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A BIFURCATION THEOREM OF SET-CONTRACTIVE MAPS*

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Abstract

In this paper, the global result on bifurcation given by P. H. Rabinowitz is extended to strictly set-contractive maps and condensing maps.

The bifurcation is an important subject in nonlinear functional analysis. There have been a lot of papers about it. The local result of M. A. Krasnosels'kii[1] and the global result of P. H. Rabinowitz[2] are very good ones in this direction. But they only discussed completely continuous maps. [8] has extended the resutl of [1] to strict-set-contractive maps. In this short paper, we extend the result of [2] to more general maps, namely, to strictly set-contractive maps and condensing maps. To do this, we need some results: the index formula of isolated fixed point of set-contractive maps (Lemma 1), the multiplicative formula of topological degree (Lemma 2) and a result of spectrum of linear set-contractive maps (Lemma 3). Our main result is Theorem 1.

In this paper we always suppose X is a Banach space and I: $X\to X$ the identity map. The concepts of strictly set-contractive maps and condensing maps, and the definition of their topological degree can be seen in [3, 6].

Lemma 1.[4] *Let F: $X\to X$ be a strictly set-contractive map, x_0 a fixed point of F. If 1 is not an eigenvalue of $F'(x_0)$ (the Frechet derivative of F at x_0), then x_0 is an isolated fixed point of F and*

$$\operatorname{index}[F, x_0] = (-1)^{\mu},$$

where μ is the sum of the algebraic multiplicities of the eigenvalues of $F'(x_0)$, lying in $(1, \infty)$.

Remark according to [5] we know that Lemma 1 is still true if F is a condensing map.

Lemma 2 (multiplicative formula): *Let Ω, D be bounded open sets in X, F: $\overline{\Omega}\to D$ be a k_1-set-contractive map, G: $\overline{D}\to X$ be a k_2-set-contractive map, $f=I-F$, $g=$*

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$I-G$. *If* $p\in X$, $p\bar{\in} g\circ f(\partial\Omega)\cup g(\partial D)$, $g^{-1}(p)$ *is a finite set*, $k=k_1+k_2+k_1k_2<1$, *then*

$$\deg(g\circ f,\ \Omega,\ p)=\sum_{\alpha}\deg(g,\ D_\alpha,\ p)\deg(f,\ \Omega,\ D_\alpha).$$

Remark. The mumber of terms in the summation is finite In the formula, D_α is a connected component of $D\backslash f(\partial\Omega)$. $\deg(f,\ \Omega,\ D_\alpha)=\deg(f,\ \Omega,\ b)$, $b\in D_\alpha$. By the property of connected region it is independent of the choice of b.

Proof We first suppose that $g^{-1}(p)=\{q\}$ only contains one point, and write $f^{-1}(q)=\{q_\beta\}$. For conciseness we may suppose $p=q=0$. If not, we discuss $F_1(x)=F(x)+q$, $G_1(x)=G(x)+p$. Write $g\circ f=h=I-H$, where H is a strictly set-contractive map because of $k=k_1+k_2+k_1k_2<1$. Then $0\bar{\in}(g\circ f)(\partial\Omega)$, $\deg(g\circ f,\ \Omega,\ 0)$ is well defined. we only need to show

$$\deg(g\circ f,\ \Omega,\ 0)=\deg(g,\ D,\ 0)\deg(f,\ \Omega,\ 0).$$

We make the elementary-sets of F relative to Ω and G relative to D:

$$\Delta_1^{(1)}=\overline{co}F(\overline{\Omega}),\ \Delta_1^{(n)}=\overline{co}F(\overline{\Omega}\cap\Delta_1^{(n-1)}),\ n\geqslant 2,\ \Delta_1=\bigcap_{n\geqslant 1}\Delta_1^{(n)};$$

$$\Delta_2^{(1)}=\overline{co}G(\overline{D}),\ \Delta_2^{(n)}=\overline{co}G(\overline{D}\cap\Delta_2^{(n-1)}),\ n\geqslant 2,\ \Delta_2=\bigcup_{n\geqslant 1}\Delta_2^{(n)}.$$

Then Δ_1, Δ_2 are compact convex sets. Let F^*: $\overline{\Omega}\to\Delta_1$ be a compact extension of F on $\Delta_1\cap\overline{\Omega}$, G^*: $\overline{D}\to\Delta_2$ be a compact extension of G on $\Delta_2\cap\overline{D}$, $f^*=I-F^*$, $g^*=I-G^*$. By [3], § 16.3, $(f^*)^{-1}(0)=f^{-1}(0)=\{q_\beta\}$, $(g^*)^{-1}(0)=g^{-1}(0)=\{0\}$ and $(g^*\circ f^*)^{-1}(0)=(g\circ f)^{-1}(0)=\{q_\beta\}$. Also by [3]

$$\deg(f,\ \Omega,\ 0)=\deg(f^*,\ \Omega,\ 0),$$

$$\deg(g,\ D,\ 0)=\deg(g^*,\ D,\ 0).$$

Next we prove $[I-(tG+(1-t)G^*)]^{-1}(0)=(I-G)^{-1}(0)$ for any $t\in[0,\ 1]$. If $[I-(tG+(1-t)G^*)](x)=0$, then $x=tG(x)+(1-t)G^*(x)$ and $x\in\overline{co}(G(x),\ \Delta_2)$ because of $G^*(x)\in\Delta_2$. By the definition of elementary-set, x belongs to Δ_2 and $G^*(x)=G(x)$. Hence $G(x)=tG(x)+(1-t)G^*(x)=x$. It is obvious that $(I-G)^{-1}(0)\subset[I-(tG+(1-t)G^*)]^{-1}(0)$. Notice $f^{-1}(0)=\{q_\beta\}$, we have obtained

$$(*)\qquad \{[I-(tG+(1-t)G^*)](I-F)\}^{-1}(0)=f^{-1}(0)=\{q_\beta\}.$$

In the same way

$$(**)\qquad \{(I-G^*)[I-(tF+(1-t)F^*)]\}^{-1}(0)=(g^*\circ f)^{-1}(0)=\{q_\beta\}.$$

Define H_1, H_2: $\overline{\Omega}\times[0,\ 1]\to X$ as follows

$$H_1(x,\ t)=F(x)+[tG+(1-t)G^*](I-F)(x),$$

$$H_2(x,\ t)=tF(x)+(1-t)F^*(x)+G^*[I-(tF+(1-t)F^*)](x).$$

We can prove H_1, H_2: $\overline{\Omega}\times[0,\ 1]\to X$ are k-set-contractive maps, where $k=k_1+k_2+k_1k_2<1$. By the condition $0\bar{\in}(g\circ f)(\partial\Omega)$ and $(*)$, $(**)$, we have $H_1(x,\ t)\neq x$ and $H_2(x,\ t)\neq x$ for any $t\in[0,\ 1]$, $x\in\partial\Omega$. Hence H_1, H_2 are homotopy maps. Obviously $g\circ f=I-H_1(x,\ 1)$, $g^*\circ f=I-H_1(x,\ 0)=I-H_2(x,\ 1)$ and $g^*\circ f^*=I-H_2(x,\ 0)$. By the homotopy invariance of topological degree, it is easy to see that

$$\begin{aligned}\deg(g\circ f,\ \Omega,\ 0) &= \deg(I-H_1(x,\ 1),\ \Omega,\ 0) = \deg(I-H_1(x,\ 0),\ \Omega,\ 0)\\ &= \deg(I-H_2(x,\ 1),\ \Omega,\ 0) = \deg(I-H_2(x,\ 0),\ \Omega,\ 0)\\ &= \deg(g^*\circ f^*,\ \Omega,\ 0).\end{aligned}$$

Obviously $0\overline{\in}(g^*\circ f^*)(\partial\Omega)\cup g^*(\partial D)$. The multiplicative formula of degree of Leray-Schauder gives

$$\deg(g^*\circ f^*,\ \Omega,\ 0) = \deg(g^*,\ D,\ 0)\deg(f^*,\ \Omega,\ 0) = \deg(g,\ D,\ 0)\deg(f,\ \Omega,\ 0).$$

Hence

$$\deg(g\circ f,\ \Omega,\ 0) = \deg(g,\ D,\ 0)\deg(f,\ \Omega,\ 0).$$

When there are more than one point in $g^{-1}(p)$, by the region addition property of topological degree, the result is easy to prove from the preceding discussion.

Remark. When one of F and G is completely continuous and the other is strictly set-contractive, the condition $k<1$ is always satisfied. By the property of condensing maps, we can prove corresponding result similarly.

Lemma 3. *If T: $X\to X$ is a linear k-set-contractive map, then the essentially spectral radius of T, $r_e(T)\leqslant k$. Therefore, out of the circular disc with radius k and center zero in camplex plane, the spectral set of T only contains finite eigenvalues and the algebraic multiplicities of these eigenvalues are all finite.*

Proof By, Theorem 1 of [7], $r_e(T)=\lim_{n\to\infty}(r(T^n))^{\frac{1}{n}}$, where $r(A)=\inf\{k \mid A \text{ is } k\text{-set-contraction}\}$. By, Lemma 1 of [7], $r(T^n)\leqslant(r(T))^n$, hence $r_e(T)\leqslant k$. Combining the definition of essential spectrum and the fact that a compact isolated points set is a finite set, we obtain this result.

Now we can prove our main results. Let X be a Banach space, Λ be a real parameter space. Consider the equation

(1) $f(x,\ \lambda)=0$,

where f: $X\times\Lambda\to X$ can be written as

$$(2)\quad \begin{cases} f(x,\ \lambda)=f_\lambda(x)=x-\lambda Ax+g(x,\ \lambda),\\ A;\ X\to X \text{ is linear } k\text{-set-contractive } (k<1),\\ g:\ X\times\Lambda\to X \text{ is a completely continuous map,}\end{cases}$$

$\|g(x,\ \lambda)\|=o(\|x\|)$ at $x=0$ for λ uniformly in a bounded interval of Λ.

So $I-\lambda A$ is Frechet derivative operator of f at $(0,\ \lambda)$ by $g(0,\ \lambda)\equiv 0$.

Theorem 1. *Let f: $X\times\Lambda\to X$ satisfy (2), $|\lambda_0|<\frac{1}{k}$, λ_0 be a characteristic value of odd algebraic multiplicity of A. Suppose S denotes the closure of the set of non-zero solutions $(x,\ \lambda)$ of equation (1), and C is a connected component of S and contains $(0,\ \lambda_0)$. If C does not intersect $\{0\}\times\left(\left[\frac{1}{k},\ \infty\right)\cup\left(-\infty,\ -\frac{1}{k}\right]\right)$, then either C is noncompact or C only contains finite points $(0,\ \lambda_j)$ $(j=0,\ 1,\ \cdots,\ n)$, where λ_j is a* *characteristic value of A, and the count of the characteristic values of odd multiplicity (including λ_0) is an even number.*

Proof Notice $x=0$ is always the trivial solution of equation (1). By Result 9.1 of [3], the λ corresponding to of these points $(0, \lambda)$ are the characteristic values of A. Because C does not intersect $\{0\}\times\left(\left[\frac{1}{k}, \infty\right)\cup\left(-\infty, -\frac{1}{k}\right]\right)$, the absolute values of these λ are all less than $\frac{1}{k}$. By Lemma 3 the count of these points are finite, we denote them by $(0, \lambda_j)$, $j=0, 1, 2, \cdots n$. Take $\eta<\frac{1}{k}$ such that $|\lambda_j|<\eta$, $j=0, 1, 2, \cdots n$.

Suppose C is compact. Since C does not intersect $\{0\}\times\left(\left[\frac{1}{k}, \infty\right)\cup\left(-\infty, -\frac{1}{k}\right]\right)$, and C is a connected component of S, there is a bounded open set Ω that contains C, such that

1) on $\partial\Omega$, there isn't any nontrivial solution of (1);

2) for any characteristic value λ of A except $\lambda_j (j=0, 1, 2, \cdots n)$ $(0, \lambda)$ does not belong to Ω.

Consider the map f_ρ: $\overline{\Omega}\to X\times\Lambda$, $f_\rho(x, \lambda)=(f(x, \lambda), \|x\|^2-\rho^2)$, it is a k-set-contractive field. Obviously, the following two facts are equivalent: f_ρ has a zero point $(\hat{x}, \hat{\lambda})$ and $f(x, \hat{\lambda})$ has a zero point $(\hat{x}, \hat{\lambda})\in\Omega$ with $\|\hat{x}\|=\rho$. By (1), $\deg(f_\rho, \Omega, (0, 0))$ is well defined for any $\rho>0$, and f_ρ is a homotopy map when ρ is changed. Hence $\deg(f_\rho, \Omega, (0, 0))$ is independent of ρ. It is easy to see that f_ρ doesn't have any zero point when ρ is very large, so $\deg(f_\rho, \Omega, (0, 0))=0$.

Next we compute $\deg(f_\rho, \Omega, (0, 0))$ when ρ is sufficiently small. Take $\varepsilon>0$ very small such that A has only one characteristic value on $[\lambda_j-\varepsilon, \lambda_j+\varepsilon]$ and write $K=[-\eta, \eta]\setminus\bigcup_{j=0}^{n}(\lambda_j-\varepsilon, \lambda_j+\varepsilon)$. By Lemma 3, $\lambda\in K$ if a regular value of A. Hence there exists a constant R such that $\|(I-\lambda A)^{-1}\|<R$ for any $\lambda\in K$. Take ρ small enough such that $\|g(x, \lambda)\|\leqslant\frac{1}{2R}\|x\|$ when $\|x\|\leqslant\rho$.

There isn't any zero point of f_ρ on $\Omega\setminus\bigcap_{j=0}^{n}\Omega_j$, where $\Omega_j\subset\Omega$ is small ball with radius $\sqrt{\rho^2+\varepsilon^2}$ and center $(0,\lambda_j)$. In fact, if $(x, \lambda)\in\Omega\setminus\bigcup_{j=0}^{n}\Omega_j$ and $f_\rho(x, \lambda)=0$, it is easy to see that $\lambda\in K$, $\|x\|=\rho$ and

$$\|x\|=\|(I-\lambda A)^{-1}g(x, \lambda)\|<R\|g(x, \lambda)\|\leqslant\frac{1}{2}\|x\|.$$

This is a contradiction.

Take

$$\begin{aligned}h_t(x, \lambda)&=(x, \lambda)-H((x, \lambda), t)\\&=((I-\lambda A)x+tg(x, \lambda), t(\|x\|^2-\rho^2)+(1-t)(\varepsilon^2-(\lambda-\lambda_j)^2)).\end{aligned}$$

It is easy to see that h_t is a homotopy map. Thus

$$\deg(f_\rho,\ \Omega_j,\ (0,\ 0))=\deg(h_1,\ \Omega_j,\ (0,\ 0))=\deg(h_0,\ \Omega_j,\ (0,\ 0)).$$

We can prove that $h_0(x,\ \lambda)=((I-\lambda A)x,\ \varepsilon^2-(\lambda-\lambda_j)^2)$ has only two isolated zero points $(x,\ \lambda)=(0,\ \lambda_j\pm\varepsilon)$ on Ω_j, and the differentials of $h_0(x,\ \lambda)$ at these two points are

$$dh_0(0,\ \lambda_j\pm\varepsilon)(\bar{x},\ \bar{\varepsilon})=((I-(\lambda_j\pm\varepsilon)A)\bar{x},\ -2(\pm\varepsilon)\bar{\varepsilon})=M\circ N(\bar{x},\ \bar{\varepsilon}),$$

where $M(\bar{x},\ \bar{\varepsilon})=((I-(\lambda_j\pm\varepsilon)A)\bar{x},\ \bar{\varepsilon})$; $N(\bar{x},\ \bar{\varepsilon})=(\bar{x},\ -2(\pm\varepsilon)\bar{\varepsilon})$. When $\varepsilon>0$ is sufficient small, $M(\bar{x},\ \bar{\varepsilon})$ is a strictly set-contractive field and $N(\bar{x},\ \bar{\varepsilon})$ is a compact continuous field. By Lemma 2

$$\text{index}\ [h_0,\ (0,\ \lambda_j+\varepsilon)]=-\text{index}[I-(\lambda_j+\varepsilon)A,\ 0]=-i^+_{(j)};$$

$$\text{index}\ [h_0,\ (0,\ \lambda_j-\varepsilon)]=\text{index}[I-(\lambda_j-\varepsilon)A,\ 0]=i^-_{(j)}.$$

Hence

$$\deg(h_0,\ \Omega_j,\ (0,\ 0))=i^-_{(j)}-i^+_{(j)}$$

and

$$0=\deg(f_\rho,\ \Omega,\ (0,\ 0))=\sum_{j=0}^{n}\deg(f_\rho,\ \Omega_j,\ (0,\ 0))=\sum_{j=0}^{n}(i^-_{(j)}-i^+_{(j)}).$$

By Lemma 1, $i^+_{(j)}=i^-_{(j)}$ when λ_j is a characteristic value of even algebraic multiplicity and $i^+_{(j)}=-i^-_{(j)}$ when λ_j is of odd algerbraic multiplicity. By these, it is easy to obtain this theorem.

Remark. Changing k into 1, Theorem 1 is still true for the condensing maps because we have the corresponding results of Lemmas 1, 2, and 3.

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