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# HIGHER COMMUTATORS OF PSEUDO-DIFFERENTIAL OPERATORS

QIAN TAO(钱 涛)*

**Abstract**

In this paper the following result is established: For $a_i, f\in\mathscr{S}(R^K)$, $i=1, \cdots, n$, and

$$T(a, f)(x)=\omega(x, D)\left(\prod_{i=1}^{n}P_{m_i}(a_i, x, \cdot)f(\cdot)\right),$$

it holds that

$$\|T(a, f)\|_q\leqslant C\|f\|_{p_0}\prod_{i=1}^{n}\|\nabla^{m_i}a_i\|_{p_i},$$

where $a=(a_1, \cdots, a_n)$, $q^{-1}=p_0^{-1}+\sum_{i=1}^{n}p_i^{-1}\in(0, 1)$, $\forall i$, $p_i\in(1, \infty)$ or $\forall i$, $p_i=\infty$, $p_0\in(1, \infty)$, for an integer $m_i\geqslant 0$,

$$P_{m_i}(a_i, x, y)=a_i(x)-\sum_{|\beta|<m_i}\frac{a_i^{(\beta)}(y)}{\beta!}\cdot(x-y)^\beta,$$

$\omega(x, \xi)$ is a classical symbol of order $|m|$, $m=(m_1, \cdots, m_n)$, $|m|=m_1+\cdots+m_n$, $m_i$ are nonnegative integers. Besides, a representation theorem is given.

The methods used here closely follow those developed by Coifman, R. and Meyer, Y. in [5] and by Cohen, J. in [3].

## § 1. A Representation Theorem

(1.1). Let $g\in C^m(R^K)$, $m\in Z$, and $Z$ denote the set of nonnegative integers. We define the remainder operator of Taylor series [3]

$$R_{-\alpha}^{m}g(\xi)=g(\xi-\alpha)-\sum_{|\beta|<m}\frac{g^{(\beta)}(\xi)}{\beta!}(-\alpha)^\beta,$$

where $\beta=(\beta_1, \cdots, \beta_K)$, $\beta_i\in Z$, $\beta!=\beta_1!\cdots\beta_K!$, $|\beta|=\beta_1+\cdots+\beta_K$, $g^{(\beta)}=\partial^\beta g$, $\xi, \alpha\in R^K$.

If $m=(m_1, \cdots, m_n)\in Z^n$, $\alpha=(\alpha_1, \cdots, \alpha_n)\in(R^K)^n$, $|m|=m_1+\cdots+m_n$, $g\in C^{|m|}(R^K)$, then the $n$-fold composition of remainder oprator of Trylor series an be introduced as

$$R_{(-\alpha)}^{(m)}g(\xi)=R_{\alpha_{-1}}^{m_1}\circ\cdots\circ R_{-\alpha_n}^{m_n}g(\xi).$$

Denote by $S^l(R^K\times R^K)$ the class of the symbols of order l:

$$S^l(R^K\times R^K)=\{\omega\in C^\infty(R^K\times R^K)\colon \{\partial_\xi^\beta\partial_x^\alpha\omega(x, \xi)|\leqslant C_{\alpha,\beta}(1+|\xi|)^{l-|\beta|}\}, l\in R^1.$$

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* Department of Mathematics, Beijing University, Beijing, China.

The following theorem is established:

**Theorem 1.** *Let* $\omega(x, \xi)\in S^{|m|}(R^K\times R^K)$ *and for every fixed* $x\in R^k, \omega(x,\cdot)\in C_0^\infty(R^K)$. *Denote*

$$L(x, y)=\int_{R^K} e^{iy\xi}\omega(x, \xi)d\xi.$$

*Then for* $f, a_i\in\mathscr{S}(R^K)$, *we have*

$$\int_{R^k}\prod_{i=1}^{n} P_{m_i}(a_i, x, y)L(x, x-y)f(y)dy$$

$$=C\int_{(R^K)^{n+1}} e^{ix\xi}R^{(m)}_{(-\alpha)}\omega(x, \xi)\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha\, d\xi,$$

*where* $\hat{a}(\alpha)=\hat{a}_1(\alpha_1)\cdots\hat{a}_n(\alpha_n)$, $[\alpha]=\alpha_1+\cdots+\alpha_n$, $d\alpha=d\alpha_1\cdots d\alpha_n$, *and* $C$ *is a constant.*

*Proof* Using the following multiple index notation

$$\mathscr{G}=\{J\subset\{1,\cdots, n\}: J=(j_1, \cdots, j_t), 1\leqslant j_1<\cdots<j_t\leqslant n\},$$

$$J'=\{1,\cdots, n\}\backslash J,$$

$$|J|=\text{number of elements in } J,$$

$$k_J=(k_{j_1}, \cdots, k_{j_t}), k_{j_i}=(k^1_{j_i},\cdots, k^K_{j_i})\in Z^K,$$

$$k_J!=k_{j_1}!\cdots k_{j_t}!, [k_J]=k_{j_1}+\cdots+k_{j_t}, |k_{j_i}|=k^1_{j_i}+\cdots+k^K_{j_i},$$

$$N_J=\{k_J: J\in\mathscr{G}, 0\leqslant k_{j_i}\leqslant m_{j_i}-1\},$$

$$(-\alpha_J)^{K_J}=(-\alpha_{j_1})^{k_{j_1}}\cdots(-\alpha_{j_t})^{k_{j_t}},$$

we have

$$R^{(m)}_{(-\alpha)}g(\xi)=\sum_{J\in\mathscr{G}}\sum_{k_J\in N_J}\frac{(-1)^{|J_g|}g^{([k_J])}(\xi-\sum_{j\in J'}\alpha_j)}{k_J!}(-\alpha_J)^{K_J}. \tag{1.1.1}$$

With the notation

$$\hat{a}_J(\alpha)=\hat{a}_{j_1}(\alpha_{j_1})\cdots\hat{a}_{j_t}(\alpha_{j_t}),$$

$$a_J^{(k_J)}(x)=-_{j_1}^{(k_{j_1})}(x)\cdots a_{j_t}^{(k_{j_t})}(x),$$

$$(a_J^{(k_J)})^\wedge_J(\alpha)=(a_{j_1}^{(k_{j_1})})^\wedge(\alpha_j)\cdots(a_{j_t}^{(k_{j_t})})^\wedge(\alpha_{j_t}), d\alpha_J=d\alpha_{j_1}\cdots d\alpha_{j_t},$$

and setting $g(\xi)=\omega(x, \xi)$ in (1.1.1), we have

$$\int_{(R^K)^{n+1}} e^{ix\xi}R^{(m)}_{(-\alpha)}\omega(x, \xi)\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha\, d\xi$$

$$=\sum_{J\in\mathscr{G}}\sum_{K_J\in N_J}(-1)^{|J|}\int_{(R^K)^{n+1}} e^{ix\xi}\frac{\omega^{([k_J])}(x, \xi-\sum_{j\in J'}\alpha_j)(-\alpha)^{k_J}}{K_J!}\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha\, d\xi$$

$$=\sum_{J\in\mathscr{G}}\sum_{k_J\in N_J}(-1)^{|J|}\int_{(R^K)} e^{ix\xi}\int_{(R^K)} e^{ix\xi}\omega^{([k_J])}(x, \xi-\sum_{j\in J'}\alpha_j)\hat{a}_{J'}(\alpha)d\alpha_{J'})$$

$$\cdot\Big(\int_{(R^K)^{|J|}}(-1)^{|[k_J]|}\Big(\frac{1}{k_J!}\frac{(a^{(k_J)})^\wedge_J(\alpha)\hat{f}(\xi-\sum_{j\in J}\alpha_j-\sum_{j\in J'}\alpha_j)}{i^{|[k_J]|}}\Big)d\alpha_J\Big)d\xi.$$

Integrating in $d\alpha_J$ the inner integral equals

$$C\Big(\frac{a_J^{(k_J)}\cdot f}{k_J!}\Big)^\wedge(\xi-\sum_{j\in J'}\alpha_j).$$

Integrating next in $d\alpha_{J'}$, and then using the following equation

$$\omega^{(k_J)}(x, \cdot)C(-a)^{|[k_J]|}((\cdot)^{[k_J]}L(x, \cdot))^\wedge,$$

we deduce

$$\int_{(R^K)^{n+1}} e^{ix\xi} R^{(m)}_{(-\alpha)}\omega(x, \xi)\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha\, d\xi$$

$$=C\sum_{J\in\mathscr{G}}\sum_{k_J\in N_J}\frac{(-1)^{|J|+|[k_J]|}}{i^{|[k_J]|}}\left(\frac{\omega^{([k_J])}(x,\cdot)(a_J^{(k_J)}\cdot f)^{\wedge}(\cdot)}{k_J!}\right)^{\vee}(x)a_{J'}(x)$$

$$=C\sum_{J\in\mathscr{G}}\sum_{k_J\in N_J}\frac{(-1)^{|J|+|[k_J]|}}{i^{|[k_J]|}}a_{J'}(x)(-i)^{|[k_J]|}$$

$$\cdot\int_{R^K}\frac{(x-y)^{[k_J]}L(x, x-y)}{k_J!}(a_J^{(k_J)}(y)f(y)dy$$

$$=C\int_{R^K}\left(\sum_{J\in\mathscr{G}}\sum_{k\in N_J}\frac{(-1)^{|J|}a_J^{(k_J)}(y)a_{J'}(x)}{k_J!}(x-y)^{[k_J]}\right)L(x, x-y)f(y)dy$$

$$=C\int_{R^K}\prod_{i=1}^{n}P_{m_i}(a_i, x, y)L(x, x-y)f(y)dy.$$

The proof is thus finished.

**Remark.** If $m_1=\cdots=m_n=1$, we have

$$\int_{(R^K)^{n+1}} e^{ix\xi}\Delta_{-\alpha_1}\circ\cdots\circ\Delta_{-\alpha_n}\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha\, d\xi$$

$$=C\int_{R^K}\prod_{i=1}^{n}(a_i(x)-a_i(y))L(x, x-y)f(y)dy$$

$$=C[a_n, \cdots, [a_1, \omega(x, D)]\cdots]f(x)$$

which is the $n$th commutator of $\omega(x, D)$, where $a_i(f)(x)=(a_i f)(x)$. Therefore we can extend the notation of commutator and call the operator in the theorem a commutator of oader $|m|$ (see [3]).

## § 2. The Boundedness of Higher Commutators The First Case: $\forall i$, $p_i\in(1,\infty)$

(2.1). For a symbol $\sigma(x, \alpha, \xi)$, we denote

$$T_\sigma(a, f)(x)=\int_{(R^K)^{n+1}} e^{ix\xi}\sigma(x, \alpha, \xi)\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha\, d\xi.$$

The main result of this paper is as follows.

**Theorem 2.** *Let* $\omega\in S^{|m|}(R^K\times R^K)$, $q^{-1}=p_0^{-1}+\sum_{i=1}^{n}p_i^{-1}\in(0, 1)$, $p_0\in(1, \infty)$, $p_i$ *satisfy one of the following two conditions*: (i) $\forall i$, $p_i\in(1 \infty)$; (ii) $\forall i$, $p_i=\infty$. *Then for* $a_i, f\in\mathscr{S}(R^K)$, *we have*

$$\|T_{R^{(m)}_{(-\alpha)}\omega}(a, f)\|_q\leqslant C\|f\|_{p_0}\cdot\prod_{i=1}^{n}\|\nabla^{m_i}a_i\|_{p_i},$$

*where* $C=C(K, n, m, C_{\alpha,\beta}, p_0, p_i)$ *is a constant,* $\|\nabla^{m_i}a_i\|_{p_i}=\sum_{|\beta|=m_i}\|\partial^\beta a_i\|_{p_i}$.

In this section, we prove the theorem for the first kind of indexes: $\forall i$, $p_i\in(1, \infty)$. Introduce the following notation:

$$\|\nabla^m a\|_p=\prod_{i=1}^n \|\nabla^{m_i} a_i\|_{p_i}, \text{ where } p=(p_1, \cdots, p_n);$$

$$M(m)=\{\sigma(x, \alpha, \xi)\in C^\infty(R^K\times(R^K)^n\times R^K): \|T_\sigma(a, f)_q\|\leqslant C\|f\|_{p_0}\|\nabla^m a\|_p,$$

$$C=C(K, n, m, \sigma, p_0, p) \text{ is a constant}\}.$$

To prove $R^{(m)}_{(-\alpha)}\omega(x, \xi)\in M(m)$ we use the induction on $|m|$. The induction hypothesis is that: For $\bar{m}=(\bar{m}_1, \cdots, \bar{m}_{\bar{n}})$, $0\leqslant|\bar{m}|<|m|$, $0\leqslant\bar{n}\leqslant n$ and $\bar{\omega}\in S^{|\bar{m}|}(R^K\times R^K)$ we have $R^{(\bar{m})}_{(-\bar{\alpha})}\bar{\omega}(x, \xi)\in M(\bar{m})$.

First we make the following observation.

Denote $J=\{i:1\leqslant i\leqslant n, m_i=0\}$ and $J'=\{1, \cdots, n\}\setminus J$. There exists

$$R^{(m)}_{(-\alpha)}\omega(x, \xi)=R^{(m_{J'})}_{(-\alpha_{J'})}\omega(x, \xi-\sum_{j\in J}\alpha_j),$$

and hence

$$T_{R^{(m)}_{(-\alpha)}\omega(x,\xi)}(a, f)(x)=Ca_J(x)T_{R^{(m_{J'})}_{(-\alpha_{J'})}\omega(x,\xi)}(a_{J'}, f)(x).$$

So we can restrict ourself to the case: $\forall i$, $m_i\geqslant 1$. And, from the above equation it follows that for $|m|=0$, $R^{(m)}_{(-\alpha)}\omega(x, \xi)\in M(m)$.

(2.2). The plan of the proof: A partition of unity of space

$$(R^K)^{n+1}=\{(\alpha_1, \cdots, \alpha_n, \xi):\alpha_i\in R^K, \xi\in R^K\}$$

permits us to decompose $R^{(m)}_{(-\alpha)}\omega(x, \xi)$ into a finite sum. There are two posibilities for the terms:

(a) The terms supported in $\{(\alpha, \xi):|\xi|>C|\alpha|\}$ lead to a kind of symbol of order 0. The estimate is obtained then by using the Coifman-Meyer's theorem ([5], Theorem 1, see (2.7), (2.8) below);

(b) The terms supported in $\{(\alpha, \xi):|\xi|\leqslant C|\alpha|\}$ lead to a subtle analysis for which a special interpolation teorem due to Coifman and Meyer is needed ([5], propotition 3, see (2.9) Lemma 5).

For the technical reasons we proceed first with some primal partitions in order to choose the biggest coordinates of the vectors $\alpha_i$, $\xi$ (see (2.3), (2.4)).

(2.3). Suppose $\varphi\in C_0^\infty(R^K)$ and $\varphi(\xi)=1$ for $|\xi|\leqslant n+1$. Writing

$$\omega(x, \xi)=\omega_1(x, \xi)+\omega_2(x, \xi),$$

where

$$\omega_1(x, \xi)=\varphi(\xi)\omega(x, \xi),$$

we can restrict ourself to the case $R^{(m)}_{(-\alpha)}\omega_2(x, \xi)\in M(m)$. In fact, let

$$\omega_1(x, \xi)=\int_{R^K} L_1(x, y)e^{-iy\xi}dy$$

as a result of regularity of $\omega_1$, $\sup_{x\in R^K}|L_1(x, y)|$ is rapidly decreasing at infinity. From Theorem 1 and the equation

$$P_{m_i}(a_i, x, x-y)=\sum_{|\alpha|=m_i}\frac{m_i}{\alpha!}(y')^\alpha\int_0^{|y|} r^{m_i-1}a_i^{(\alpha)}(x-ry')dr, \tag{2.3.1}$$

where $y'=\frac{y}{|y|}$, and using Minkowski's and Hölder's inequalities, we get

$$R^{(m)}_{(-\alpha)}\omega_1(x, \xi)\in M(m).$$

(2.4) Choose $\varphi_1, \cdots, \varphi_K \in C^\infty(R^K\setminus\{0\})$ such that $\forall_j$, $\varphi_j$ is homogeneous of degree 0, $1=\varphi_1+\cdots+\varphi_K$ on $R^K\setminus\{0\}$ and

$$\varphi_j(\xi)\neq 0 \Rightarrow |\xi_j|\geqslant \frac{1}{2}\sup(|\xi_1|, \cdots, |\xi_K|).$$

Since $|\xi|\leqslant n+1\Rightarrow \omega(x, \xi)=0$, we can write

$$\omega(x, \xi)=\varphi_1(\xi)\omega(x, \xi)+\cdots+\varphi_K(\xi)\omega(x, \xi)$$

and $\varphi_j(\xi)\omega(x, \xi)=\xi_j\tau_j(x, \xi)$, where $\tau_j(x, \xi)\in S^{|m|-1}(R^K\times R^K)$.

The folowing lemma is established.

**Lemma 1.** *Under the induction hypothesis shown in* (2.1), *for* $\tau\in S^{|m|-1}(R^K\times R^K)$ *we have*

$$R^{(m)}_{(-\alpha)}(\xi_j\tau(x, \xi))-\xi_j R^{(m)}_{(-\alpha)}\tau(x, \xi)\in M(m).$$

*Proof* Keeping in mind $m_i\geqslant 1$, and letting $F(t)=G(t)H(t)$, $0\leqslant t\leqslant 1$, $G(t)=\xi_j - t\alpha_{n,j}$, $H(t)=\tau(x, \xi-t\alpha_n)$, we have

$$\begin{aligned}R^{m_n}_{-\alpha_n}(\xi_j\tau(x, \xi)) &= F(1)-F(0)-F'(0)-\cdots-\frac{1}{(m_n-1)!}F^{(m_n-1)}(0)\\ &= G(0)(H(1)-H(0)-\cdots-\frac{1}{(m_n-1)!}H^{(m_n-1)}(0))\\ &\quad + G'(0)(H(1)-H(0)-\cdots-\frac{1}{(m_n-2)!}H^{(m_n-2)}(0))\\ &= \xi_j R^{m_n}_{-\alpha_n}\tau(x, \xi)-\alpha_{n,j}R^{m_n-1}_{-\alpha_n}\tau(x, \xi).\end{aligned}$$

Repeating this programme up to a total of $n$ times, we derive the formula

$$R^{(m)}_{(-\alpha)}(\xi_j\tau(x, \xi))=\xi_j R^{(m)}_{(-\alpha)}\tau(x, \xi)-\sum_{i=1}^{n}\alpha_{i,j}R^{(m^i)}_{(-\alpha)}\tau(x, \xi),$$

where $m^i=(m_1, \cdots, m_i-1, \cdots, m_n)$. By denoting $a^i=\left(a_1, \cdots, \frac{\partial a_i}{\partial x_j}, \cdots, a_n\right)$, there exists

$$T_{\alpha_{i,j}R^{(m^i)}_{(-\alpha)}\tau}(a, f)(x)=(-i)T_{R^{(m^i)}_{(-\alpha)}\tau}(a^i, f)(x),$$

and then the induction hypothesis can be used to $m^i$.

Now we have to prove that $\tau\in S^{|m|-1}(R^K\times R^K)\Rightarrow \xi_j R^{(m)}_{(-\alpha)}\tau(x, \xi)\in M(m)$. By introducing the class $\overline{M}(m)=\{\sigma(x, \alpha, \xi):\sigma(x, \alpha, \xi-[\alpha])\in M(m)\}$ and applying the following lemma, it is reduced to proving $\xi_j R^{(m)}_{(-\alpha)}\tau(x, \xi+[\alpha])\in \overline{M}(m)$.

**Lemma 2.** *Under the induction hypothesis shown in*(2.1), *for* $\tau(x, \xi)\in S^{|m|-1}(R^K\times R^K)$; *we have*

$$\alpha_{i,j}R^{(m)}_{(-\alpha)}\tau(x, \xi+[\alpha])\in\overline{M}(m).$$

*Proof* Without loss of generality we can suppose $i=j=1$. Since $m_i\geqslant 1$, we have

$$\alpha_{1,1}R^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha])=\alpha_{1,1}R^{m_1-1}_{-\alpha_1}\cdots R^{m_n}_{-\alpha_n}\tau(x,\ \xi+[\alpha])$$
$$-\frac{\alpha_{1,1}}{(m_1-1)!}\sum_{|\beta|=m_1-1}R^{m_2}_{-\alpha_2}\cdots R^{m_n}_{-\alpha_n}\tau^{(\beta)}(x,\ \xi+[\alpha])(-\alpha_1)^{\beta}=I_1-I_2,$$

and then the induction hypothesis can be used to $I_1$ and to each term in $I_2$.

Make a further partition of unity

$$\xi_jR^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha])=\Big(\sum_1^K\varphi_j(\xi)\Big)\Big(\sum_1^K\varphi_j(\alpha_1)\Big)\cdots\Big(\sum_1^K\varphi_j(\alpha_n)\Big)\xi_jR^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha])$$
$$=\sum\varphi_{j_0}(\xi)\varphi_{j_1}(\alpha_1)\cdots\varphi_{j_n}(\alpha_n)\xi_jR^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha]).$$

By symmetry we can restict ourtelf to the case $j_0=\cdots=j_n=1$ and prove

$$\pi(x,\ \alpha,\ \xi)=\varphi_1(\xi)\varphi_1(\alpha_1)\cdots\varphi_1(\alpha_n)\xi_jR^{(m)}_{(-\alpha)}\tau(x,\xi+[\alpha])\in\overline{M}(m),$$

where $\tau\in S^{|m|-1}(R^K\times R^K)$ and $|\xi|\leqslant n+1\Rightarrow\tau(x,\ \xi)=0$.

(2.5) We need the following formula.

**Lemma 3.** *Let $m\in Z$, $m\geqslant1$. Then for $F\in C^{m-1}(R^K)$ and $\alpha,\ \xi\in R^K$,*

$$R^m_{-\alpha}F(\xi)=\sum_{0\leqslant|k|<m}\frac{(-\alpha)^k}{k!}\sum_{r=j_k+1}^{K}R^{m-|k|}_{-\alpha_r}F^{(k)}(\xi-\bar\alpha(r+1)),$$

*where $k=(k_1,\ \cdots,\ k_{K-1},\ 0)$, $k_i\in Z$, $i=1,\ \cdots,\ K-1$,*

$$j_k=\begin{cases}0, & k=\bar0=(0,\ \cdots,\ 0)\\ \sup\{j:k_j>0\}, & k\neq\bar0,\end{cases}$$

$\bar\alpha_j=(0,\ \cdots,\ \alpha_j,\ 0,\ \cdots,\ 0)$, $1\leqslant j\leqslant K$, $\bar\alpha(s)=\bar\alpha_k+\cdots+\bar\alpha_s$, $1\leqslant s\leqslant K$, *and* $\bar\alpha(K+1)=0$.

*Proof* We use the induction on $m$. For $m=1$, the formula is clearly correct. Now, suppose the formula is correct for $m_1$: $1\leqslant m_1\leqslant m$. We have only to prove

$$-\sum_{|k|=m}\frac{F^{(k)}(\xi)}{k!}(-\alpha)^k=I_1+I_2,\tag{2.5.1}$$

where

$$I_1=-\sum_{0\leqslant|l|<m}\sum_{r=j_l+1}^{K}\frac{1}{l!(m-|l|)!}F^{(l+(m-|l|)\delta_r)}(\xi-\bar\alpha(r+1))(-\alpha)^{l+(m-|l|)\delta_r},$$
$$I_2=\sum_{|k|=m}\frac{(-\alpha)^k}{k!}(F^{(k)}(\xi-\bar\alpha(j_k+1))-F^{(k)}(\xi)),$$

and $\delta_r=(0,\ \cdots,\ 0,\ 1,\ 0,\ \cdots,\ 0)$, 1 occupying the $r$th place.

To prove (2.5.1) we take a fixed $\bar k$: $|\bar k|=m$ and examine that in the both sides of the equation the terms related to $F^{(\bar k)}$ are equal. There are two cases:

(i). $(\bar k)_K=0$. Then in $I_2$ the terms related to $F^{(\bar k)}$ exist and are

$$\frac{(-\alpha)^{\bar k}}{\bar k!}(F^{(k)}(\xi-\bar\alpha(j_{\bar k}+1))-F^{(\bar k)}(\xi)).\tag{2.5.2}$$

To see $I_1$ we decompose $\bar k$ in $\bar k=(\bar k-(\bar k)_{j_{\bar k}})+(\bar k)_{j_{\bar k}}=\bar l+(m-|\bar l|)\ \delta_{j_{\bar k}}$ and the term is

$$\frac{(-1)(-\alpha)^{\bar k}}{\bar l!(m-|\bar l|)!}F^{(\bar k)}(\xi-\bar\alpha(j_{\bar k}+1)).\tag{2.5.3}$$

Since $\bar k!=\bar l!(m-|\bar l|)!$, by adding (2.5.2) to (2.5.3) we see the term of $F^{(\bar k)}$ in the right hand of (2.5.1) is $-\dfrac{(-\alpha)^{\bar k}}{\bar k!}F^{(\bar k)}(\xi)$, which equals the corresponding term in the left hand of(2.5.1).

(ii). $(\bar{k})_K>0$. Then there is no corresponding term in $I_2$. To see $I_1$ we decompose $\bar{k}=\bar{l}+(m-|\bar{l}|)\delta_K$ and the terms of $F^{(\bar{k})}$ in the both sides equal $-\frac{(-\alpha)^{\bar{k}}}{\bar{k}!}F^{(\bar{k})}(\xi)$.

The formula permits us to write $R^{(m)}_{(-\alpha)}F(\xi)$ as

$$R^{(m)}_{(-\alpha)}F(\xi)=\sum_{\substack{0\leqslant|k_i|<m_i\\1\leqslant i\leqslant n}}\frac{(-\alpha_1)^{k_1}\cdots(-\alpha_n)^{k_n}}{k_1!\cdots k_n!}\sum_{\substack{j_{k_i}+1\leqslant r_i\leqslant k\\1\leqslant i\leqslant n}}R^{m_1-|k_1|}_{-\alpha_1,r_1}\circ\cdots$$

$$\circ R^{m_n-|k_n|}_{-\alpha_n,r_n}F^{(k_1+\cdots+k_n)}(\xi-\bar{\alpha}_1(r_1+1)-\cdots-\bar{\alpha}_n(r_n+1)). \tag{2.5.4}$$

With the notation

$$k=(k_1,\cdots,k_n)\in(Z^K)^n,\ k!=k_1!\cdots k_n!,\ \alpha^k=\alpha_1^{k_1}\cdots\alpha_n^{k_n},\ \alpha_j\in R^K,$$

$$\bar{0}\leqslant k<m\Leftrightarrow\forall j,\ 0\leqslant k_j<m_j,\ j_k=(j_{k_1},\cdots,j_{k_n}),\ \bar{r}=(r_1,\cdots,r_n),$$

$$\bar{K}=(K,\cdots,K),\ j_k+1\leqslant\bar{r}\leqslant\bar{K}\Leftrightarrow\forall j,\ j_{k_j}+1\leqslant r_j\leqslant K,$$

$$(|k|)=(|k|_1,\cdots,|k_n|),\ [k]=k_1+\cdots+k_n,\ \bar{\alpha}_{\bar{r}}=(\bar{\alpha}_{1,r_1},\cdots,\bar{\alpha}_{n,r_n})\in(R^k)^n,$$

$$\bar{\alpha}_i(l)=(0,\cdots,0,\alpha_{i,l},\cdots,\alpha_{i,K})=\sum_{j=l}^{K}\bar{\alpha}_{i,j}\in R^K,\ 0\leqslant l\leqslant K,$$

$$\bar{\alpha}(l)=\sum_{i=1}^{n}\bar{\alpha}_i(l_i)\in R^K,\ \bar{0}\leqslant l\leqslant\bar{K},$$

we can rewrite(2.5.4)as

$$R^{(m)}_{(-\alpha)}F(\xi)=\sum_{0\leqslant k<m}\sum_{j_k+1\leqslant\bar{r}\leqslant\bar{K}}R^{(m)-(|k|)}_{(-\bar{\alpha}_{\bar{r}})}F^{([k])}(\xi-\bar{\alpha}(\bar{r}+1)). \tag{2.5.5}$$

Ussng(2.5.5)to $F(\xi)=\tau(x,\xi+[\alpha])$, there follows

$$\pi(x,\alpha,\xi)=\sum_{0\leqslant k<m}\frac{(-\alpha)^k}{k!}\sum_{j_k+1\leqslant\bar{r}\leqslant\bar{K}}\xi_j\tau_{k,\bar{r}}(x,\alpha,\xi)\varphi_1(\xi)\varphi_1(\alpha_1)\cdots\varphi_1(\alpha_n),$$

where

$$\tau_{k,\bar{r}}(x,\alpha,\xi)=R^{(m)-(|k|)}_{(-\bar{\alpha}_{\bar{r}})}\tau^{([k])}(x,\xi+[\alpha]-\bar{\alpha}(\bar{r}+1)). \tag{2.5.6}$$

(2.6). Let $\lambda\in C_0^\infty(R^1)$and $s\in[-10kn,10kn]\Rightarrow\lambda(s)=1$. Denote

$$\Omega(x,\alpha,\xi)=\Big(1-\lambda\Big(\frac{\xi_1}{\alpha_{1,1}}\Big)\Big)\cdots\Big(1-\lambda\Big(\frac{\xi_1}{\alpha_{n,1}}\Big)\Big)\pi(x,\alpha,\xi),$$

We have to prove

$$\Omega(x,\alpha,\xi)\in\bar{M}(m),\ \prod_{i\in J}\lambda\Big(\frac{\xi_1}{\alpha_{i,1}}\Big)\pi(x,\alpha,\xi)\in\bar{M}(m),$$

where $\emptyset\neq J\subset\{1,\cdots,n\}$.

For the first assertion we need to prove that

$$\Omega_{k,\bar{r}}(x,\alpha,\xi)=(-\alpha)^k\Big(1-\lambda\Big(\frac{\xi_1}{\alpha_{1,1}}\Big)\Big)\cdots\Big(1-\lambda\Big(\frac{\xi_1}{\alpha_{n,1}}\Big)\Big)\xi_j\tau_{k,\bar{r}}(x,\alpha,\xi)\varphi_1(\xi)\varphi_1(\alpha_1)$$

$$\cdots\varphi_1(\alpha_n)\in\bar{M}(m).$$

(2.7). Suppose $\sigma\in C^\infty(R^K\times(R^K)^n)$, and for $\forall\beta\in R^K$, $\forall\alpha\in(R^K)^n$, $\exists C_{\alpha,\beta}$ snch that for $\forall(x,\xi)\in R^K\times(R^K)^n$, $\xi=(\xi_1,\cdots,\xi_n)$,

$$|D_X^\beta\partial_\xi^\alpha\sigma(x,\xi)|\leqslant C_{\alpha,\beta}(1+|\xi|)^{l-|\alpha|},\ l\in R^1.$$

Then we call $\sigma$ a symbol of order $l$ and type(1, $n$), denoted by $\sigma\in S^l(R^K\times(R^K)^n)$.

The following theorem was proved in[5].

**Theorem A.** *If* $\sigma\in S^0(R^K\times(R^K)^n)$ *and* $\forall j$, $p_j\in(1,\infty)$,

$$q^{-1}=\sum_{i=1}^n p_i^{-1}\in(0,1),$$

*then for* $f_i\in\mathscr{S}(R^K)$ *and*

$$T(f_1,\cdots,f_n)(x)=\int_{(R^K)^n}e^{jx[\xi]}\sigma(x,\xi)\hat{f}_1(\xi_1)\cdots\hat{f}_n(\xi_n)d\xi,$$

it holds that

$$\|T(f_1,\cdots,f_n)\|_q\leqslant C\prod_{i=1}^n\|f_i\|_{p_i},$$

*where* $C=C(K,n,C_{\alpha,\beta},p_j)$ *is a constant.*

(2.8). To deal with $\Omega_{k,\bar{r}}\in\overline{M}(m)$, we see that for a fixed $x\in R^K$,

$$\operatorname{supp}\Omega_{k,\bar{r}}\subset\Delta=\{(\alpha,\xi):|\xi_1|>10Kn\cdot|\alpha_{j,1}|,$$
$$|\alpha_{j,1}|\geqslant\frac{1}{2}|\alpha_{j,l}|,\ |\xi_1|\geqslant\frac{1}{2}|\xi_l|,\ 1\leqslant l\leqslant K,\ 1\leqslant j\leqslant n\}.$$

Let

$$\tilde{\Delta}=\left\{(\alpha,\xi):|\xi_1|>\frac{1}{3n}|\xi|,\ |\xi_1|>4n|\alpha_j|,\ 1\leqslant j\leqslant n\right\}.$$

It is easy to see that $\Delta\setminus\{0\}\subset\tilde{\Delta}$. Choose $\theta_1(\alpha,\xi)\in C^\infty((R^K)^{n+1}\setminus\{0\})$, homogeneous of degaee 0, $\theta_1=1$ on $\Delta\setminus\{0\}$ and supp $\theta_1\subset\tilde{\Delta}\cup\{0\}$. To smooth $\theta_1$ we choose again a $\theta_2(\alpha,\xi)\in C^\infty((R^K)^{n+1})$ such that $\theta_2=0$ if $|(\alpha,\xi)|<\frac{1}{2}$ and $\theta_2=1$ if $|(\alpha,\xi)|>1$. Make $\theta=\theta_1\cdot\theta_2$, It follows that $\theta=1$ on the support of $\Omega_{k,\bar{r}}$. Therefore

$$\begin{aligned}\Omega_{k,\bar{r}}(x,\alpha,\xi)&=\theta^{n+1}\Omega_{k,\bar{r}}(x,\alpha,\xi)\\&=\prod_{i=1}^n\left(\theta(\alpha,\xi)\left(1-\lambda\left(\frac{\xi_1}{\alpha_{i,1}}\right)\right)\right)\theta(\alpha,\xi)\xi_j(-\alpha)^k\cdot\tau_{k,\bar{r}}(x,\alpha,\xi))\cdot(\varphi_1(\xi)\varphi_1(\alpha_1)\\&\quad\cdots\varphi_1(\alpha_n)).\end{aligned}$$

We make the followimg observation:

If $m_i$, $1\leqslant i\leqslant n$, and m are $L^p$ Fourier multipliers, $1<p<\infty$, and $\sigma(x,\alpha,\xi)\in\overline{M}(m)$, then $m(\xi)m_1(\alpha_1)\cdots m_n(\alpha_n)\sigma(x,\alpha,\xi)\in\overline{M}(m)$.

Since $\varphi_1$ is an $L^p$ Fourier multiplier([6], Ch. VI, 3.2), we need only to prove that

(i). $\forall i$, $\theta(\alpha,\xi)\left(1-\lambda\left(\frac{\xi_1}{\alpha_{i,1}}\right)\right)$is a symbol of order 0 and type(1, $n+1$);

(ii). $\dfrac{\theta(\alpha,\xi)\xi_j\tau_{k,\bar{r}}(x,\alpha,\xi)}{\prod_{i=1}^n(-\alpha_{i,r_i})^{m_i-|k_i|}}$ is a symbol of order 0 and type (1, $n+1$).

To see (i) we make the following observation:

If $\sigma(\alpha,\xi)\in C^\infty((R^K)^{n+1})$and it is homogeneous of degree O outside a neighborhood of the origin, then $\sigma(\alpha,\xi)\in S^0(R^K\times(R^K)^{n+1})$.

To see(ii), first, we have the following equation

$$R^{(m)-(|k|)}_{(-\bar{\alpha}_{\bar{r}})}F(\xi)=\frac{\prod_{i=1}^{n}(-\alpha_{j,r_i})^{m_i-|k_i|}}{\prod_{i=1}^{n}(m_i-|k_i|-1)!}\int\cdots\int_{\substack{0\le t_i\le 1\\1\le i\le n}}\prod_{i=1}^{n}t_i^{m_i-|k_i|-1}.$$

$$\cdot F^{(\Sigma(m_i-|k_i|)\delta_{r_i})}\Big(\xi-\sum_{i=1}^{n}\bar{\alpha}_{i,r_i}+\sum_{i=1}^{n}t_i\bar{\alpha}_{i,r_i}\Big)dt_1\cdots dt_n, \qquad (2.8.1)$$

which can be proved from (2.3.1). Let $F(\xi)=\tau_{k,\bar{r}}(x,\alpha,\xi)$ in (2.8.1). We need to prove that

$$\theta(\alpha,\ \xi)\xi_j\tau^{([k]+\Sigma(m_i-|k_i|)\delta_{r_i})}\Big(x,\ \xi+[\alpha]-\bar{\alpha}(\bar{r}+1)-\sum_{i=1}^{n}\bar{\alpha}_{i,r_i}+\sum_{i=1}^{n}t_i\bar{\alpha}_{i,r_i}\Big)$$
$$\in S^0(R^K\times(R^K)^{n+1}), \qquad (2.8.2)$$

and the corresponding constants $C_{\alpha,\beta}$ are independent of $t\in[0,1]^n$.

To see this we have, firstly

$$\tau\in S^{|m|-1}(R^K\times R^K)\Rightarrow\tau^{([k]+\Sigma(m_i-|k_i|)\delta_{r_i})}\in S^{-1}(R^K\times R^K).$$

Since

$$\Big|\bar{\alpha}(\bar{r}+1)-[\alpha]+\sum_{i=1}^{n}\bar{\alpha}_{i,r_i}-\sum_{i=1}^{n}t_i\bar{\alpha}_{i,r_i}\Big|\le 3(|\alpha_1|+\cdots+|\alpha_n|),$$

in the support of $\theta$ there exists

$$\Big|\xi+[\alpha]-\bar{\alpha}(\bar{r}+1)-\sum_{i=1}^{n}\bar{\alpha}_{i,r_i}+\sum_{i=1}^{n}t_i\bar{\alpha}_{i,r_i}\Big|\ge|\xi|-3(|\alpha_1|+\cdots+|\alpha_n|)$$
$$\ge\frac{1}{24n}|(\alpha,\ \xi)|.$$

Now it is easy to see that (2.8.2) holds.

(2.9) To deal with

$$\Big(\prod_{i\in J}\lambda\Big(\frac{\xi_1}{\alpha_{i,1}}\Big)\Big)\pi(x,\ \alpha,\ \xi)\in\bar{M}(m),$$

keeping in mind that $\frac{\xi_j}{|\xi_1|}\varphi_1(\xi)$, $\varphi_1(\alpha_i)$ are $L^p$. Fourier multipliers, from the observation made in (2.8) we need to prove that

$$|\xi_1|\cdot\Big(\prod_{i\in J}\lambda\Big(\frac{\xi_1}{\alpha_{i,1}}\Big)\Big)R^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha])\in\bar{M}(m).$$

**Lemma 4.** *If $\lambda\in C_0^\infty(R^1)$ and $\lambda$ is even, then for $\forall\varepsilon>0$, there is $\eta\in\mathscr{G}(R^1)$ such that for $\forall t\ne0$,*

$$|t|^\varepsilon\lambda(t)=\int_{-\infty}^{\infty}|t|^{iu}\eta(u)du.$$

*Proof* In fact, denoting $\varphi(x)=e^{\varepsilon x}\lambda(e^x)$ we put $\eta=\varphi^V$.

Uging the lemma

$$|\xi_1|\Big(\prod_{i\in J}\lambda\Big(\frac{\xi_1}{\alpha_{i,1}}\Big)\Big)R^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha])$$
$$=\Big(\prod_{i\in J}|\alpha_{i,1}|^{\frac{1}{|J|}}\Big)R^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha])\cdot\int_{R^J}\prod_{i\in J}\Big(\frac{|\xi_1|}{|\alpha_{i,1}|}\Big)^{iu_i}\eta(u_i)du_J,$$

since $|\xi_1|^{iu_i}$ and $|\alpha_{i,1}|^{-iu_i}$ are $L^p$ Fourier multipliers and $\eta\in\mathscr{G}(R)$, it is sufficient to show that

$$(\prod_{i\in J}|\alpha_{i,1}|^{\frac{1}{|J|}})R^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha])\in\overline{M}(m).$$

The last assertion comes from Lemma 2 and the following lemma.

**Lemma 5.** *Uuder the induction hypothesis shown in*(2.1), *for*

$$\tau\in S^{|m|-1}(R^K\times R^K),\ 0\leqslant t_i\leqslant 1,\ \sum_{i=1}^n t_i=1,$$

*and all the choices of* $\{j_1,\cdots,j_n\}\subset\{1,\cdots,n\}$, *we have*

$$|\alpha_{1,j_1}|^{t_1}\cdots|\alpha_{n,j_n}|^{t_n}\cdot R^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha])\in\overline{M}(m).$$

The proof is similar to the one in [5], Proposition 3 except that the definition of $\omega_z(x,\alpha,\xi)$ is substituted here by

$$\omega_z(x,\alpha,\xi)=|\alpha_{1,j_1}|^{zs_1+(1-z)t_1}\cdots|\alpha_{n,j_n}|^{zs_n+(1-z)t_n}\cdot R^{(m)}_{(-\alpha)}\tau(x,\ \xi+[\alpha]).$$

## § 3. The Boundedness of Higher Commutators The Second case $\forall i,\ p_i=\infty$

(3.1). We are going to show that in this case

$$T_{R^{(m)}_{(-\alpha)}\omega}(a,f)=T(f)$$

is a Calderón-Zygmund operator ([4], Ch.IV, Definition 1), and henc it is bounded on $L^p$, $1<p\infty$, and maps $L^1$ into weak $L^1$([4], Ch. IV)..

For $q\in[1,\infty)$, $f\in L^q_{loc}(R^K)$and a cube $Q$, we define

$$M_q(f;Q)=\left(\frac{1}{|Q|}\int_Q|f(x)|^q dx\right)^{\frac{1}{q}}.$$

The following proposition was established([5]).

**Propositioe A.** *Suppose* $K(x,y)$ *defined on* $\{(x,y)\in R^K\times R^K:x\neq y\}$ *satisfies the following conditions*:

(i) $$|K(x,y)|\leqslant C|x-y|^{-K},\tag{3.1.1}$$

(ii) $$|\nabla_x K(x,y)|\leqslant C|x-y|^{-K-1},\tag{3.1.2}$$

$$|\nabla_y K(x,y)|\leqslant C|x-y|^{-K-1},\tag{3.1.3}$$

(iii) $\forall f\in C_0^\infty(R^K)$,

$$T(f)(x)=\lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon}K(x,y)f(y)dy\quad \textit{exists a. e..}\tag{3.1.4}$$

Then $T$ can be extended to a bounded operator on $L^2(R^K)$ if and only if there is a pair of real numbers: $q,\ r,\ 1<q\leqslant r<\infty$, such that for every cube $Q$ and $f\in C_0^\infty(R^K)$, supp $f\subset Q$, we have

$$M_q(T(f);Q)\leqslant CM_r(f;Q).\tag{3.1.5}$$

Furthermore

$$\|T\|_{2,2}\leqslant C(K,q,r)\cdot C,\tag{3.1.6}$$

where $\|T\|_{2,2}$ denotes the norm of $T$: $L^2\to L^2$, $C$ is the largest constant in (3.1.1)—(3.1.3)and(3.1.5).

(3.2). Take $\varphi\in C_0^\infty(R^K)$, $\varphi_v(\xi)=\varphi\left(\frac{\xi}{v}\right)$, $v\geqslant 1$. Let

$$\omega_v(x,\ \xi)=\varphi_v(\xi)\omega(x,\ \xi)\quad \omega_v(x,\ \xi)=\int_{R^K}e^{-iy\xi}L_v(x,\ y)\mathrm{d}y$$

and

$$K_v(x,\ y)=\prod_{i=1}^{n}P_{m_i}(a_i,\ x,\ y)L_v(x,\ x-y).$$

It holds that

$$T(f)(x)=\lim_{v\to\infty}T_v(f)(x)=\lim_{v\to\infty}\int K_v(x,\ y)f(y)dy.$$

It is sufficient to examine (i)—(iii) and (3.1.5) for $K_v$ and $T_v$ with some constants independent of $v$. For the assertion that T is a Calderón-Zygmund operaior, refer to [4], the proofs of Theorem 19 and Theorem 18.

(3.3). Suppose $\forall i$, $\|\nabla^{m_i}a_i\|_\infty=1$. By using (2.3.1), Leibnitz formula, the following formulas

$$\nabla_x P_{m_i}(a_i,\ x,\ y)=P_{m_i-1}(\nabla_x a_i,\ x,\ y),\tag{3.3.1}$$

$$\nabla_y P_{m_i}(a_i,\ x,\ y)=\frac{-1}{(m_i-1)!}\left(\sum_{j=1}^{n}(x_j-y_j)D_j\right)^{m_i-1}\nabla a_i(y)\tag{3.3.2}$$

and by a standard argument on the kernel corresponding to a symbol of order $|m|$ ([4], Ch. IV), we get (3.1.1)—(3.1.3) for $K_v$ with some constants independent of $v$.

Now we are going to show (3.1.5). Take a cube $Q$ and denote by $\tilde{Q}$ the double of $Q$. Take $\chi\in C_0^\infty(R^K)$, which equals 1 on $Q$, *supp* $\chi\subset\tilde{Q}$ and

$$\|\nabla^l\chi\|_\infty\leqslant C_l(\mathrm{diam}(Q))^{-l},\ l\in Z.\tag{3.3.3}$$

For the existence of such a $\chi$, refer to [6], Ch. VI, 1.3. Now let

$$A_j(x)=P_{m_j}(a_j,\ x,\ x_0)\chi(x),$$

where $x_0$ is the center of $Q$. It is easy to see that for $x,\ y\in Q$, we have

$$P_{m_j}(A_j,\ x,\ y)=P_{m_j}(a_j,\ x,\ y).$$

Therefore, for $x\in Q$ and $f\in C_0^\infty(R^K)$, supp $f\subset Q$, it follows that

$$T_v(f)(x)=T_{v,Q}(f)(x),$$

where

$$T_{v,Q}(f)(x)\triangleq\int_{R^K}\left(\prod_{i=1}^{n}P_{m_i}(A_i,\ x,\ y)\right)L_v(x,\ x-y)f(y)dy.$$

According to the result obtained in § 2, for a choice of $p$, $q$, $r$ such that $p$, $q$, $r\in(1,\infty)$ and $q^{-1}=np^{-1}+r^{-1}$, we have

$$\left(\int_Q|T_v(f)|^q\right)^{\frac{1}{q}}=\left(\int_Q|T_{v,Q}(f)|^q\right)^{\frac{1}{q}}\leqslant C\|f\|_r\prod_{i=1}^{n}\|\nabla^{m_i}A_i\|_p,$$

where the constant $C$ is independent of $v$.

By using Leibnitz formula, (3.3.3), (3.3.1), (2.3.1) and keeping in mind that supp $A_i\subset\tilde{Q}$, $x\in\tilde{Q}\Rightarrow|x-x_0|\leqslant\sqrt{K}\cdot\mathrm{diam}(Q)$, we have

$$\|\nabla^{m_i}A_i\|_\infty \leqslant C,$$

and therefore

$$\|\nabla^{m_i}A_i\|_p \leqslant C|Q|^{\frac{1}{q}}.$$

So we finally obtain

$$\left(\int_Q T_v(f)|^q\right)^{\frac{1}{q}} \leqslant C\,|Q|^{\frac{n}{p}}\|f\|_r \leqslant C|Q|^{\frac{1}{q}-\frac{1}{r}}\|f\|_r,$$

i, e., (3.1.5), and the constant $C$ is independent of $v$.

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