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A NOTE ON GENERALIZED DERIVATION RANGES

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Abstract

This paper is devoted to the generalized derivations determined by compact operators acting on Banach spaces. The author introduces the concept of "family of approximately linearly indepedent vectors" and employs this concept to prove the following result: Let X, Y be Banach spaces, $A\in B(X)$, $B\in B(Y)$ be compact operators. Then $\mathscr{T}_{AB}$ has closed range if and only if both A and B have closed range.

Let X, Y be Banach spaces over complex field C (throughout, we assume $\dim X=\dim Y=\infty$). Let $B(X, Y)$ denote the set of all bounded linear operators from X to Y. If $X=Y$, we write $B(X)$ for $B(X, Y)$. For $A\in B(X)$ and $B\in B(Y)$, the generalized derivation $\mathscr{T}_{AB}$ is an operator belonging to $B(B(Y, X))$ defined by the equation

$$\mathscr{T}_{AB}S=AS-SB,\ (\forall)S\in B(Y, X).$$

If $X=Y$ and $A=B=T$, the generalized derivation is reduced to the inner derivation

$$\Delta_T S=TS-ST,\ (\forall)S\in B(X).$$

Similarly, as in the case of inner derivation, one of the problems on generalized derivations is: "under what conditions on A and B, is the range of $\mathscr{T}_{AB}$ norm closed?". In 1975 J. Anderson and C. Foias [1] proved that if A and B are scalar operators acting on Banach spaces X, Y respectively, then $\mathscr{T}_{AB}$ has closed range if and only if $\lambda=0$ is an isolated point of $\sigma(\mathscr{T}_{AB})$. In 1980, L. A. Fialkow[2] gave some conditions under which $\mathscr{T}_{AB}$ has closed range where A, B are hyponormal operators, nilpotent operators of order 2 and compact operators in a Hilbert space respectively.

The aim of this note is to prove the following

Theorem. *Let X, Y be Banach spaces and $A\in B(X)$, $B\in B(Y)$ be compact operators, then $\mathscr{T}_{AB}$ has closed range if and only if both A and B have closed range.*

In order to prove this theorem we give some notation first. N will be the set

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of natural numbers. For a Banach space X, X^* will denote the conjugate space of X. For a subset $M\subset X$, $M^\perp$ will denote the annihilator of M

$$M^\perp=\{x^*\in X^*, \langle x, x^*\rangle=0, (\forall)x\in M\},$$

$S_P\{M\}$ will denote the subspace spanned by M

$$S_P\{M\}=\text{closure}\left\{\sum_{i=1}^{n} a_i x_i,\ a_i\in C,\ x_i\in M,\ n\in N\right\}.$$

For an operator $T\in B(X, Y)$, $\ker T$, $R(T)$ will denote the null space and the range of T respectively. nul$T$ will denote the nullity of T, i. e., the dimension of ker$T$. nul$'T$ will denote the approximate nullity of T, i. e., the greatest number $m \leqslant\infty$ with the following property: for any $\varepsilon>0$, there exists an m dimensional closed linear manifold M_ε such that $\|Tx\|\leqslant\varepsilon\|x\|$ for every $x\in M_\varepsilon$ (see [3], p. 232 for details). T^* will denote the conjugate operator of T. $\sigma(T)$, $\rho(T)$ will denote the spectrum and the resolvent set respectively.

For any $x\in X$ and $y^*\in Y^*$, define the rank-one operator $x\otimes y^*\in B(Y, X)$ by the equation

$$(x\otimes y^*)y=\langle y, y^*\rangle x, (\forall)y\in Y.$$

We recall a result which will be important for our purpose.

Theorem IV. 5. 2 of [3]. *Let X, Y be Banach spaces, then $T\in B(X, Y)$ has closed range if and only if $\gamma(T)>0$, where*

$$\gamma(T)=\inf\{\|Tx\|, \text{dist}(x, \ker T)=1\}$$

is called reduced minimum modulus.

Let us begin with lemmas.

Lemma I. *Let X, Y be Banach spaces and $A\in B(X)$, $B\in B(Y)$.*

i) *If there exists $\lambda\in C$ such that $R(A-\lambda)$ is not closed and there exist unit vectors $y_k\in Y$, $y_k^*\in Y^*$ such that*

$$(B-\lambda)y_k=\lim_{k\to\infty}(B-\lambda)^*y_k^*=0,\ \langle y_k, y_k^*\rangle=\lambda_k,\ \min_k|\lambda_k|=\lambda>0,$$

then $\gamma(\mathscr{T}_{AB})=0$.

ii) *If there exists $\lambda\in C$ such that $R(B-\lambda)$ is not closed and there exist unit vectors $x_k\in X$, $x_k^*\in X^*$ such that*

$$\lim_{k\to\infty}(A-\lambda)x_k=(A-\lambda)^*x_k^*=0,\ \langle x_k, x_k^*\rangle=\lambda_k,\ \min_k|\lambda_k|=\lambda_0>0,$$

then $\gamma(\mathscr{T}_{AB})=0$.

Proof Since $\mathscr{T}_{AB}=\mathscr{T}_{(A-\lambda)(B-\lambda)}$, we may assume that $\lambda=0$.

i) Since $R(A)$ is not closed, there exists a sequence $\{x_k\}_{k\in N}\subset X$ such that

$$\text{dist}(x_k, \ker A)=1>\|x_k\|-\frac{1}{k},\ \lim_{k\to\infty}\|Ax_k\|=0.$$

Define

$$S_k=x_k\otimes y_k^*.$$

Then we have

$$\|AS_k - S_kB\| = \|Ax_k \otimes y_k^* - x_k \otimes B^* y_k^*\|$$
$$\leqslant \|Ax_k\| + \left(1+\frac{1}{k}\right)\|B^*y_k^*\| \to 0, \text{ as } k\to\infty.$$

On the other hand, for any $T\in\ker\mathscr{T}_{AB}$, we have

$$\|S_k - T\| \geqslant \|(S_k - T)y_k\| \text{ (via } T\ker B\subset\ker A) \geqslant \operatorname{dist}(\lambda_k x_k, \ker A) \geqslant \lambda_0 > 0.$$

Therefore $\operatorname{dist}(S_k, \ker\mathscr{T}_{AB}) \geqslant \lambda_0$ and hence $\gamma(\mathscr{T}_{AB}) = 0$.

ii) Since $R(B^*)$ is closed, there exists a sequence $\{y_k^*\}_{k\in N}\subset y^*$ such that

$$\operatorname{dist}(y_k^*, \ker B^*) = 1 \geqslant \|y_k^*\| - \frac{1}{k}, \quad \lim_{k\to\infty}\|B^*y_k^*\| = 0.$$

Define

$$S_k = x_k \otimes y_k^*.$$

Then we have

$$\|AS_k - S_kB\| = \|Ax_k \otimes y_k^* - x_k \otimes B^*y_k^*\|$$
$$\leqslant \left(1+\frac{1}{k}\right)\|Ax_k\| + \|B^*y_k^*\| \to 0, \text{ as } k\to\infty.$$

On the other hand, for $T\in\ker\mathscr{T}_{AB}$ we have

$$\|S_k - T\| = \|S_k^* - T^*\| \geqslant \|(S_k^* - T^*)x_k^*\| \text{ (via } T^*\operatorname{kre} A^*\subset\ker B^*) \geqslant \operatorname{dist}(\lambda_k y_k^*, \ker B^*)$$
$$\geqslant \lambda_0 > 0$$

Therefore $\operatorname{dist}(S_k, \ker\mathscr{T}_{AB}) \geqslant \lambda_0 > 0$ and hence $\gamma(\mathscr{T}_{AB}) = 0$.

Lemma 2. *Let X, Y be Banach spaces and let $A\in B(X)$, $B\in B(Y)$ be compact. If one of the ranges $R(A)$ and $R(B)$ is closed, while another is not closed, then $R(\mathscr{T}_{AB})$ is not closed.*

Proof First we suppose $R(A)$ is not closed and $R(B)$ is closed. Then B is a finite dimensional operator and hence has the following matrix form

$$B = \begin{pmatrix} 0 & 0 \\ 0 & B' \end{pmatrix} \text{ on } M \dotplus N = Y,$$

where $\dim N < \infty$. In this case, by Lemma IV. 2.3 and Theorem III. 1.22 of [3], there exist unit vectors $y_k\in Y$, $y_k^*\in Y^*$ such that $By_k = B^*y_k^* = 0$, $\langle y_k, y_k^*\rangle = 1$. Thus $\gamma(\mathscr{T}_{AB}) = 0$ results from i) of Lemma 1.

The proof for the other case is similar.

Definition. *A sequence of vectors $\{u_k\}_{k\in N}$ is called approximately linearly independent, if for any $n\in N$, there exists $m\in N$ such that if $k_n > k_{n-1} > \cdots > k_1 \geqslant m$, then any n vectors $u_{k_1}, \cdots, u_{k_n}$ are linearly independent.*

Lemma 3. *Let X, Y be Banach spaces and $A\in B(X)$, $B\in B(Y)$ be compact. If both $R(A)$ and $R(B)$ are not closed and there exist two sequences $\{v_k\}_{k\in N}\subset Y$ and $\{\lambda_k\}_{k\in N}\subset C$ such that*

i) $\{v_k\}_{k\in N}$ is approximately linearly independent,

ii) $Bv_k = \lambda_k v_k$,

then $R(\mathscr{T}_{AB})$ is not closed.

Proof By condition i), we may assume that (eventually discarding a finite number of $v_k's$) $v_i\neq 0$ and this fact will imply $\lambda_k\in\sigma(B)$. In order to prove this lemma, we distinguish the following two subcases:

a) If $\lambda_k=0$ for infinitely many k's, then we have $\text{nul}B=\infty$. By the proof of Lemma 2 we have $\gamma(\mathscr{T}_{AB})=0$.

b) If $\lambda_k=0$ for finitely many k's, by discarding those λ_k we may suppose all of λ_k are not zero. Let M_k, N_k be the eigenspaces of A and B corresponding to $\{\lambda_k\}$ respectively (if $\lambda_k\notin\sigma(A)$, then $M_k=\{0\}$). Since A and B are compact, we have dim $M_k<\infty$, $\dim N_k<\infty$. By condition i) we may assume that $\lambda_k\neq\lambda_{k'}$ for $k\neq k'$ and this fact will imply that $N_k\neq N_{k'}$ for $k\neq k'$. Let $y_1\in N_1$, $y_1^*\in Y^*$ such that

$$\|y_1\|=\|y_1^*\|=\langle y_1,\ y_1^*\rangle=1$$

and choose $y_1'\in Y$ such that

$$|\langle y_1',\ B^*y_1^*\rangle|>\|B^*y_1^*\|-1,\ \|y_1'\|=1.$$

Let n_2 be a natural number such that $\dim S_P\{N_2,\ \cdots,\ N_{n_2}\}>2$. By Lemma IV. 2.3 of [3], there exists a vector $y_2\in S_P\{N_2,\ \cdots,\ N_{n_2}\}$ such that

$$\|y_2\|=\text{dist}(y_2,\ S_P\{y_1,\ By_1'\})=1.$$

Hence there exists a functional $y_2^*\in Y^*$ such that

$$\|y_2^*\|=\langle y_2,\ y_2^*\rangle=1,\ y_2^*\in(S_P\{y_1,\ By_1'\})^\perp.$$

Continuing in the same way, we can construct three sequences, $\{y_k\}_{k\in N}\subset Y$, $\{y_k'\}_{k\in N}\subset Y$, $\{y_k^*\}_{k\in N}\subset Y^*$ such that

$$y_k\in S_P\{N_{n_{k-1}+1},\ \cdots,\ N_{n_k}\},\ n_0=0,\ n_1=1,$$
$$\|y_k\|=\text{dist}(y_k,\ S_P\{y_1,\ By_1',\ \cdots,\ y_{k-1},\ By_{k-1}'\})=1,$$
$$\|y_k^*\|=\langle y_k,\ y_k^*\rangle=1,\ y_k^*\in(S_P\{y_1,\ By_1',\ \cdots,\ y_{k-1},\ By_{k-1}'\})^\perp,$$
$$|\langle y_k',\ B^*y_k^*\rangle|>\|B^*y_k^*\|-\frac{1}{k},\ \|y_k'\|=1.$$

By the proof of Lemma 3.2 of [4] we may suppose $\|B^*y_k^*\|\to 0$. Since $R(A)$ is not closed and $\dim M_k<\infty$, by Lemma 1.1 of [4] there exists a sequence $\{x_k\}_{k\in N}\subset X$ such that

$$\|x_k\|=\text{dist}(x_k,\ S_P\{M_{n_{k-1}+1},\ \cdots,\ M_{n_k}\})=1,\ \|Ax_k\|<\frac{1}{k}.$$

Define

$$Q_k=x_k\otimes y_k^*.$$

We have $\|\mathscr{T}_{AB}Q_k\|\to 0$. On the other hand, since $SM_k\subset M_k$ for $S\in\ker\mathscr{T}_{AB}$, we derive that

$$\begin{aligned}\text{dist}(Q_k,\ \ker\mathscr{T}_{AB})&=\inf\{\|Q_k-S\|,\ S\in\ker\mathscr{T}_{AB}\}\\&\geqslant\inf\{\|(Q_k-S)y_k\|,\ S\in\ker\mathscr{T}_{AB}\}\\&\geqslant\text{dist}(x_k,\ S_P\{M_{n_{k-1}+1},\ \cdots,\ M_{n_k}\})=1.\end{aligned}$$

Thus the lemma is proved.

In the sequel, we shall consider the Banach space

$$Z = X \dot{+} Y,$$

where the norm of the element $z=(x, y)\in Z$ is defined as follows

$$\|z\| = \max\{\|x\|, \|y\|\}.$$

We define the projections P, Q by the following equations

$$Pz=x,\ Qz=y,\ \text{for } z=x+y,\ x\in X,\ y\in Y.$$

Lemma 4. *Let X, Y, Z, P, Q be defined as above. Let $\{P_k\}_{k\in N}$ $\{E_k\}_{k\in N}\subset B(Z)$ be sequences of projections with the following properties:*

i) *$\{\|P_k\|\}_{k\in N}$ is bounded,*

ii) $\lim\limits_{k\to\infty}\|E_k-P_k\|=0$,

let $\{x_k\}_{k\in N}\subset X$, $\{p_k\}_{k\in N}$, $\{z_k\}_{k\in N}\subset Z$ be sequences such that

iii) *$P_kp_k=p_k$, $\{\|p_k\|\}_{k\in N}$ is bounded,*

iv) $x_k=Pp_k$, $\|x_k\|=1$, $\operatorname{dist}(x_{k+1}, S_P\{x_i\}_{i=1}^k)=1$ *for* $k\in N$,

v) $z_k=E_kp_k$.

Then for any $n\in N$ and $0<\delta<1$, there exists a natural number $N(n, \delta)$ such that

$$\operatorname{dist}(Pz_{k_{n+1}}, S_P\{Pz_{k_i}\}_{i=1}^n)\geqslant\delta,$$

provided $k_{n+1}>k_n>\cdots>k_1\geqslant N(n, \delta)$.

Proof We proceed by induction. Let $n=1$. Suppose the conclusion is false, then there exist two subsequences $\{z_{k_1(m)}\}_{m\in N}$, $\{z_{k_2(m)}\}_{m\in N}\subset Z$ such that

$$\operatorname{dist}(Pz_{k_2(m)}, S_P\{Pz_{k_1(m)}\})<\delta, \text{ as } m\to\infty.$$

Hence there exists a sequence $\{C_{k_1(m)}\}_{m\in N}\subset C$ such that

$$\|Pz_{k_2(m)}-C_{k_1(m)}Pz_{k_1(m)}\|<\delta, \text{ as } m\to\infty.$$

Since

$$\lim_{k\to\infty}\|Pz_k\|=\lim_{k\to\infty}\|PE_kp_k\|=\lim_{k\to\infty}\|PP_kp_k\|=\lim_{k\to\infty}\|x_k\|=1,$$

we have

$$|C_{k_1(m)}|\leqslant 1+\delta, \text{ as } m\to\infty.$$

By condition ii), we can derive

$$\delta\geqslant\varliminf_{m\to\infty}\|Pz_{k_2(m)}-C_{k_1(m)}Pz_{k_1(m)}\|=\varliminf_{m\to\infty}\|P(E_{k_2(m)}p_{k_2(m)}-C_{k_1(m)}E_{k_1(m)}p_{k_1(m)})\|$$

$$=\varliminf_{m\to\infty}\|P(P_{k_2(m)}p_{k_2(m)}-C_{k_1(m)}P_{k_1(m)}p_{k_1(m)})\|=\varliminf_{m\to\infty}\|x_{k_2(m)}-C_{k_1(m)}x_{k_1(m)}\|\geqslant 1.$$

This contradiction shows that the conclusion is true for $n=1$.

Now assume that the conclusion is true for $n-1$ and it is not true for n. Then there exist sequences $\{z_{k_i(m)}\}_{m\in N}(i=1, \cdots, n+1)\subset Z$, $\{C_{k_i(m)}\}_{m\in N}(i=1, \cdots, n)\subset C$ such that

$$\|Pz_{k_{n+1}(m)}-C_{k_1(m)}Pz_{k_1(m)}-\cdots-C_{k_n(m)}Pz_{k_n(m)}\|<\delta, \text{ as } m\to\infty.$$

We shall prove that $\{|C_{k_n(m)}|\}_{m\in N}$ is bounded. Otherwise, we may assume that

$$|C_{k_n(m)}|\to\infty, \text{ as } m\to\infty.$$

Then we have

$$\lim_{m\to\infty}\|Pz_{k_n(m)}-C_{k_n(m)}^{-1}C_{k_1(m)}Pz_{k_1(m)}-\cdots-C_{k_n(m)}^{-1}C_{l_{n-1}(m)}Pz_{k_{n-1}(m)}\|=0.$$

This fact will imply that

$$\operatorname{dist}(Pz_{k_n(m)},\ S_P\{Pz_{k_i(m)}\}_{i=1}^{n-1})\to 0,$$

which is a contradiction. Continuing in the same way we can prove in turn that $\{|C_{k_{n-1}(m)}|\}_{m\in N}$ is bounded, $\cdots$, $\{|C_{k_1(m)}|\}_{m\in N}$ is bounded. Similarly, as in the proof for $n=1$, we can derive

$$\delta\geqslant\varliminf_{m\to\infty}\|Pz_{k_{n+1}(m)}-C_{k_1(m)}Pz_{k_1(m)}-\cdots-C_{k_n(m)}Pz_{k_n(m)}\|$$
$$=\varliminf_{m\to\infty}\|x_{k_{n+1}(m)}-C_{k_1(m)}x_{k_1(m)}-\cdots-C_{k_n(m)}x_{k_n(m)}\|\geqslant 1.$$

Again we obtain a contradiction and the proof is complete.

Lemma 5. *Let X, Y be Banach spaces, $A\in B(X)$, $B\in B(Y)$ be compact. If both $R(A)$ and $R(B)$ are not closed, then $R(\mathscr{T}_{AB})$ is not closed.*

Proof By the symmetry, Lemma 5 is equivalent to the following:

If both $R(A)$ and $R(B)$ are not closed, then $R(\mathscr{T}_{BA})$ is not closed.

Therefore we need only to prove $\gamma(\mathscr{T}_{AB})=0$ or $\gamma(\mathscr{T}_{BA})=0$. By the proof of Lemma 3.2 of [4] we can construct four sequences

$$\{x_k\}_{k=1}^{\infty}\subset X,\ \{x_k^*\}_{k=1}^{\infty}\subset X^*,\ \{y_k\}_{k=1}^{\infty}\subset Y,\ \{y_k^*\}_{k=1}^{\infty}\subset Y^*,$$

such that

$$\|x_k\|=\|x_k^*\|=\|y_k\|=\|y_k^*\|=1,$$
$$\langle x_k,\ x_{k'}^*\rangle=\langle y_k,\ y_{k'}^*\rangle=\delta_{kk'}\ \text{for}\ k\leqslant k',$$
$$\lim_{k\to\infty}\|Ax_k\|=\lim_{k\to\infty}\|A^*x_k^*\|=\lim_{k\to\infty}\|By_k\|=\lim_{k\to\infty}\|B^*y_k^*\|=0,$$

where $\delta_{kk'}$ is the Kronecker delta. Then we define operators

$$U_k=x_k\otimes y_k^*,\ V_k=y_k\otimes x_k^*.$$

Obviously,

$$\lim_{k\to\infty}\|\mathscr{T}_{AB}U_k\|=\lim_{k\to\infty}\|\mathscr{T}_{BA}V_k\|=0.$$

If $\operatorname{dist}(U_k,\ \ker\mathscr{T}_{AB})\geqslant\delta$ for some $\delta>0$, then $\gamma(\mathscr{T}_{AB})=0$. If $\operatorname{dist}(V_k,\ \ker\mathscr{T}_{BA})\geqslant\delta$ for some $\delta>0$, then $\gamma(\mathscr{T}_{BA})=0$. Thus the lemma is proved. Therefore we assume that $\operatorname{dist}(U_k,\ \ker\mathscr{T}_{AB})=\operatorname{dist}(V_k,\ \ker\mathscr{T}_{BA})=0$ and hence there exist two sequences $\{A_k\}_{k=1}^{\infty}\subset\ker\mathscr{T}_{AB}$ and $\{B_k\}_{k=1}^{\infty}\subset\ker\mathscr{T}_{BA}$ such that

$$\|U_k-A_k\|\to 0,\ \|V_k-B_k\|\to 0.$$

Consider the Banach space

$$Z=X\dotplus Y.$$

Define operators

$$T=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix},\ W_k=\begin{pmatrix}0 & U_k\\ V_k & 0\end{pmatrix},\ C_k=\begin{pmatrix}0 & A_k\\ B_k & 0\end{pmatrix}.$$

It is easy to see that

$$\operatorname{rank} W_k=2,\ C_k\in\ker\Delta_T,\ \|W_k-C_k\|\to 0.$$

Since spaces X and Y can be decomposed as follows

$$X=S_P\{x_k\}\dot{+}X_k,\ Y=S_P\{y_k\}\dot{+}Y_k,$$

where $X_k=\ker x_k^*$, $Y_k=\ker y_k^*$. Therefore Z can be written as

$$Z=S_P\{x_k,\ y_k\}\dot{+}Z_k,$$

where $Z_k=X_k\dot{+}Y_k$. Put

$$p_k=x_k+y_k,\quad q_k=x_k-y_k,$$

then $\|p_k\|=\|q_k\|=1$, $\{p_1,\ p_2,\ \cdots\}$, $\{q_1,\ q_2,\ \cdots\}$ are linearly independent respectively and

$$W_kp_k=p_k,\ W_kq_k=-q_k,\ W_kZ_k=\{0\}.$$

Hence, corresponding to the decomposition

$$Z=S_P\{p_k\}\dot{+}S_P\{q_k\}\dot{+}Z_k,$$

W_k has the following matrix form

$$W_k=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Put

$$P_k=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad Q_k=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Then P_k and Q_k are rank-one projections and

$$W_k=P_k-Q_k.$$

More precisely P_k, Q_k have the following forms

$$P_k=\frac{1}{2}\,p_k\otimes(x_k^*+y_k^*),\quad Q_k=\frac{1}{2}\,q_k\otimes(x_k^*-y_k^*),$$

where $\langle x+y,\ x_k^*+y_k^*\rangle=\langle x,\ x_k^*\rangle+\langle y,\ y_k^*\rangle$. Hence we have

$$\|P_k\|=\|Q_k\|=1,\quad k\in N,$$

$$P_kp_k=p_k,\ Q_kq_k=q_k,\ k\in N.$$

By simple calculation, for $\lambda\notin\{-1,\ 0,\ 1\}$ we derive that

$$\begin{aligned}(\lambda-W_k)^{-1}&=(\lambda-P_k+Q_k)^{-1}=(\lambda-P_k)^{-1}[+Q_k(\lambda-P_k)^{-1}]^{-1}\\ &=\lambda(\lambda-P_k)^{-1}(\lambda+Q_k)^{-1}=-\lambda\left(\frac{1-P_k}{\lambda}-\frac{P_k}{1-\lambda}\right)\left(\frac{1-Q_k}{-\lambda}-\frac{Q_k}{1+\lambda}\right)\\ &=\frac{(1-P_k)(1-Q_k)}{\lambda}-\frac{P_k}{1-\lambda}+\frac{Q_k}{1+\lambda}.\end{aligned}$$

Let Γ be the circle with center $\lambda=1$, radius $\Gamma=\frac{1}{2}$ and counter clockwise direction. Then we have

$$P_k=\frac{1}{2\pi i}\int_\Gamma(\lambda-W_k)^{-1}d\lambda.$$

On the other hand, for $\lambda\notin\{-1,\ 0,\ 1\}$ we have

$$\lambda-C_k=(\lambda-W_k)[1+(\lambda-W_k)^{-1}(W_k-C_k)].$$

Since $\left\{\left\|\frac{(1-P_k)(1-Q_k)}{\lambda}\right\|,\ \left\|\frac{P_k}{1-\lambda}\right\|,\ \left\|\frac{Q_k}{1+\lambda}\right\|,\ \lambda\in\Gamma,\ k=1,\ 2,\ \cdots\right\}$ is uniformly bounded and $\|W_k - C_k\| \to 0$, we may suppose $\Gamma \subset P(C_k)$. Put

$$E_k = \frac{1}{2\pi i}\int_\Gamma (\lambda - C_k)^{-1} d\lambda.$$

Obviously, E_k is a projection and since $C_k \in \ker \Delta_T$, we have $E_k \in \ker \Delta_T$. Moreover we can derive that

$$\|E_k - P_k\| \leqslant \frac{1}{2\pi i}\int_\Gamma \|(\lambda - C_k)^{-1} - (\lambda - W_k)^{-1}\| d\lambda$$

$$\leqslant \frac{1}{2\pi i}\int_\Gamma \|[1 + (\lambda - W_k)^{-1}(W_k - C_k)]^{-1} - 1\| \, \|(\lambda - W_k)^{-1}\| d\lambda.$$

Hence we may assume that

$$\lim_{k\to\infty} \|E_k - P_k\| = 0.$$

Applying Lemma I. 4.10 of [3] we have rank E_k+1.

We now construct those three sequences defined in Lemma 3. Put

$$Z_k = E_k Z.$$

Since $\dim Z_k = 1$ and Z_k is an invariant subspace of T (i. e., $TZ_k \subset Z_k$), there exists an eigenvalue λ_k and for any vector $z \in Z_k$, we have

$$Tz = \lambda_k z.$$

Put $z_k = E_k p_k$. Since $z_k \in Z_k$, z_k satisfies the above equation. Let $z_k = u_k + v_k$, where $u_k \in X$, $v_k \in Y$. Then we have

$$Au_k = \lambda_k u_k, \ Bv_k = \lambda_k v_k.$$

We shall prove that both $\{u_k\}_{k\in N}$ and $\{v_k\}_{k\in N}$ are approximately linearly independent In fact, since $\{P_k\}_{k\in N}$, $\{E_k\}_{k\in N}$, $\{x_k\}_{k\in N}$, $\{p_k\}_{k\in N}$, $\{z_k\}_{k\in N}$ satisfy the conditions of Lemma 4, for any $n \in N$, there exist $\delta > 0$, $m \in N$ such that for any $C_{k_1}, \cdots, C_{k_n} \in C$,

$$\|u_{k_{n+1}} - C_{k_1}u_{k_1} - \cdots - C_{k_n}u_{k_n}\| \geqslant \delta,$$

provided $k_{n+1} > k_n > \cdots > k_1 \geqslant m$. Passing to sequence $\{y_k\}_{k\in N}$ and projection Q we obtain

$$\|v_{k_{n+1}} - C_{k_1}v_{k_1} - \cdots - C_{k_n}v_{k_n}\| \geqslant \delta.$$

Obviously, these facts will imply that $\{u_k\}_{k\in N}$, $\{v_k\}_{k\in N}$ are approximately linearly independent. Thus the lemma follows from Lemma 3.

Proof of Theorem The "only if" part of the theorem follows from Lemma 2 and Lemma 5, while the "if" part is a routine exercise, so we omit it. Thus the theorem is proved.

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