\mathscr{r}(A)^\lambda = l(A)^\lambda = (\bigcap_{\substack{i=1,\cdots,n_k \\ k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)}))^\lambda,\ \lambda > 2^m;$$

this is equivalent to saying that $\mathfrak{p}'(b_j^{(k)})$ and $\mathfrak{p}''(b_r^{(k)})$ all belong to the set $\{\mathfrak{p}(a_i^{(k)})\}_{\substack{i=1,\cdots,n_k \\ 0=k,\cdots,m}}$. Now we ask whether R has Artinian quotient ring if the Artinian radiaal A of R satisfies the above stated condition. To reply this quastion we need to establish the following theorem which expands Lenagan's theorem (see [1], p. 73, Theorem 5.6).

Theorem 10. *Let R be a left and right Noetherian ring with Krull dimension one and let A be the Artinian radical. Then R has Artinian quotient ring if and only if* $\mathscr{r}(A)^\lambda = l(A)^\lambda = (\bigcap_{\substack{i=1,\cdots,n_k \\ k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)}))^\lambda$.

Proof The necessity of the condition has been shown by Theorems 5.1 and 7 in [1]. Now we need only to show the sufficiency of the condition. If

$$\mathscr{r}(A)^\lambda = l(A)^\lambda = (\bigcap_{\substack{i=1,\cdots,n_k \\ k=0,\cdots,m}} \mathfrak{p}(a_i^{(k)}))^\lambda,$$

then from Theorem 8 it follows that $R = A \oplus \mathscr{r}(A)^\lambda$. Write $S = A \oplus \mathscr{r}(A)^\lambda$. Then we can show that the Krull dimension of S is one. In fact, the minimal left and right ideals of S are also the left and right ones of R, and $R/A\oplus L \cong \mathscr{r}(A)^\lambda/L$, where L is a right (left) ideal which contains nilpotent radical N of S. Set $\bar{S} = S/N$. Then $\bar{L}$ is essential right (left) ideal of $\bar{S}$ if and only if $A\oplus L/\tilde{N}$ is essential right (left) ideal of $R/\tilde{N}$, where $\tilde{N}$ is nilpotent radical of R. It is clear that $\tilde{N} \supset N$. Because the Krull dimension of R is one, the Krull dimension of S is also one. Since the Artinian radical of S is zero, by Lenagan's theorem S has Artinian quotient ring. It is well-known that the Artinian quotient ring of A is itself. On the other hand, if c is regular element of R, then $c = c_1 + c_2$, $c_1 \in A$, $c_2 \in S$, c_1 and c_2 are regular elements of A and S respectively. It is easy to see that the Artinian quotient ring of R is the sum of the Artinian quotient rings of A and S. This proves the theorem.

Theorem 11. *Let R be a left and right Noetherian ring with Krull dimension one, let A be the Artinian radical of R,*

$$\mathscr{r}(A)^\lambda = l(A)^\lambda = (\bigcap_{\substack{i=1,\cdots,n_k \\ k=0,\cdots,m}} \mathfrak{p}(_i^{(k)}))^\lambda,$$

and let K be a right ideal of R, $K \supset A$. Then R/K is Artinian as right R-module if and only if K contains regular elements.

Proof From the hypothesis it follows that $R = A \oplus \mathscr{r}(A)^\lambda$. Hence $K = A \oplus B$, $B = K \cap \mathscr{r}(A)^\lambda$, and $R/K \cong \lambda(A)^\lambda/B$. Since the Artinian radical of ring $\mathscr{r}(A)^\lambda$ is zero, by Lenagan's theorem (see p. 74 in [1]) $\mathscr{r}(A)^\delta/B$ is Artinian as right $\mathscr{r}(A)^\lambda$-module if and only if B contains the regular element c' of $\mathscr{r}(A)$. Hence R/K is Artinian as R-module if and only if K contains regular element $e + c'$, where $A = eR$, e is the central idempotent element of R. This proves the theorem.

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