Chin. Ann. of Math.
6B (2) 1985

OSCILLATORY AND ASYMPTOTIC BEHAVIORS OF FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper the author discusses the following first order functional differential equations:

$$x'(t)+\int_a^b p(t,\xi)x[g(t,\xi)]d\sigma(\xi)=0, \tag{1}$$

$$x'(t)+\int_a^b f(t,\xi,x[g(t,\xi)])d\sigma(\xi)=0. \tag{2}$$

Some sufficient conditions of oscillation and nonoscillation are obtained, and two asymptotic properties and their criteria are given. These criteria are better than those in [1, 2], and can be used to the following equations:

$$x'(t)+\sum_{i=1}^n p_i(t)x[g_i(t)]=0, \tag{3}$$

$$x'(t)+\sum_{i=1}^n f_i(t,x[g_i(t)])=0. \tag{4}$$

§ 1. Introduction

In this paper we consider oscillatory and asymptotic behaviors for the following first order functional differential equations:

$$x'(t)+\int_a^b p(t,\xi)x[g(t,\xi)]d\sigma(\xi)=0 \quad (b>a), \tag{1}$$

$$x'(t)+\int_a^b f(t,\xi,x[g(t,\xi)])d\sigma(\xi)=0 \quad (b>a). \tag{2}$$

We first make the following assumptions:

(R_1) $g(t,\xi)\leqslant t$, $\xi\in[a,b]$, $\mathbf{R}^*=[t_0,+\infty)$, g: $\mathbf{R}^*[a,b]\to\mathbf{R}^+$ is continuous, $g(t,\xi)$ is a nondecreasing function with respect to t and ξ, respectively.

$$\lim_{t\to+\infty}\min_{\xi\in[a,b]}\{g(t,\xi)\}=+\infty.$$

Moreover, there exists a continuous function φ: $\mathbf{R}^*\times[a,b]\to\mathbf{R}^+$ such that

$$\varphi(\varphi(t,\xi)\xi)=g(t,\xi),$$

Manuscript received January 17, 1983. Revised April 14, 1983.

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$\varphi(t, \xi)$ is a nondecreasing function with respect to t and ξ respectively.

$$\lim_{t\to+\infty} \min_{\xi\in[a,b]} \{\varphi(t, \xi)\} = +\infty. \ t \geqslant \varphi(t, \xi) \geqslant g(t, \xi);$$

(R_2) p: $\mathbf{R}^* \times [a, b] \to \mathbf{R}^+$ is continuous;

(R_3) σ: $[a, b] \to \mathbf{R}$ is a nondecreasing function;

(R_4) The nonlinear function $f(t, \xi, v)$ in (2) satisfies "the bounded sublinear" condition: if $|v| \leqslant c_0$ ($c_0 > 0$), then

$$|f(t, \xi, v)| \geqslant p(t, \xi)|v|.$$

Furthermore, suppose

$$f(t, \xi, 0) \equiv 0; \ f(t, \xi, v)v > 0 \quad (v \neq 0);$$

(R_5) The integral in (1) or (2) is a Stieltjes integral.

Recently studies on oscillatory and asymptotic behaviors for functional differential equations as (1) or (2) are noticed, for example, David L. Lovelady[3] considered oscillatory and asymptotic behaviors for second order functional differential equations which are analogous to (1) or (2).

In § 2 we first consider the following first order functional differential inequalities:

$$x'(t) + \int_a^b p(t, \xi) x[g(t, \xi)] d\sigma(\xi) \leqslant 0, \tag{3}$$

$$x'(t) + \int_a^b p(t, \xi) x[g(t, \xi)] d\sigma(\xi) \geqslant 0 \tag{4}$$

and establish some sufficient conditions for (3), (4) having no ultimate positive solution (ultimate negative solution). Since the integral in (1) or (2) is a Stieltjes integral, we easily know that (1) or (2) contains the following kinds of equations:

$$x'(t) + \sum_{i=1}^n p_i(t) x[g_i(t)] = 0, \tag{5}$$

$$x'(t) + \sum_{i=1}^n p_i(t) f_i(x[g_i(t)]) = 0, \tag{6}$$

where $p_i(t)$ is nonnegative and not identically zero in any subinterval $[t_1, \infty)$ of $[t_0, \infty)$. There exists $\varphi_i(t)$ which satisfies

$$\varphi_i(\varphi_i(t)) = g_i(t), \quad \lim_{t\to+\infty} \varphi_i(t) = +\infty, \ i = 1, 2, \cdots, n.$$

So results of this paper generalize and modify the corresponding results in [1].

In § 3 we establish some sufficient conditions for (1) having nonoscillatory solutions and we point out that asymptotic behaviors of nonoscillatory solutions to (1) or (2) belong only to one of types A_0 and A_1. Also we give some sufficient conditions for (1) or (2) having a solution of the typc A_0 or A_1 respectively.

In §4 we give some examples.

If assumptions (R_1) and (R_2) are respectively modified by the following:

(R_1') $g(t, \xi) \geqslant t$ ($t \geqslant t_1$, $\xi \in [a, b]$). There exists a continuous function $\psi(t, \xi)$

from $\mathbf{R}^*\times[a, b]$ into $\mathbf{R}^*$, which satis fies $\psi(\psi(t), \xi)=g(t, \xi)$, $t\leqslant\psi(t, \xi)\leqslant g(t, \xi)$ $(t\geqslant t_0, \xi\in[a, b])$,

$$\lim_{t\to+\infty}\min_{\xi\in[a,b]}\{g(t, \xi)\}=+\infty;$$

(R_2') $\mathbf{R}^{**}=(-\infty, t_0]$, p: $\mathbf{R}^*\times[a, b]\to\mathbf{R}^{**}$; i. e. (1) and (2) are functional differential equations with advanced argument. We can use the methods of this paper to obtain analogous results which will be considered in a forth coming paper.

§2. Oscillation Criteria

Set $P(s)=\int_a^b p(s, \xi)d\sigma(\xi)$.

(H_1) $\underline{\lim}_{t\to+\infty}\int_{g(t,b)}^t P(s)ds>\frac{1}{e}$;

(H_2) $\underline{\lim}_{t\to+\infty}\int_{g(t,b)}^t P(s)ds>0$.

Theorem 1. *Suppose that* (H_1), (H_2) *hold.*

Then there is not any ultimate positive solution to (3).

Proof If conditions (H_1), (H_2) are satisfied and there exists an ultimate positive solution to (3), $x(t)>0$ $(t\geqslant t_1)$, then $x'(t)\leqslant 0$ $(t\geqslant t_1)$ and there exists $t_2\geqslant t_1$ such that $x[g(t, \xi)]\geqslant x(t)$ $(t\geqslant t_2, \xi\in[a, b])$. We have

$$\ln x(t)-\ln x[g(t, b)]+\int_{g(t,b)}^t\left[\int_a^b p(s, \xi)\frac{x[g(s, \xi)]}{x(s)}d\sigma(\xi)\right]ds\leqslant 0.$$

Set $W(t)=\frac{x[g(t, a)]}{x(t)}$. It is easy to see that $W(t)\geqslant 1$,

$$\ln W(t)\geqslant\int_{g(t,b)}^t W(s)\left[\int_a^b p(s, \xi)d\sigma(\xi)\right]ds=\int_{g(t,b)}^t W(s)P(s)ds$$

Set $l=\underline{\lim}_{t\to+\infty}W(t)$. The existence of l is assured by $W(t)\geqslant 1$.

(1) In the case of $l<+\infty$. There exists $t_n\to+\infty$ such that $W(t_n)\to l$ $(n\to+\infty)$. It is easy to see that

$$\ln W(t_n)\geqslant W(\xi_n)\int_{g(t_n,b)}^{t_n}P(s)ds,\ \xi_n\in[g(t_n, b), t_n],\ n=1, 2, \cdots,$$

$$\ln l=\lim_{n\to+\infty}\ln W(t_n)\geqslant\underline{\lim}_{n\to+\infty}\left[W(\xi_n)\int_{g(t_n,b)}^{t_n}P(s)ds\right]$$

$$\geqslant\underline{\lim}_{n\to+\infty}W(\xi_n)\ \underline{\lim}_{n\to+\infty}\int_{g(t_n,b)}^{t_n}P(s)ds,$$

$$\frac{\ln l}{l}\geqslant\underline{\lim}_{t\to+\infty}\int_{g(t,b)}^t P(s)ds.$$

By $\max_{l\geqslant 1}\frac{\ln l}{l}=\frac{1}{e}$, we have

$$\frac{1}{e}\geqslant\underline{\lim}_{t\to+\infty}\int_{g(t,b)}^t P(s)ds.$$

This is a contradiction.

(1) In the case of $l=+\infty$. From (3) we have

$$x(t)-x[\varphi(t,b)]+\int_{\varphi(t,b)}^{t}\int_{a}^{b}p(s,\xi)x[g(t,\xi)]d\sigma(\xi)ds\leqslant 0,$$

$$x(t)-x[\varphi(t,b)]+x[g(t,b)]\int_{\varphi(t,b)}^{t}P(s)ds\leqslant 0, \tag{7}$$

$$\varliminf_{t\to+\infty}\frac{x[\varphi(t,b)]}{x(t)}\geqslant 1+\left[\varliminf_{t\to+\infty}\frac{x[g(t,b)]}{x(t)}\right]\left[\varliminf_{t\to+\infty}\int_{\varphi(t,b)}^{t}P(s)ds\right].$$

By (H_1) and $l=+\infty$, we obtain

$$\varliminf_{t\to+\infty}\frac{x[\varphi(t,b)]}{x(t)}=+\infty.$$

From (7), we also can obtain

$$\frac{x(t)}{x[\varphi(t,b)]}-1+\left[\frac{x[g(t,b)]}{x[\varphi(t,b)]}\right]\left[\int_{g(t,b)}^{t}P(s)ds\right]\leqslant 0,$$

$$\left\{\varliminf_{t\to\infty}\frac{x[g(t,b)]}{x[\varphi(t,b)]}\right\}\cdot\left\{\varlimsup_{t\to\infty}\int_{\varphi(t,b)}^{t}P(s)ds\right\}\leqslant 1.$$

Using (H_2) and

$$\varliminf_{t\to\infty}\frac{x[g(t,b)]}{x[\varphi(t,b)]}=\varliminf_{t\to\infty}\frac{x[\varphi(\varphi(t,b),b)]}{x[\varphi(t,b)]}\geqslant\varliminf_{t\to+\infty}\frac{x[\varphi(\eta,b)]}{x(\eta)}=+\infty,$$

We can obtain

$$\varlimsup_{t\to+\infty}\int_{\varphi(t,b)}^{t}P(s)ds>0.$$

This is a contradiction. So there is not any ultimate positive solution to (3). Theorm 1 is proved.

We can easily obtain the following results:

Theorem 2. *Suppose (H_1), (H_2) hold. Then there is not any ultimate negative solution to* (4).

Theorem 3. *Suppose (H_1), (H_2) hold. Then all solutions to* (1) *are oscillatory.*

For equation (5) we suppose

$$\begin{aligned}&g_1(t)\leqslant g_2(t)\leqslant\cdots\leqslant g_n(t)\leqslant t,\\&\varphi_1(t)\leqslant\varphi_2(t)\leqslant\cdots\leqslant\varphi_n(t),\\&\varphi_i(\varphi_i(t))=g_i(t),\ \lim_{t\to+\infty}g_i(t)=\lim_{t\to+\infty}\varphi_i(t)=\infty,\ i=1,2,\cdots,n.\end{aligned}\tag{8}$$

Set

(H_1') $\varliminf_{t\to+\infty}\sum_{i=1}^{n}\int_{g_n(t)}^{t}p_i(s)ds>\frac{1}{e}$,

(H_2') $\varlimsup_{t\to+\infty}\sum_{i=1}^{n}\int_{\varphi_n(t)}^{t}p_i(s)ds>0$.

Corollary 1. *Suppose (H_1'), (H_2') hold. Then all solutions to* (5) *are oscillatory.*

Theorem 4. *Suppose that f in* (2) *is "bounded sublinear" and (H_1'), (H_2') hold. Then all solutions to* (2) *are oscillatory.*

For equation (6) we suppose (8) hold and

$$f_i(0)=0,\ |f_i(x)|\geqslant\beta|x|>0\ (|x|\leqslant c_0),$$
$$xf_i(x)>0(x\neq 0),\ i=1,2,\cdots n. \tag{9}$$

Corollary 2. *Suppose* (H_1'), (H_2') *hold. Then all solutions to* (6) *are oscillatory.*

§ 3. Typies of Asymptotic Behaviors of Nonoscillatory Solutions and their Criteria

In what follows we remove the assumption that $g(t, \xi)$ is a nondecreasing function with respect to ξ.

Theorem 5. *A sufficient condition for* (1) *having a nonoscillatory solution is that* $\exists\ T^*\geqslant t_1$ *such that*

$$\int_{g(t,c)}^{t} P(s)ds\leqslant\frac{1}{e}\ \text{for}\ c\in[a,\ b],\ t\geqslant T^*. \tag{10}$$

Proof 1. We establish the integral equation

$$\lambda(t)=-\int_a^b\varphi(t,\ \xi)\exp\left(-\int_{g(t,\xi)}^{t}\lambda(s)ds\right)d\sigma(\xi). \tag{11}$$

We shall prove that there exists a solution to (11). From (1) we have

$$[\ln|x(t)|]'+\int_a^b p(t,\ \xi)\frac{x[g(t,\ \xi)]}{x(t)}\,d\sigma(\xi)=0\quad(t\geqslant t_2). \tag{12}$$

It is easy to see that

$$\int_{t_2}^{t}\lambda(s)ds=\ln|x(t)|,$$

$$x(t)=\pm\exp\left[\int_{t_2}^{t}\lambda(s)ds\right].$$

2. We make a sequence:

$$\lambda_0(t)=-eP(t),$$
$$\lambda_1(t)=-\int_a^b p(t,\ \xi)\cdot\exp\left(\int_{g(t,\xi)}^{t}-\lambda_0(s)ds\right)d\sigma(\xi),$$
$$\cdots\cdots,$$
$$\lambda_n(t)=-\int_a^b p(t,\ \xi)\exp\left(\int_{g(t,\xi)}^{t}-\lambda_{n-1}(s)ds\right)d\sigma(\xi).$$

Using the induction, we can prove that $\lambda_n(t)$ is a nondecreasing sequence and $-eP(t)\leqslant\lambda_n(t)\leqslant 0$. By (10) we have

$$\lambda_1(t)\geqslant-e\int_a^b p(t,\ \xi)d\sigma(\xi)=\lambda_0(t).$$

Suppose $\lambda_{n-1}(t)\geqslant\lambda_{n-2}(t)\geqslant\cdots\geqslant\lambda_1(t)\geqslant\lambda_0(t)$. Then

$$\lambda_n(t)\geqslant-\int_a^b p(t,\ \xi)\exp\left(-\int_{g(t,\xi)}^{t}\lambda_{n-2}(s)ds\right)d\sigma(\xi)=\lambda_{n-1}(t).$$

3. Set

$$I(\lambda_n)=\int_{g(t,\xi)}^{t}\lambda_n(s)ds.$$

Using Fatou Lemma, we see that there exists $\lambda(t)$ such that $\lambda_n(t)\to\lambda(t)$ $(n\to+\infty)$ and

$$\lim_{n\to+\infty}\int_{g(t,\xi)}^{t}\lambda_n(s)ds=\int_{g(t,\xi)}^{t}\lambda(s)ds.$$

Hence

$$\lim_{n\to+\infty}\int_a^b p(t,\ \xi)\exp\left(\int_{g(t,\xi)}^{t}-\lambda_n(s)ds\right)d\sigma(\xi)=\int_a^b p(t,\ \xi)\exp\left(\int_{g(t,\xi)}^{t}-\lambda(s)ds\right)d\sigma(\xi).$$

So $\lambda(t)$ is a solution to (11).

4. Set $x(t)=\exp\left(\int_{t_2}^{t}\lambda(s)ds\right)$. We have

$$\frac{x'(t)}{x(t)}=\lambda(t).$$

It is easy to see that $x(t)$ satisfies (1) and $x(t)>0$. Then $x(t)$ is a nonoscillatory solution to (1). Theorem 5 is proved.

Remark. In that proof of Theorem 5 we set $\lambda_0(t)=-eP(t)<0$, because $\lambda(t)\leqslant 0$ from (11), (12). It is easy to see that

$$\lambda_n(t)<0,\quad -eP(t)\leqslant\lambda(t)\leqslant 0.$$

Any solution $x(t)$ to (1) must satisfy $x(t)x'(t)\leqslant 0$. So we take

$$x(t)=\exp\left(\int_{t_2}^{t}\lambda(s)ds\right).$$

Theorem 6. *If f in (2) only satisfies $f(t,\ \xi,\ v)v>0(v\neq 0)$, $f(t,\ \xi,\ 0)=0$, then any nonoscillatory solution to (2) belongs to one of the following typies*:

A_0: $x(t)\to 0(t\to+\infty)$, (13)

A_1: $x(t)\to c\neq 0(t\to+\infty)$. (14)

Proof By $x(t)x'(t)<0$ we know that $x(t)$ must have an asymptotic behavior. It is easy to see that $\lim\limits_{t\to+\infty}x(t)=\infty$ is not true. Theorem 6 is proved.

Theorem 7. *A sufficient coudition for (1) having a nonoscillatory solution of type A_0 is that (10) and*

$$\int^{+\infty}P(s)ds=+\infty \tag{15}$$

are true.

A sufficient condition for (1) having a nonoscillatory solution of type A_1 is that

$$0<\int^{+\infty}P(s)ds<+\infty. \tag{16}$$

Proof 1. If (10) is satisfied, then there exrsts a nonoscillatory solution $x(t)$ to (1). By (11) we have

$$\lambda(t)=-\int_a^b p(t,\ \xi)\exp\left[-\int_{g(t,\xi)}^{t}\lambda(s)ds\right]d\sigma(\xi)\leqslant-\int_a^b p(t,\ \xi)d\sigma(\xi)=-P(t).$$

By (9) and $-eP(t)\leqslant\lambda(t)\leqslant 0$ we obtain

$$\int_{t_1}^{+\infty}\lambda(s)ds=-\infty.$$

So $x(t)=\exp\left(\int_{t_1}^{t}\lambda(s)ds\right)\to 0\ (t\to+\infty)$.

2. If (16) is satisfied, then there exists $T\geqslant t_2$ such that

$$\int_{T}^{+\infty}P(s)ds<\varepsilon \text{ for } \varepsilon\in\left(0,\ \frac{1}{2e}\right).$$

Hence

$$\left|\int_{g(t,c)}^{t}P(s)ds\right|\leqslant\left|\int_{T}^{t}P(s)ds\right|+\left|\int_{T}^{g(t,c)}P(s)ds\right|<2\varepsilon<\frac{1}{e}.$$

It is easy to see that there exists a nonoscillatory solution $x(t)$ to (1) and $x(t)$ satisfies

$$\exp\left(-e\int^{t}P(s)ds\right)\leqslant x(t)=\exp\left(\int^{t}\lambda(s)ds\right)\leqslant\exp\left(-\int^{t}P(s)ds\right).$$

By (16) we have

$$+\infty>\exp\left(-e\int^{+\infty}P(s)ds\right)>0,$$

$$\lim_{t\to+\infty}x(t)=c\neq 0.$$

Theorem 7 is proved.

Corollary 3. *For* (5) *we suppose*

$$g_1(t)\leqslant g_2(t)\leqslant\cdots\leqslant g_n(t)\leqslant t.$$

If there exists $T^*\geqslant t_2$ such that

$$\sum_{i=1}^{n}\int_{g_i(t)}^{t}P_i(s)ds\leqslant\frac{1}{e}\quad(t\geqslant T^*),\tag{17}$$

then there exists a nonoscillatory solution to (5).

Defination. *$f(t,\ \xi,\ u)$ in* (2) *is said to be a sublinear function, if*

$$\frac{f(t,\ \xi,\ u_1)}{u_1}\leqslant\frac{f(t,\ \xi,\ u_2)}{u_2}\ (u_1\geqslant u_2>0),$$
$$\frac{f(t,\ \xi,\ v_1)}{v_1}\leqslant\frac{f(t,\ \xi,\ v_2)}{v_2}\ (v_1\leqslant v_2<0).\tag{18}$$

Theorem 8. *Set*

$$Q(t)=\int_{a}^{b}f(t,\ \xi,\ 1)d\sigma(\xi).$$

Suppose f in (2) *is a sublinear function and there exists T_1^* such that for $t\geqslant T_1^*$, $c\in[a,\ b]$,*

$$\int_{g(t,c)}^{t}Q(s)ds\leqslant\frac{1}{e}.\tag{19}$$

Then there exists a nonoscillatory solution to (2).

Proof We consider the equation

$$\lambda(t) = -\int_a^b \frac{f\left(t,\ \xi,\ \exp\left[\int_t^{g(t,\xi)} \lambda(s)ds\right]\right)}{\exp\left[\int_{t_2}^t \lambda(s)ds\right]} d\sigma(\xi). \tag{20}$$

We also establish a sequence:

$$\lambda_0(t) = -eQ(t),$$

$$\lambda_1(t) = -\int_a^b \frac{f\left(t,\ \xi,\ \exp\left[\int_{t_2}^{g(t,\xi)} \lambda_0(s)ds\right]\right)}{\left[\exp\int_{t_2}^t \lambda_0(s)ds\right]} d\sigma(\xi),$$

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$$\lambda_n(t) = -\int_a^b \frac{f\left(t,\ \xi,\ \exp\left[\int_{t_2}^{g(t,\xi)} \lambda_{n-1}(s)ds\right]\right)}{\exp\left[\int_{t_2}^t \lambda_{n-1}(s)ds\right]} d\sigma(\xi).$$

It is easy to see that $\lambda_n(t)$ is a nondecreasing sequence and there exists $\lambda(t)$ such that $\lambda_n(t) \to \lambda(t)\ (n \to +\infty)$. So there exists a positive solution to (20) Set.

$$x(t) = \exp\left[\int_{t_2}^t \lambda(s)ds\right] > 0.$$

Then it is a nonoscillatory solution to (2). Theorem 8 is proved.

Theorem 9 *Suppose f in (2) is a sublinear function aud (19) is true. Then a sufficient condition for (2) having a nonoscillatory solution of type A_0 is that*

$$\int_{t_2}^{+\infty} Q(s)ds = +\infty. \tag{21}$$

A sufficient condition for (2) having a nonoscillatory solution of type A_1 is that

$$0 < \int_{t_2}^{+\infty} Q(s)ds < +\infty. \tag{22}$$

Corollary 4. *Suppose f in (6) is a sublinear function and $g_1(t) \leqslant g_2(t) \leqslant \cdots \leqslant g_n(t) \leqslant t$. Then a sufficient condition for (6) having a nonoscillatory solution is that*

$$\int_{g_1(t)}^t Q^*(s)ds \leqslant \frac{1}{e}, \tag{23}$$

where

$$Q^*(t) = \sum_{i=1}^n p_i(t) f_i(1).$$

Remark. If $n=1$ in (5) and (6), then these equations are contained in (1) and (2). It is easy to see that results of corollaries 1—4 are better than [2].

§ 4. Some Examples

Now we give some examples as applications of the results in § 2, § 3.

Example 1.

$$x'(t) + \int_{-2}^{-1} px(t+\xi)d\xi = 0, \tag{24}$$

where $p>0$, $g(t, \xi)=t+\xi$, we have

$$P(t)=\int_{-2}^{-1} p d\xi=p.$$

(1) If $p>\frac{1}{e}$, then all solutions to (24) are oscillatory;

(2) If $p\leq\frac{1}{2e}$, then there exists anonoscillatory solution of type A_0 for (24).

Example 2.

$$x'(t)+\int_{-2}^{-1}\frac{\xi^2}{t^2}x(t+\xi)d\xi=0, \tag{25}$$

where

$$p(t, \xi)=\frac{\xi^2}{t^2},\ P(t)=\frac{7}{3t^2}.$$

We have $0<\int^{+\infty}P(s)ds<+\infty$, so we can see that there exists a nonoscillatory solution of type A_1 for (25).

Example 3.

$$x'(t)+\int_{-2}^{-1}px^{\gamma}(t+\xi)d\xi=0, \tag{26}$$

where $p>0$, $0<\gamma<1$, γ is a ratio of two relatively prime odd numbers. It is easy to see that f is "bounded sublinear" and sublinear.

(1) If $p>\frac{1}{e}$, then all solutions to (26) are oscillatory;

(2) If $p\leq\frac{1}{2e}$, then there exists a nonoscillatory solution to (26) and it is of type A_0.

Example 4.

$$x'(t)+\int_{1}^{2}px(k\xi t)d\xi=0\left(p>0,\ \frac{1}{2}\geq k>0\right), \tag{27}$$

where

$$g(t, \xi)=k\xi t\leq t,\ \varphi(t, \xi)=\sqrt{k\xi}\,t.$$

(1) If $k\in\left(0, \frac{1}{2}\right)$, then all solutions to (27) are oscillatory.

Comment 1. The existence of function $\varphi(t, \xi)$ and the relation between $\varphi(t, \xi)$ and $g(t, \xi)$ in assumption (R_1) are seldom discussed. Here we give some examples to show it.

(1) If $g(t, \xi)=t+\tau\xi$, $\xi\in[-1, 1]$, then $\varphi(t, \xi)=t+\frac{\tau\xi}{2}$;

(2) If $g(t, \xi)=kt\xi$, $k\in(0, 1)$, $\xi\in[1, 2]$, then $\varphi(t, \xi)=\sqrt{k}\sqrt{\xi}\,t$;

(3) If $g(t, \xi)=\sqrt[4]{t\xi}$, $\xi\in[1, 2]$, then $\varphi(t, \xi)=\sqrt{t\xi}$;

(4) If $g(t, \xi)=k(t+\xi)$, $k\in(0, 1)$, $\xi\in(-2, -1)$, then

$$\varphi(t, \xi)=\sqrt{k}\,t+\frac{k}{\sqrt{k}+1}\xi;$$

(5) If $g(t, \xi)=\ln t$, then we can not find $\varphi(t, \xi)$,

Comment 2. If the assumption that $g(t, \xi)$ in [1, 2] is a nondecreasing function with ξ is not satisfied, then all results of this paper (except Theorem 1) still hold.

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