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THEORY OF SPECTRAL DECOMPOSITIONS WITH RESPECT TO THE IDENTITY FOR CLOSED OPERATORS*

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Abstract

In this paper, the auther discusses some properties of closed operators acting on a Banach space with the spectral decomposition property with respect to the identity (abbrev. SDI). First, some equivalent conditions are given for a closed operator T to have the SDI. Next, for every hyperinvariant subspace Y of T with the SDI, it is proved that the coinduced operator $\hat{T}=T/Y$ has the SDI. Finally, properties of maximal nets of hyperinvariant subspaces are discussed.

In the present paper, the author discusses some properties of maximal nets of hyperinvariant subspaces for a given closed operator T with the spectral decomposition property with respeot to the identity (abbrev. SDI). Let C be the complex plane, X a complex Banach space. The sign $C(X)$ denotes the set of all closed operators T acting in X and $B(X)$ denotes the algebra of all bounded operators acting on X. A set $E\subset C$ is called a neighborhood of ∞, denoted by $E\in V_\infty$, if for $r>0$ sufficiently large

$$\{\lambda\in C: |\lambda|>r\}\subset E.$$

An open set $\Delta\subset C$ is called a Cauchy domain if it has a finite number of components and its boundary $\partial\Delta$ is a positively oriented finite system of closed, nonintersecting, rectifiable Jordan curves. The following definition was given in [2, 9].

Definition 1. *Given $T\in C(X)$ and a positive integer $n\geqslant 1$. We say that T has the n-spectral decomposition property with respect to the identity (n-SDI), if for every open cover $\{G_i\}_{i=0}^n$ of $\sigma(T)$, where G_0 is a neighborhood of ∞, there exists a system $\{X_i\}_{i=0}^n$ of invariant subspaces of T with the following properties:*

(i) $\sigma(T|X_i)\subset G_i$ *for* $i=0, 1, 2, \cdots, n$;

(ii) *if* $G_i(1\leqslant i\leqslant n)$ *is relatively compact, then* $X_i\subset D_T$;

(iii) *there exists* $P_i\in B(X)(0\leqslant i\leqslant n)$ *commuting with* T, *such that*

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$$I=\sum_{i=0}^{n} P_i,\ R(P_i)\subset X_i \quad (0\leqslant i\leqslant n).$$

If for every $n\geqslant 1$, T has the n-SDI, then we say that T has the spectral decomposition property with respect to the identity (SDI).

It is easily seen that if T has the n-SDI, then it has the n-SDP and hence for every closed $F\subset C$, $X_T(F)$ is closed (see [1] Theorem 3). If F is compact, then [1, Theorem 4] implies that

$$X_T(F)=X_T^0(F)\oplus X_T(\emptyset),$$
$$\sigma(T\,|\,X_T^0(F))=\sigma(T\,|\,X_T(F)).$$

Hence, in Definition 1, X_0 can be replaced by $X_T(\overline{G}_0)$ and $X_i(1\leqslant i\leqslant n)$ can be replaced by $X_T^0(\overline{G}_i)$. Furthermore, using a similar argument given in [2], we can prove that if T has the n-SDI, then it has the $(n+1)$-SDI. Thus, if T has the 1-SDI, then it has the n-SDI for every $n\geqslant 1$ and hence it has the SDI. As for the open cover $\{G_i\}_{i=0}^n$ of $\sigma(T)$ in Definition 1, it is easily shown that $\{G_i\}_{i=0}^n$ can be changed as the cover of C.

The following Theorem is an extension of [3, Theorem 2.2] to the unbounded case, so we only sketch out the proof.

Theorem 2. *Given T, then the following assertions are equivelent:*

(i) *T has the SDI;*

(ii) (a) *T has the SDP,*

(b) *for every closed $F\subset C$ and every open $G\in V_\infty$, if $G\supset F$, then there exists an operator $P\in B(X)$ commuting with T such that*

$$Px=x \text{ for every } x\in X_T(F),\ R(P)\subset X_T(\overline{G});$$

(iii) (a) *T has the SDP,*

(b) *for every closed $F\subset C$ and every open $G\in V_\infty$, if $G\supset F$, then there exists a $B(X)$-valued analytic function R_λ for $\lambda\notin\overline{G}$ commuting with T such that*

$$(\lambda-T)R_\lambda x=x \text{ for every } x\in X_T(F),$$
$$R(R_\lambda)\subset X_T(\overline{G})\cap D_T.$$

Proof (i)$\Rightarrow$(ii). Put $G_0=G$ and let open G_1 be relatively compact and

$$\overline{G}_1\cap F=\emptyset,\ G_0\cup G_1\supset\sigma(T),$$

then there exists $P_i\in B(X)$ $(i=0, 1)$ commuting with T such that

$$I=P_0+P_1,\ R(P_0)\subset X_T(\overline{G}_0),\ R(P_1)\subset X_T^0(\overline{G}_1). \tag{1}$$

For $x\in X_T(F)$, we have $P_1x=0$ and hence $P_0x=x$. Let $P=P_0$, it follows from (1) that P satisfies the request.

(ii)$\Rightarrow$(iii). Let P be the operator given in (ii). Then the operator

$$R_\lambda=(\lambda-T\,|\,X_T(\overline{G}))^{-1}P$$

for $\lambda\notin\overline{G}$ satisfies all the properties given in (iii).

(iii)$\Rightarrow$(i). Let $\{G_0, G_1\}$ be an open cover of $\sigma(T)$ with $G_0\in V_\infty$ and $\overline{G}_0\neq C$, G_1

relatively compact. Let $H_0 \in V_\infty$ be another open subset such that

$$\overline{H}_0 \subset G_0, \ H_0 \cup G_1 \supset \sigma(T).$$

Then there exists a $B(X)$-valued analytic function R_λ for $\lambda \notin \overline{G}_0$, commuting with T such that

$$(\lambda - T)R_\lambda x = x \text{ for } x \in X_T(\overline{H}_0), \ R(R_\lambda) \subset X_T(\overline{G}_0) \cap D_T. \tag{2}$$

Let $\lambda_0 \notin \overline{G}_0$ be fixed. Put $P_0 = (\lambda_0 - T)R_{\lambda_0}$, $P_1 = I - P_0$, then $P_i \in B(X)$ $(i=0, 1)$ and commutes with T. Let $x \in X$, there corresponds a decomposition

$$x = x_0 + x_1 \text{ with } x_0 \in X_T(\overline{H}_0), \ x_1 \in X_T^0(\overline{G}_1).$$

It follows from (2) that $P_0 x_0 = x_0$ and hence

$$P_1 x = (I - P_0)X_0 + P_1 x_1 \in X_T^0(\overline{G}_1)$$

or equivalently, $R(P_1) \subset X_T^0(\overline{G}_1)$. (2) also implies that $R(P_0) \subset X_T(\overline{G}_0)$. Then for $X_0 = X_T(\overline{G}_0)$, $X_1 = X_T^0(\overline{G}_1)$ and P_0, P_1, all the conditions in Definition 1 are satisfied. T thus has the *SDI*.

Remara. If $F \subset C$ is compact, then the conditions (ii) and (iii) given in Theorem 3 can be replaced by the following ones respectively:

(ii′) (a) T has the SDP,

(b) for every compact F and every relatively compact open G, if $G \supset F$, then there exists $P \in B(X)$ commuting with T such that

$$Px = x \text{ for } x \in X_T^0(F), \ R(P) \subset X_T^0(\overline{G});$$

(iii′) (a) T has the SDP,

(b) for every compact F and every relatively compact open G, if $G \supset F$, then there exists $B(X)$-valued analytic function R_λ for $\lambda \notin \overline{G}$, commuting with T and

$$(\lambda - T)R_\lambda x = x \text{ for } x \in X_T^0(F), \ R(R_\lambda) \subset X_T^0(\overline{G}).$$

Lemma 3. *Let $T: D_T \to X$ be a linear operator. $Y_i (i=0, 1)$ is invariant under T and satisfies*

$$X = Y_0 + Y_1, \ Y_1 \subset D_T, \ T|Y_1 \in B(Y_1). \tag{3}$$

Then T is closed iff $T|Y_0$ is closed.

Proof The "only if" part is evident.

"If". It follows from (3) that there exists a number $M > 0$ such that for every $x \in X$, there exists $y_i \in Y_i (i=0, 1)$ satisfying

$$x = y_0 + y_1, \ \|y_0\| + \|y_1\| \le M\|x\|. \tag{4}$$

To prove the closedness of T, let $\{x_n\}_{n=1}^\infty \subset D_T$ satisfy

$$\{x_n\} \to x, \ \{Tx_n\} \to z.$$

Without loss of generality, we may suppose that

$$\sum_{n=1}^\infty \|x_{n+1} - x_n\| < +\infty.$$

It follows from (4) that for every x_n, there exists $y_{ni} \in Y_i (i=0, 1)$ such that

$$x_n = y_{n0} + y_{n1};$$

$$\|y_{n+10}-y_{n0}\|+\|y_{n+11}-y_{n1}\|\leqslant M\|x_{n+1}-x_n\|$$

and hence $\{y_{ni}\}_{n=1}^{\infty}$ converges. Let

$$\lim y_{ni}=y_i (i=0,1).$$

Since $T|Y_1$ is bounded, we have $Ty_{n1}\to Ty_1$ and hence

$$Ty_{n0}\to z-Ty_1. \tag{5}$$

(5) and the closedness of $T|Y_0$ imply that $y_0\in Y_0\cap D_T$ and $Ty_0=z-Ty_1$. Thus $x=y_0+y_1\in D_T$, $Tx=Ty_0+Ty_1=z$ and hence T is closed.

Theorem 4. *Given T with the SDI. If $Z\subset X$ is hyperinvariant under T, then the coinduced operator $\hat{T}$ on X/Z of T is closed.*

Proof The proof consists of three stages.

A. Let $F\subset C$ be a closed subset and let $G\supset F$ be open and $\overline{G}\neq C$. It follows from Theorem 2 that there exists an operator $P\in B(X)$ commuting with T such that

$$Px=x \text{ for every } x\in X_T(F),\ \mathscr{R}(P)\in X_T(\overline{G}).$$

Put $R_\lambda=(\lambda-T|X_T(\overline{G}))^{-1}P$ for $\lambda\notin\overline{G}$, then $\mathscr{R}(R_\lambda)\subset X_T(\overline{G})\cap D_T$ and for every $x\in X$,

$$(\lambda-T)R_\lambda x=Px. \tag{6}$$

(6) implies that $T R_\lambda\in B(X)$ and hence $\hat{T}\hat{R}_\lambda=\widehat{TR_\lambda}\in B(X/Z)$, furthermore, we have for every $\hat{x}\in X/Z$,

$$(\lambda-\hat{T})\hat{R}_\lambda\hat{x}=\hat{P}\hat{x}. \tag{7}$$

It follows from $\mathscr{R}(R_\lambda)\subset X_T(\overline{G})\cap D_T$ that

$$\mathscr{R}(PR_\lambda)\subset X_T(\overline{G})\cap D_T,$$

then

$$\begin{aligned}PR_\lambda&=(\lambda-T|X_T(\overline{G}))^{-1}(\lambda-T)PR_\lambda\\&=(\lambda-T|X_T(\overline{G}))^{-1}P(\lambda-T)R_\lambda=R_\lambda P.\end{aligned} \tag{8}$$

Put

$$\hat{X}_G=\{\hat{x}:\hat{x}\in X/Z,\ \hat{P}\hat{x}=\hat{x}\},$$

evidently,

$$\hat{X}_G\supset\widehat{X_T(F)}=\{\hat{x}:\hat{x}\cap X_T(F)\neq\emptyset\}. \tag{9}$$

Since $\hat{P}$ commutes with $\hat{T}$, we have $\hat{P}\hat{T}\hat{x}=\hat{T}\hat{P}\hat{x}=\hat{T}\hat{x}$ for every $\hat{x}\in\hat{X}_G\cap D_{\hat{T}}$ and henec $\hat{X}_G$ is invariant under $\hat{T}$. It follows from (8) that $\hat{P}$ commutes with $\hat{R}_\lambda$, so $\hat{X}_G$ is invariant under $\hat{R}_\lambda$. (7) and the commutability of $\hat{T}$ with $\hat{R}_\lambda$ imply that

$$(\lambda-\hat{T})\hat{R}_\lambda\hat{x}=\hat{x} \text{ for } \hat{x}\in\hat{X}_G,$$

$$\hat{R}_\lambda(\lambda-\hat{T})\hat{x}=\hat{x} \text{ for } \hat{x}\in\hat{X}_G\cap D_{\hat{T}}$$

and hence $(\lambda-\hat{T}|\hat{X}_G)^{-1}=\hat{R}_\lambda|\hat{X}_G$. Since $\hat{R}_\lambda|\hat{X}_G\in B(\hat{X}_G)$, we obtain that $\lambda-\hat{T}|\hat{X}_G$ is closed and so is $\hat{T}|\hat{X}_G$. Furthermore, we have

$$\sigma(\hat{T}|\hat{X}_G)\subset\overline{G}. \tag{10}$$

B. Let F be compact and let open G be relatively compact and $F\subset G$. It follows from the Remark of Theorem 2 that there exists $P\in B(X)$ such that

$$Px=x \text{ for every } x\in X_T^0(F),\ \mathscr{R}(P)\subset X_T^0(\overline{G}).$$

Put

$$\hat{X}_G^0=\{\hat{x}: \hat{x}\in X/Z,\ \hat{P}\hat{x}=x\}, \tag{11}$$

evidently, we have

$$\hat{X}_G^0\supset\widehat{X_T^0(F)}=\{\hat{x}: \hat{x}\cap X_T^0(F)\neq\emptyset\}. \tag{12}$$

Using the similar manner given in stage A, we can prove that $\hat{X}_G^0$ is invariant under $\hat{T}$, $\hat{T}|\hat{X}_G^0$ is closed and

$$\sigma(\hat{T}|\hat{X}_G^0)\subset\overline{G}. \tag{13}$$

It follows from $\mathscr{R}(P)\subset X_T^0(\overline{G})\subset D_T$ that $\mathscr{R}(\hat{P})\subset D_{\hat{T}}$ and hence $\hat{X}_G^0\subset D_{\hat{T}}$ by the equality (11). Thus $\hat{T}|\hat{X}_G^0$ is bounded.

C. In this final stage, we prove that $\hat{T}$ is closed. Let $\{G_0, G_1\}$ be an open cover of $\sigma(T)$ with $G_0\in V_\infty$, $\overline{G}_0\neq C$ and G_1 relatively compact. Let $\{H_0, H_1\}$ be another open cover of $\sigma(T)$ such that $H_0\in V_\infty$, $\overline{H}_0\subset G_0$ and $\overline{H}_1\subset G_1$. Then we have

$$X=X_T(\overline{H}_0)+X_T^0(\overline{H}_1)$$

and hence

$$\hat{X}=\widehat{X_T(\overline{H}_0)}+\widehat{X_T^0(\overline{H}_1)}.$$

Applying (9) and (12) to the pairs $\overline{H}_0$, G_0 and $\overline{H}_1$, G_1 respectively, we have

$$\widehat{X_T(\overline{H}_0)}\subset\hat{X}_{G_0},\ \widehat{X_T^0(\overline{H}_1)}\subset\hat{X}_{G_1}^0$$

and hence

$$\hat{X}=\hat{X}_{G_0}+\hat{X}_{G_1}^0.$$

It follows from stage A that $\hat{T}|\hat{X}_{G_0}$ is closed and form stage B that $\hat{T}|\hat{X}_{G_1}^0$ is bonuded. Thus $\hat{T}$ is closed by Lemma 3.

Theorem 5. *Given T with the SDI. If Y and Z are hyperinvariant under T and $Y\supset Z$. Then the restriction operator $\hat{T}|\hat{Y}$ has the SDI, where $\hat{T}$ is the coinduced operator on X/Z and $\hat{Y}=Y/Z$.*

Proof First we prove that for open $G\subset C$, $\overline{G}\neq C$,

$$\sigma(\hat{T}|\hat{Y}\cap\hat{X}_G)\subset\overline{G}, \tag{14}$$

if G is relatively compact, then

$$\sigma(\hat{T}|\hat{Y}\cap\hat{X}_G^0)\subset\overline{G}, \tag{15}$$

where $\hat{X}_G$, $\hat{X}_G^0$ are defined in stage A and stage B of Theorem 4 respectively. We confine the proof to (14). Since Y is hyperinvariant under T, it is invariant under R_λ given in stage A of Theorem 4 and hence $\hat{Y}$ is invariant under $\hat{R}_\lambda$. It follows from (7) and the commutability of $\hat{T}$ with $\hat{R}_\lambda$ that

$$(\lambda-\hat{T})\hat{R}_\lambda\hat{x}=\hat{x} \text{ for } \hat{x}\in\hat{Y}\cap\hat{X}_G,$$

$$\hat{R}_\lambda(\lambda-\hat{T})\hat{x}=\hat{x} \text{ for } \hat{x}\in\hat{Y}\cap\hat{X}_G\cap D_T,$$

thus (14) is proved.

Next, assume that $\{G_0, G_1\}$ is an open cover of C with $G_0\in V_\infty$, $\overline{G}_0\neq C$ and G_1

relatively compact. Let $\{H_0, H_1\}$ be another open cover of C such that $H_0 \in V_\infty$, $\overline{H}_0 \subset G_0$, $\overline{H}_1 \subset G_1$. Then there exists $P_i \in B(X)(i=0, 1)$ commuting with T and satisfying

$$I = P_0 + P_1, \ \mathscr{R}(P_0) \subset X_T(\overline{H}_0), \ \mathscr{R}(P_1) \subset X_T^0(\overline{H}_1),$$

thus we have

$$\hat{I} = \hat{P}_0 + \hat{P}_1, \ \mathscr{R}(\hat{P}_0) \subset \widehat{X_T(\overline{H}_0)} \subset \hat{X}_{G_0},$$

$$\mathscr{R}(\hat{P}_1) \subset \widehat{X_T^0(\overline{H}_1)} \subset \hat{X}_{G_1}^0. \tag{16}$$

(16) implies that

$$\hat{I}|\hat{Y} = \hat{P}_0|\hat{Y} + \hat{P}_1|\hat{Y}, \ \mathscr{R}(\hat{P}_0|\hat{Y}) \subset \hat{Y} \cap X_{G_0}, \ \mathscr{R}(\hat{P}_1|\hat{Y}) \subset \hat{Y} \cap \hat{X}_{G_1}^0 \tag{17}$$

and (14), (15) imply that

$$\sigma(\hat{T}|\hat{Y} \cap \hat{X}_{G_0}) \subset \overline{G}_0, \ \sigma(\hat{T}|\hat{Y} \cap \hat{X}_{G_1}^0) \overline{G}_1. \tag{18}$$

(17), (18) conclude that $\hat{T}|\hat{Y}$ has the SDI.

Corollary 1. *If T has the SDI, then every hyperinvariant subspace Z of T is analytically invariant under T.*

Proof Let $f: \omega_f \to D_T$ be analytic on an open connected ω_f and

$$(\lambda - T) f(\lambda) \in Z.$$

Then $(\lambda - \hat{T})\hat{f}(\lambda) = \hat{O}$. It follows from Theorem 5 that $\hat{T}$ has the SDI and hence it has the SVEP. Thus we have $\hat{f}(\lambda) = \hat{O}$ and hence $f(\lambda) \in Y$.

Corollary 2. *If T has the SDI, then for every hyperinvariant subspace Y of $T|Y$ has the SDI.*

Proof Put $Z = \{0\}$, then the SDI of $T|Y$ is a consequence of Theorem 5.

Corolloary 3. *If T has the SDI, then for every two hyperinvariant subspaces Y and Z, $Y \supset Z$ implies that $\sigma(T|Y) \supset \sigma(T|Z)$.*

Proof It follows from Corollary 1 that Z is analytically invariant under T and hence is analytically invariant under $T|Y$. Thus we have $\sigma(T|Y) \supset \sigma(T|Z)$.

Proposition 6. *If the densely defined operator T has the SDP and if for every relatively compact open G, there exists an operator $P \in B(X)$ commuting with T and satisfying $\mathscr{R}(P) \subset X_T^0(\overline{G})$, then for the operator T^*, P^* commutes with it and satisfies $\mathscr{R}(P^*) \subset X_{T^*}^{*0}(\overline{G})$.*

Proof Let the open G be relatively compact. Since T has the SDP, it follows from [8, Theorem IV 5.5] that T^* has the SDP and

$$X_{T^*}^{*0}(\overline{G}) = [X_T(H)]^{\perp}, \tag{19}$$

where $H = C \backslash \overline{G}$. By the hypothesis, for the open G, there exists an operator $P \in B(X)$ such that P commutes with T and that $\mathscr{R}(P) \subset X_T^0(\overline{G})$. Let $x \in X_T(H)$, then $\sigma_T(Px) \subset \sigma_T(x)$. Since $\sigma_T(x) \cap \overline{G} = \emptyset$, we have

$$Px \in X_T^0(\overline{G}) \cap X_T(\sigma_T(x)) = X_T^0(\overline{G} \cap \sigma_T(x)) = X_T^0(\emptyset) = \{0\} \tag{20}$$

and hence $Px = 0$. Let $x^* \in X^*$, then

$$\langle x, P^* x^* \rangle = \langle Px. x^* \rangle = 0 \quad (x \in X_T(H_0))$$

and hence (19) implies that $P^*x^* \in X^{*0}_{T^*}(\bar{G})$, or equivalently,

$$\mathscr{R}(P^*) = X^{*0}_{T^*}(\bar{G}). \tag{21}$$

Proposition is thus proved.

Corollary. *If T is densely defined and has the SDI, then T^* has the SDI.*

Proof Let $\{G_0, G_1\}$ be an open cover of C with $G_0 \in V_\infty$ and G_1 relatively compact, then there exists $P_i \in B(X)$ commuting with T and satisfying

$$I = P_0 + P_1,\ \mathscr{R}(P_0) \subset X_T(\bar{G}_0),\ \mathscr{R}(P_1) \subset X^0_T(\bar{G}_1).$$

It follows from (21) that

$$\mathscr{R}(P_1^*) \subset X^{*0}_{T^*}(\bar{G}_1). \tag{22}$$

By [8, Theorem IV 5.5], T^* has the SDP and in addition to (19), we have

$$X^*_{T^*}(\bar{G}_0) = [\overline{X^0_T(H_1)}]^\perp,$$

where $H_1 = C \backslash \bar{G}_0$ and $\overline{X^0_T(H_1)} = \bigvee_{\substack{F\subset H_1\\ F \text{compact}}} X^0_T(F)$. By a similar argument used in Proposition 7, we have

$$\mathscr{R}(P_0^*) \subset X^*_{T^*}(\bar{G}_0). \tag{23}$$

(22), (23) and the evident equality $I^* = P_0^* + P_1^*$ imply that T^* has the SDI.

Definition 7. *Let T have the SDP. If there exists a sequence of relatively compact open sets $\{G_n\}_{n=1}^\infty$ and a sequence $\{P_n\}_{n=1}^\infty$ of bounded linear operators commuting with T such that $\mathscr{R}(P_n) \subset X^0_T(\bar{G}_n)$ and that for every $x \in X$ and every $x^* \in X^*$,*

$$\langle P_n x, x^* \rangle \to \langle x, x^* \rangle,$$

then we say that T has property (δ).

Theorem 8. *Let T have the SDP and property (δ), then for every family of hyperinvariant su bspaces $\{X_\alpha\}_{\alpha\in A}$ of T, $Y = \bigvee_{\alpha\in A} X_\alpha$ is also hyperinvariant under T.*

Proof Let $S \in B(X)$ commute with T. Since S is bounded, it is easily seen that Y is invariant under S and hence it is sufficient to prove that Y is invariant under T.

Let $x \in X_\alpha$, then $P_n x \in X_\alpha \cap X^0_T(\bar{G}_n)$ and

$$\lim_{n\to\infty} \langle P_n x, x^* \rangle = \langle x, x^* \rangle,$$

consequently, by the Hahn-Banach Theorem, we have

$$X_\alpha = \bigvee_{n=1}^\infty X_\alpha \cap X^0_T(\bar{G}_n).$$

Since $X_\alpha \cap X^0_T(\bar{G}_n) \subset D_T$, $T | X_\alpha$ is densely defined. It follows from the same reason that T is densely defined. Let $X_\alpha^\perp$ be the annihilator of X_α in X^*. Let $x \in X_\alpha \cap D_T$, $x^* \in X_\alpha^\perp \cap D_{T^*}$, then $\langle x, T^*x^* \rangle = \langle Tx, x^* \rangle = 0$ *and* $\overline{X_\alpha \cap D_T} = X_\alpha$ imply that $T^*x^* \in X_\alpha^\perp$ and hence $X_\alpha^\perp$ is invariant under T^*.

Since $Y^\perp = (\bigvee_{\alpha\in A} X_\alpha)^\perp = \bigcap_{\alpha\in A} X_\alpha^\perp$ and $X_\alpha^\perp$ is invariant under T^*, we have that $Y^\perp$ is invariant under T^*. Now, suppose that $x \in Y \cap D_T$, $x^* \in Y^\perp$. It follows from

Proposition 6 that $P_n^* x^* \in X_{T^*}^{*0}(\overline{G}_n)$ *and hence*

$$P_n^* x^* \in Y^\perp \cap X_{T^*}^{*0}(\overline{G}_n) \subset Y^\perp \cap D_{T^*};\ T^* P_n^* x^* \in Y^\perp.$$

Thus for every n,

$$0 = \langle x, T^* P_n^* x^* \rangle = \langle Tx, P_n^* x^* \rangle = \langle P_n Tx, x^* \rangle = \lim_{n\to\infty} \langle P_n Tx, x^* \rangle = \langle Tx, x^* \rangle,$$

which concludes that Y is invariant under T.

Now we are in a position to discuss the properties of a given maximal net of hyperinvariant subspaces of T. Let A be a totally ordered set, a family $N = \{X_\alpha\}_{\alpha\in A}$ of hyperinvariant subspaces of T is called a net, if for $\alpha, \beta \in A$, $\alpha < \beta$ implies $X_\alpha \subset X_\beta$. In virtue of Zorn's Lemma, we can show that N is contained in a maximal net of hyperinvariant subspaces of T. Without loss of generality, we may suppose that N itself is maximal. Thus we have that $\{0\}$, $X \in N$ and hence N is nonempty.

Lemma 9. *Let T have the SDP and property (δ) and let $N = \{X_\alpha\}_{\alpha\in A}$ be a maximal net of hyperinvariant subspaces of T. Then for every $\alpha \in A$, there exists a $\beta \in A$ such that*

$$X_\beta = \bigvee_{\gamma<\alpha} X_\gamma.$$

Proof Put $Y = \bigvee_{\gamma<\alpha} X_\gamma$, by Theorem 9, Y is hyperinvariant under T, furthermore, we have

$$X_\gamma \subset Y \subset X_{\gamma'} \text{ for } \gamma < \alpha \leqslant \gamma'. \tag{24}$$

If $Y \notin N$, it follows from (24) that $N' = \{X_\alpha, Y\}_{\alpha\in A}$ is a net of hyperinvari ant subspaces of T which contains N as a proper subset, this contradicts the maximal property of N. Thus the lemma follows.

Denote β by $\alpha - 0$, then $\alpha - 0 \leqslant \alpha$. If $\alpha - 0 = \alpha$, we will say that N is cotinuous at α.

Theorem 10. *Given T with the SDI and property (δ). Let $N = \{X_\alpha\}_{\alpha\in A}$ be a maximal net of hyperinvariant subspaces of T, then*

(i) *for every $\alpha \in A$, we have*

$$\sigma(T|X_\gamma) \subset \sigma(T|X_{\alpha-0}) \text{ for } \gamma < \alpha, \tag{25}$$

$$\sigma(T|X_{\alpha-0}) = \bigvee_{\gamma<\alpha} \sigma(T|X_\gamma), \tag{26}$$

$$\sigma(T|X_{\alpha-0}) \subset \sigma(T|X_\alpha); \tag{27}$$

(ii) *if N is discontinuous at α, let $\hat{T}_\alpha$ be the coinduced operator of $T|X_\alpha$ on $\hat{X}_\alpha = X_\alpha / X_{\alpha-0}$, then either*

(1°) *$\hat{T}_\alpha$ is unbounded and $\sigma(\hat{T}_\alpha) = \emptyset$*

or

(2°) *$\hat{T}_\alpha \in B(\hat{X}_\alpha)$ and $\sigma(\hat{T}_\alpha)$ consists of exactly one point ξ_α, furthermore, either $\hat{T}_\alpha = \xi_\alpha \hat{T}_\alpha$ or $\hat{T}_\alpha - \xi_\alpha \hat{I}_\alpha$ is a quasinilpotent;*

(iii) *N is discontinuous at α, if either $\sigma(T|X_{\alpha-0}) \neq \sigma(T|X_\alpha)$ or $\hat{X}_\alpha$ is finitly* *dimensional, then there exists a hyperinvariant subspace* Y_α *of* T *such that*

$$X_\alpha = X_{\alpha-0} \oplus Y_\alpha,\ Y_\alpha \subset D_T$$

and $\sigma(T|Y_\alpha)$ *consists of exactly one point* ξ_α*, furthermore, the conclusion given in* (ii, 2°) *remains true and in the case of* $\hat{X}_\alpha$ *being finitly dimensional, we have*

$$T|Y_\alpha = \xi_\alpha I|Y_\alpha.$$

Proof Since the proof is similar to that of [3, Theorem 3.2], we only sketch out it.

(i) (25) and (27) are the consequence of Corollary 3 of Theorem 5. To prove (26), it is easy to see that

$$\sigma(T|X_{\alpha-0}) \supset \bigvee_{\gamma<\alpha} \sigma(T|X_\gamma).$$

In virtue of the reduction to absurdity, we can prove the opposite inclusion.

(ii) Theorem 5 implies that $\hat{T}_\alpha$ has the SDI. If $\hat{T}_\alpha$ is unbounded and if $\sigma(\hat{T}_\alpha)$ consists at least one point, then there exists for $\hat{T}_\alpha$, a nontrivial $\hat{T}_\alpha$-bounded spectral maximal space $\hat{Z}$. Let

$$Z = \{x: x \in \hat{x} \in \hat{Z}\},$$

then $X_{\alpha-0} \subsetneqq Z \subsetneqq X_\alpha$ and Z is hyperinvariant under T, this is impossible, since N is a maximal net. Thus $\sigma(\hat{T}_\alpha)$ is empty. If $\hat{T}_\alpha$ is bounded and if $\sigma(\hat{T}_\alpha)$ consists of more than one point, then a similar argument used above shows that there is a contradiction. Hence $\sigma(\hat{T}_\alpha)$ consists of exactly one point ξ_α. By the reduction to absurdity, it follows the second conclusion of (ii, 2°).

(iii) First, suppose that $\sigma(T|X_{\alpha-0}) \subsetneqq \sigma(T|X_\alpha)$, it follows from [4, Theorem 2.1] that $\sigma(\hat{T}_\alpha)$ is nonempty and then (ii) implies that $\sigma(\hat{T}_\alpha)$ consists of exactly one point ξ_α and $\hat{T}_\alpha$ is bounded. Evidently, $\{\xi_\alpha\}$ and $\sigma(T|X_{\alpha-0})$ are spectral set of $T|X_\alpha$. Then the application of [5, Theorem V. 9.1] concludes the first case of (iii).

Next, suppose that $\hat{X}_\alpha$ is finitly dimensional, let $\{\hat{x}_1, \cdots, \hat{x}_n\}$ be a base of $\hat{X}_\alpha$. Then $\{x_1, \cdots, x_n\}$ is a linear independent system contained in X_α, where $x_i \in \hat{x}_i (1 \leqslant i \leqslant n)$. Let Y_α be the subspace spanned by $\{x_1, \cdots, x_n\}$, then

$$X_\alpha = X_{\alpha-0} \oplus Y_\alpha$$

and $T|Y_\alpha = \xi_\alpha I|Y_\alpha$.

Given T. Suppose that there exists a function $f: G \to C$ analytic on a neighborhood G of $\sigma(T) \cup \{\infty\}$ and assuming zero at most at $\lambda=0$ and at $\lambda=\infty$ and being non-constant on every component of G such that $f(T)$ is completely continuous. Then $\sigma(T)$ has no non-zero cluster point on C and hence, by applying [5, Theorem V. 9.1], T has the SDI. Furthermore, we suppose that T satisfies the property (δ), then it follows from Theorem 8 that for a family $\{X_\alpha\}_{\alpha \in A}$ of hyperinvariant subspaces of T, $Y = \bigvee_{\alpha \in A} X_\alpha$ is hyperinvariant under T.

Theorem 11. *Suppose that T satisfies all the conditions mentioned above. Let $N=\{X_\alpha\}_{\alpha\in A}$ be a maximal net of hyperinvariant subspaces of T, if for every discontinuous point α of N, the coinduced operator $\hat{T}_\alpha$ on $X_\alpha/X_{\alpha-0}$ satisfies $\sigma(\hat{T}_\alpha)=\{0\}$ or $\sigma(\hat{T}_\alpha)=\emptyset$, then T is a quasinilpotent, i. e. T is bounded and $\sigma(T)=\{0\}$.*

Proof First, we show that $\sigma(T)=\{0\}$ or $\sigma(T)=\emptyset$. Assume the contrary, then there exists a point $\xi_0\neq 0$ such that $\xi_0\in\sigma(T)$. By the hypothesis, ξ_0 is an isolated point of $\sigma(T)$. Since $\{0\}$, $X\in N$ and $\sigma(T|\{0\})=\emptyset$, $\sigma(T|X)=\sigma(T)$, we may divide A into two parts:

$$A^-=\{\gamma:\ \xi_0\notin\sigma(T|X_\gamma),\ \gamma\in A\};\tag{28}$$

$$A^+=\{\gamma:\ \xi_0\in\sigma(T|X_\gamma),\ \gamma\in A\}.\tag{29}$$

A^-, A^+ are nonempty and

$$A=A^-\cup A^+,\ A^-\cap A^+=\emptyset.$$

Put

$$X^-=\bigvee_{\gamma\in A^-}X_\gamma,\ X^+=\bigcap_{\gamma\in A^+}X_\gamma,$$

then $X^\pm$ is hyperinvariant under T and satisfies

$$X_\gamma\subset X^-\subset X^+\subset X_{\gamma'}\ \text{for}\ \gamma\in A^-,\ \gamma'\in A^+.$$

Since N is a maximal net and T has property (δ), there exists $\alpha\in A$ such that

$$X^-=X_{\alpha-0},\qquad X^+=X_\alpha.$$

Since ξ_0 is an isolated point of $\sigma(T)$, it follows from (26) and (28) that

$$\xi_0\notin\bigvee_{\gamma\in A^-}(T|X_\gamma)=\sigma(T|X_{\alpha-0}).\tag{30}$$

Next, we show that $\xi_0\in\sigma(T|X_\alpha)$, by Corollary 1 of Theorem 5, for eveay $\gamma\in A$, X_γ is analytically invariant under T, then

$$X_\gamma\cap X_T^0(\{\xi_0\})=X^0_{\gamma,T|X_\gamma}(\{\xi_0\})\tag{31}$$

for $\gamma\in A$. Since $\xi_0\neq 0$ and $\xi_0\neq\infty$, we have $f(\xi_0)\neq 0$. Since $f(T)$ is completely continuous $X_{f(T)}(f(\{\xi_0\}))$ is finitely dimensional, [1, Theorem 2.1] implies that

$$X_{f(T)}(f(\{\xi_0\}))=X_T^0(f^{-1}(f(\{\xi_0\}))\supset X_T^0(\{\xi_0\})$$

and hence $X_T^0(\{\xi_0\})$ is finitly dimensional. Since, for every $\gamma\in A^+$, we have $\xi_0\in\sigma(T|X_\gamma)$ and $\sigma(T|X^0_{\gamma,T|X_\gamma}(\{\xi_0\}))=\{\xi_0\}\neq\emptyset$, it follows that

$$X^0_{\gamma,T|X_\gamma}(\{\xi_0\})\neq\{0\}\ \text{for}\ \gamma\in A^+.$$

$X_T^0(\{\xi_0\})$ being finite dimensional, (31) implies that there exists a $\gamma_0\in A^+$ such that, for every $\gamma\in A^+$ with $\gamma\leqslant\gamma_0$ we have

$$X_T^0(\{\xi_0\})\cap X_\gamma=X_T^0(\{\xi_0\})\cap X_{\gamma_0}\neq\{0\}.\tag{32}$$

It follows from (31), (32) and $X^+=X_\alpha$ that

$$X^0_{\alpha,T|X_\alpha}(\{\xi_0\})=X_T^0(\{\xi_0\})\cap X_\alpha=X_T^0(\{\xi_0\})\cap(\bigcap_{\gamma\in A^+}X_\gamma)=\bigcap_{\gamma\in A^+}[X_T^0(\{\xi_0\})\cap X_\gamma]$$

$$=X_T^0(\{\xi_0\})\cap X_{\gamma_0}\neq\{0\}$$

and hence

$$\xi_0 \in \sigma(T \mid X^0_{\alpha, T|X_\alpha}(\{\xi_0\})) \subset \sigma(T \mid X_\alpha). \tag{33}$$

(30) and (33) imply that N is discontinuous at α and $\sigma(\hat{T}_\alpha) = \{\xi_0\}$. By hypothesis, we have either $\sigma(T_\alpha) = \{0\}$ or $\sigma(T_\alpha) = \emptyset$. This contradicts the assumption $\xi_0 \neq 0$, therefore $\sigma(T_\alpha) = \{0\}$ or $\sigma(T_\alpha) = \emptyset$. Next, to prove that $X_T(\emptyset) = \{0\}$, let $x \in X_T(\emptyset)$, by the property (δ), there exist relatively compact open G_n and operator

$$P_n \in B(X)$$

such that P_n commutes with T and satisfies $\mathscr{R}(P_n) \subset X^0_T(\overline{G}_n)$, then

$$P_n x \in X^0_T(\overline{G}_n) \cap X_T(\emptyset) = X^0_T(\emptyset) = \{0\}$$

and hence $P_n x = 0$. It follows from the equality

$$\langle x, x^* \rangle = \lim_{n \to \infty} \langle P_n x, x^* \rangle = 0$$

for every $x^* \in X^*$ that $x = 0$ and hence $X_T(\emptyset) = \{0\}$. Thus $\sigma(T) = \{0\}$ and the decomposition

$$X = X_0 \oplus X_T(\emptyset)$$

with $X_0 \subset D_T$ implies that T is bounded and thus T is a quasinilpotent.

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