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GLOBAL SMOOTH SOLUTIONS OF DISSIPATIVE BOUNDARY VALUE PROBLEMS FOR FIRST ORDER QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract

This paper discusses the following initial-boundary value problems for the first order quasilinear hyperbolic systems:

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=0, \tag{1}$$

$$u^{\rm II}=F(u^{\rm I}), as \quad x=0, \tag{2}$$

$$u^{\rm I}=G(u^{\rm II}), as \quad x=L, \tag{3}$$

$$u=u^0(x), \ as \quad t=0, \tag{4}$$

where the boundary conditions (2), (3) satisfy $F(0)=0$, $G(0)=0$ and the dissipative conditions, that is, the spectral radii of matrices $B_1=\frac{\partial F}{\partial u^{\rm I}}(0)\frac{\partial G}{\partial u^{\rm II}}(0)$ and $B_2=\frac{\partial G}{\partial u^{\rm II}}(0)\frac{\partial F}{\partial u^{\rm I}}(0)$ are less than unit.

Under certain assumptions it is proved that the initial-boundary problem (1)—(4) admits a unique global smooth solution $u(x, t)$ and the C^1-norm $|u(t)|_{C^1}$ of $u(x, t)$ decays exponentially to zero as $t\to\infty$, provided that the C^1-norm $|u^0|_{C^1}$ of the initial data is sufficiently small.

§ 1. Introduction

It is well known that for the initial value problems and boundary value problems of first order quasilinear hyperbolic systems, in general, the smooth solutions can exist only locally. That is to say, the solutions may develop singularities in a finite time (for example, cf. [1, 2, 3]). But if we add a dissipative influence in the systems or in the boundary conditions, then it can guarantee the existence on $t\geqslant 0$ for global smooth solutions of Cauchy problems or boundary problems at least for small smooth initial data (cf. [4, 5, 6, 7]).

Greenberg and Li Ta-tsien discussed in [7] the boundary value problem for quasilinear wave equation with an elastic damping boundary condition on one end. In practice, they discussed in [7] the following general bounary value problem with

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two unknown functions

$$\frac{\partial u_1}{\partial t}+\lambda_1(u_1,u_2)\frac{\partial u_1}{\partial x}=0,\quad 0<t<\infty, \tag{1}$$

$$\frac{\partial u_2}{\partial t}+\lambda_2(u_1,u_2)\frac{\partial u_2}{\partial x}=0,\quad 0<x<L, \tag{2}$$

$$\lambda_1<0<\lambda_2$$

$$u_1(t,L)=f(u_2(t,L)), \tag{3}$$

$$u_2(t,0)=g(u_1(t,0)), \tag{4}$$

$$t=0:\ u_1=u_1^0(x),\ u_2=u_2^0(x). \tag{5}$$

Under the assumption that boundary conditions (3), (4) and initial condition (5) satisfy compatibility conditions, they proved that if f, g satisfy

$$f(0)=g(0)=0, \tag{6}$$

$$|f'(0)g'(0)|<1, \tag{7}$$

then boundary value problem (1)—(5) admits a unique global C^1 smooth solution on $t\geqslant 0$ and the C^1 norm of the solution decays exponentially to zero as $t\to\infty$, provided that $|u_1^0|_{C^1}+|u_2^0|_{C^1}$ is sufficiently small.

The condition (7) expresses the dissipative effect on boundary. The role of this condition in guaranteeing the existence of global smooth solution for boundary value problem (1)—(5) is similar to that of dissipative term added in system (for example, as in [4, 5, 6]) in guaranteeing the existence of global smooth solution for Cauchy problem. Therefore, the condition (7) is essential. If it is not satisfied (for example, $|f'(0)g'(0)|=1$), then using the methods in [8], it is not difficult to construct an example in which the solution develops a singularity in a finite time.

In this paper, our aim is to generalize these results to general boundary value problems for first order quasilinear hyperbolic systems with several unknown functions.

§ 2. Main results

Consider the following first order quasilinear system

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=0, \tag{8}$$

where u is an unknown function vector with n components, and $A(u)$ is an $n\times n$ suitably smooth function matrix which depends only on the unknown function u.

First, we state the asummptions on system (8), and then discuss how to pose the boundary conditions.

(A_1). Assume that system (8) is hyperbolic for sufficiently small $|u|_{C^0}$ in following sense:

1°) The matrix $A(u)$ has n smooth real eigenvalues $\lambda_1(u),\cdots,\lambda_n(u)$, and

$$\lambda_1(0) \leqslant \cdots \leqslant \lambda_m(0) < 0 < \lambda_{m+1}(0) \leqslant \cdots \leqslant \lambda_n(0). \tag{9}$$

2°) $A(u)$ has n linearly independent left eigenvectors corresponding to n real eigenvalues

$$\zeta_i(u) = (\xi_{i1}(u), \cdots, \zeta_{in}(u)) \quad (i=1, \cdots, n).$$

Without loss of generality, we suppose that matrix

$$\zeta(u) = \begin{pmatrix} \zeta_{11}(u)\cdots\zeta_{1n}(u) \\ \cdots\cdots\cdots\cdots \\ \zeta_{n1}(u)\cdots\zeta_{nn}(u) \end{pmatrix}$$

is identity when $u=0$, that is

$$\zeta(0) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \tag{10}$$

Now, we discuss how to pose the boundary conditions. When the condition (10) holds, the general initial-boundary conditions for system (8) should have following forms:

$$u^{\mathrm{II}} = F(u^{\mathrm{I}}), \text{on} \quad x=0, \tag{11}$$

$$u^{\mathrm{I}} = G(u^{\mathrm{II}}), \text{ on } x=L, \tag{12}$$

$$t=0: u=\varphi(x), \tag{13}$$

where

$$u^{\mathrm{I}} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad u^{\mathrm{II}} = \begin{pmatrix} u_{m+1} \\ \vdots \\ u_n \end{pmatrix},$$

$$F = \begin{pmatrix} F_{m+1} \\ \vdots \\ F_n, \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ \vdots \\ G_m \end{pmatrix}.$$

Furthermore, we write

$$B_1 = \frac{\partial F}{\partial u^{\mathrm{I}}}(0) \cdot \frac{\partial G}{\partial u^{\mathrm{II}}}(0),$$

$$B_2 = \frac{\partial G}{\partial u^{\mathrm{II}}}(0) \cdot \frac{\partial F}{\partial u^{\mathrm{I}}}(0),$$

where

$$\frac{\partial F}{\partial u^{\mathrm{I}}} = \frac{\partial(F_{m+1}, \cdots, F_n)}{\partial(u_1, \cdots, u_m)}, \quad \frac{\partial G}{\partial u^{\mathrm{II}}} = \frac{\partial(G_1, \cdots, G_m)}{\partial(v_{m+1}, \cdots, u_n)}.$$

(A_2). Assume that the initial-boundary conitions (11), (12), (13) satisfy the following

1°) F, $G \in C^1$, and

$$F(0) = G(0). \tag{14}$$

2°) The Apectral radii of B_1, B_2 are less than 1.

3°) Compatibity conditions

$$\varphi^{\mathrm{II}}(0) = F(\varphi^{\mathrm{I}}(0)), \ \varphi^{\mathrm{I}}(L) = G(\varphi^{\mathrm{II}}(L)), \tag{15}$$

$$\left[A^{21}(\varphi(0))-\frac{\partial F}{\partial u^{\mathrm{I}}}(\varphi(0))A^{11}(\varphi(0))\right]\frac{\partial\varphi^{\mathrm{I}}(0)}{\partial x}$$

$$+\left[A^{22}(\varphi(0))-\frac{\partial F}{\partial u^{\mathrm{I}}}(\varphi(0))A^{12}(\varphi(0))\right]\frac{\partial\varphi^{\mathrm{II}}(0)}{\partial x}=0, \tag{16}$$

$$\left[A^{11}(\varphi(L))-\frac{\partial G}{\partial u^{\mathrm{II}}}(\varphi(L))A^{21}(\varphi(L))\right]\frac{\partial\varphi^{\mathrm{I}}(L)}{\partial x}$$

$$+\left[A^{12}(\varphi(L))-\frac{\partial G}{\partial u^{\mathrm{II}}}(\varphi(L))A^{22}(\varphi(L))\right]\frac{\partial\varphi^{\mathrm{II}}(L)}{\partial x}=0, \tag{17}$$

where corresponding to $u=\begin{pmatrix}u^{\mathrm{I}}\\ u^{\mathrm{II}}\end{pmatrix}$, we write φ and $A(u)$ in the forms of block matrices, for example

$$A(u)=\begin{pmatrix}A^{11}(u) & A^{12}(u)\\ A^{21}(u) & A^{22}(u)\end{pmatrix}.$$

Here, the assumption $(A_2)3°)$ is used only to guarantee the local existence for smooth solutions. It is not essential for the existence of global smooth solutions. The assumption $(A_2)2°)$ which corresponds to condition (7) is a dissipative condition on boundary, it is essential for our following discussion. If only the condition $(A_2)2°)$ is satisfied, from the related results in matrix theory we know that there exists a positive integer p such that

$$\sigma_0=\max(\|B_1^p\|,\ \|B_2^p\|)<1, \tag{18}$$

where norm $\|\cdot\|$ of a matrix denotes the maximum of the row sums of the absolutes of the entries of the matrix.

Now, we can state the main results in the present paper.

Theorem. *Suppose that (A_1) and (A_2) hold. Then there exists $\delta>0$ such that the mixed initial-boundary problem (8) (11) (12) (13) admits a unique global smooth solution $u(t, x)$ on $t\geqslant 0$ and $|u(t)|_{C^1}$ decays exponentially to zero as. $t\to\infty$, the decay rate being*

$$\exp\left(\frac{\lambda_{\min}\ln\sigma}{2pL}t\right), \tag{19}$$

provided that $|\varphi|_{C^0}+\left|\frac{\partial\varphi}{\partial x}\right|_{C^0}<\delta$. Here

$$\lambda_{\min}=\min_{1\leqslant i\leqslant n}|\lambda_i|_{C^0}, \tag{20}$$

and σ is any real number in $(\sigma_0, 1)$.

Remark 1. The important and useful case is that when $p=1$ in (18), that is

$$\sigma_0=\max(\|B_1\|,\ \|B_2\|)<1. \tag{21}$$

If (21) is satisfied, then it is easy to see that $(A_2)2°)$ holds, that is, the absolutes of eigenvalues of B_1 and B_2 are less than 1. For this case, the initial-boundary problem (8) (11) (12) (13) admits a global smooth solution $u(t,x)$, and the exponentially decay rete of $|u(t)|_{C^1}$ as $t\to\infty$ is

$$\exp\left(\frac{\lambda_{\min}\ln\sigma}{2L}t\right). \tag{22}$$

Remark 2. From the decay rates (19) and (22) of solutions we can see that, for fixed p, the smaller the σ_0 is, i. e. the stronger the dissipative effectis, the more rapidly the solutions decay, and that the smaller the interval L of mixed initial boundary problem is or the greater the absolutes of eigenvalues λ_i are, the more sufficiently the boundary dissipative effect can be used and the more rapidly the solutions decay. This is reasonable.

§ 3. Proof of Theorem

Multiplying both sides of system (8) by $\zeta(u)$ from the left, we obtain

$$\zeta(u)\frac{\partial u}{\partial t}+\Lambda(u)\zeta(u)\frac{\partial u}{\partial x}=0, \tag{23}$$

where

$$\Lambda(u)=\operatorname{diag}(\lambda_1(u),\cdots,\lambda_n(u)).$$

Differentiating system (8) with respect to x yields

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)+A(u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=-\frac{\partial A(u)}{\partial x}\cdot\frac{\partial u}{\partial x}. \tag{24}$$

Multiplying above equation by $\zeta(u)$ from the left, we obtain

$$\zeta(u)\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)+\Lambda(u)\zeta(u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=-\zeta(u)\frac{\partial A(u)}{\partial x}\frac{\partial u}{\partial x}. \tag{25}$$

Setting

$$U(t,x)=\zeta(u)u, \tag{26}$$

$$V(t,x)=\Lambda(0)\zeta(u)\frac{\partial u}{\partial x}, \tag{27}$$

we can write (23) and (25) in following forms:

$$\frac{\partial U}{\partial t}+\Lambda(u)\frac{\partial U}{\partial x}=\left[\frac{\partial\zeta(u)}{\partial t}+\Lambda(u)\frac{\partial\zeta(u)}{\partial x}\right]u, \tag{28}$$

$$\frac{\partial V}{\partial t}+\Lambda(u)\frac{\partial V}{\partial x}=\Lambda(0)\left[\frac{\partial\zeta(u)}{\partial t}+\Lambda(u)\frac{\partial\zeta(u)}{\partial x}-\zeta(u)\frac{\partial A(u)}{\partial x}\right]\frac{\partial u}{\partial x}. \tag{29}$$

Taking into account that

$$\frac{\partial\zeta(u)}{\partial t}=\frac{\partial\zeta(u)}{\partial u}\frac{\partial u}{\partial t}, \tag{30}$$

$$\frac{\partial u}{\partial t}=-A(u)\frac{\partial u}{\partial x}, \tag{31}$$

$$u=\zeta^{-1}(u)U, \tag{32}$$

$$\frac{\partial u}{\partial x}=\zeta^{-1}(u)\Lambda^{-1}(0)V, \tag{33}$$

(28) and (29) can be written as

$$\frac{\partial U_i}{\partial t}+\lambda_i(u)\frac{\partial U_i}{\partial x}=\sum_{k,l=1}^{n}a^i_{kl}(u)U_kV_l, \tag{34}$$

$$\frac{\partial V_i}{\partial t}+\lambda_i(u)\frac{\partial V_i}{\partial x}=\sum_{k,l=1}^{n} b_{kl}^i(u)V_kV_l, \quad (i=1,\cdots,n). \tag{35}$$

Now we consider the initial and boundary conditions which are satisfied by U and V. It is evident that

$$U=\zeta(\varphi(x))\varphi(x),\ V=\Lambda(0)\zeta(\varphi(x))\frac{\partial\varphi(x)}{\partial x}, \text{ as } t=0. \tag{36}$$

From (26) and (10), we know that

$$U_i=u_i+\sum_{k,l=1}^{n} c_{kl}^i(u)u_ku_l \quad (i=1,\cdots,n). \tag{37}$$

Using boundary condition (11) and taking into account the assumption (14), we know that at $x=0$,

$$u_s=\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)u_j+\sum_{k,l=1}^{n} d_{kl}^s u_ku_l, \quad (s=m+1,\cdots,n). \tag{38}$$

From (37), (38) and by using (32), (10), we obtain immediately the boundary condition at $x=0$, which is satisfied by U,

$$U_s=\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)U_j+\sum_{k,l=1}^{n}\alpha_{kl}^s U_kU_l, \text{ as } x=0, \quad (s=m+1,\cdots,n). \tag{40}$$

In the same we can obtain the boundary condition at $x=L$, which is satisfied by U,

$$U_r=\sum_{j=m+1}^{n}\frac{\partial G_r}{\partial u_j}(0)U_j+\sum_{k,l=1}^{n}\alpha_{kl}^r U_kU_l, \text{ as } x=L \quad (r=1,\cdots,m). \tag{41}$$

Differentiating (40) in t yields

$$\frac{\partial U_s}{\partial t}=\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)\frac{\partial U_j}{\partial t}+\sum_{k,l=1}^{n}\left(\sum_{i=1}^{m}\frac{\partial\alpha_{kl}^s}{\partial u_i}\frac{\partial u_i}{\partial t}U_kU_l+\alpha_{kl}^s\frac{\partial U_k}{\partial t}U_l+\alpha_{kl}^s U_k\frac{\partial U_l}{\partial t}\right). \tag{42}$$

Using (26), (31), (32), and (33), the above equation can be written as

$$\frac{\partial U_s}{\partial t}=\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)\frac{\partial U_j}{\partial t}+\sum_{k,l=1}^{n}\delta_{kl}^s U_kV_l. \tag{43}$$

Using (34) and taking into account

$$\Lambda(0)\frac{\partial U}{\partial x}=\Lambda(0)\frac{\partial\zeta(u)}{\partial u}\zeta^{-1}(u)\Lambda^{-1}(0)\cdot V\cdot\zeta^{-1}(u)U+V,$$

we see that V at $x=0$ satisfies the following boundary condition

$$V_s=\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)\frac{\lambda_s(0)}{\lambda_s(u)}\cdot\frac{\lambda_j(u)}{\lambda_j(0)}V_j+\sum_{k,l=1}^{n}\gamma_{kl}^s U_kV_l, \quad (s=m+1,\cdots,n). \tag{44}$$

It is easy to see that (44) can be written as

$$V_s=\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)V_j+\sum_{k,l=1}^{n}\beta_{kl}^s U_kV_l, \text{ as } x=0, \quad (s=m+1,\cdots,n). \tag{45}$$

In the same way, we can see that V at $x=L$ satisfies the following boundary condition

$$V_r=\sum_{j=m+1}^{n}\frac{\partial G_r}{\partial u_j}(0)V_j+\sum_{k,l=1}^{n}\beta_{kl}^r U_kV_l, \text{ as } x=L, \quad (r=1,\cdots,m). \tag{46}$$

Now, we discuss the system (34) (35) which is satisfied by U and V with the initial conditions (36) and boundary conditions (40), (41), (45) and (46). From above deduction it is not difficult to see that the coefficients a_{kl}^i, b_{kl}^i, α_{kl}^i and β_{kl}^i in (34), (35), (40), (41), (45) and (46) are uniformly bounded for bounded $|u|_{C^0}$.

We write

$$X_i(t)=\sup_{\tau\geq t}|U_i(\tau, x)|_{C^0},\ Y_i(t)=\sup_{\tau\geq t}|V_i(\tau, x)|_{C^0},\quad (i=1, \cdots, n),$$

$$X(t)=\max_{1\leq i\leq n} X_i(t),\ Y(t)=\max_{1\leq i\leq n} Y_i(t),\quad (i=1, \cdots, n),$$

$$W(t)=X(t)+Y(t),\quad (i=1, \cdots, n),$$

$$T_1=\frac{2L}{\lambda_{\min}}.$$

Lemma 1. *There exists $\delta_1>0$ such that when $X(t)<\delta_1$ the following estimates hold:*

$$X(t)\leq\sigma_0 X(t-pT_1)+H_p\int_{t-pT_1}^{t} X(\tau)Y(\tau)d\tau+M_p X^2(t-pT_1),\tag{47}$$

$$Y(t)\leq\sigma_0 Y(t-pT_1)+H_p\int_{t-pT_1}^{t} X(\tau)Y(\tau)d\tau+M_p X(t-pT_1)Y(t-pT_1),\tag{48}$$

where H_p and M_p are constants.

Proof Suppose that δ_1 is chosen so that the assumption (A_1) in §2 holds and U, V satisfy the boundary conditions in the forms (40), (41), (45) and (46) for small $|u|_{C^0}$ which is derived from $X(t)<\delta_1$.

Set

$$B_1^p=(b_{ij}^{(p)}).$$

First, we prover that for any positive integer p the following estimates hold:

$$X_s(t)\leq\sum_{j=m+1}^{n} b_{sj}^{(p)} X_j(t-pT_1)$$
$$+H_p\int_{t-pT_1}^{t} X(\tau)Y(\tau)d\tau+M_p X^2(t-pT_1),\quad (s=m+1, \cdots, n).\tag{49}$$

We consider the case when $p=1$ at first. Let (t, x) is any point in region $0<t<\infty$, $0<x<L$ and the sth backward characteristic through it hits the line $x=0$ in a point $(t_s, 0)$. Integrating the s th equation in system (34) along this characteristic curve and taking into account boundary condition (40), we can obtain

$$U_s(t, x)=\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)U_j(t_s,0)+\sum_{k,l=1}^{n}\alpha_{kl}^{s} U_k U_l(t_s, 0)+\int_{t_s}^{t}\sum_{k,l=1}^{n} a_{kl}^{s} U_k V_l d\tau.\tag{50}$$

Let the j th $(1\leq j\leq m)$ backward characteristic curve passing through $(t_s, 0)$ hit the line $x=L$ in a point (τ_j, L). Integrating the j th equation in system (34) along this characteristic curve and taking into account boundary condition (41), we have

$$U_j(t_s, 0)=\sum_{i=m+1}^{n}\frac{\partial G_j}{\partial u_i}(0)U_i(\tau_j,L)+\sum_{k,l=1}^{n}\alpha_{kl}^{j} U_k U_l(\tau_j, L)+\int_{\tau_j}^{t_s}\sum_{k,l=1}^{n} a_{kl}^{j} U_k V_l d\tau.\tag{51}$$

Substituting (51) into (50), we obtain

$$U_s(t, x)=\sum_{j=1}^{m}\sum_{i=m+1}^{n}\frac{\partial F_s}{\partial u_j}(0)\frac{\partial G_j}{\partial u_i}(0)\,U_i(\tau_j, L)$$
$$+\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)\int_{\tau_j}^{st}\sum_{k,l=1}^{n} a_{kl}^{j} U_k V_l d\tau+\int_{t_s}^{t}\sum_{k,l=1}^{n} a_{kl}^{s} U_k V_l d\tau$$
$$+\sum_{j=1}^{m}\frac{\partial F_s}{\partial u_j}(0)\sum_{k,l=1}^{n}\alpha_{kl}^{j} U_k U_l(\tau_j, L)+\sum_{k,l=1}^{n}\alpha_{kl}^{s} U_k U_l(t_s, 0).\tag{52}$$

Taking $$|U_i(\tau_j, L)| \leqslant X_i(t-T_1),$$
into acount, we obtain immediately

$$X_s(t) \leqslant \sum_{j=m+1}^{n} b_{sj}^{(0)} X_j(t-T_1) + H_1 \int_{t-T_1}^{t} X(\tau) Y(\tau) d\tau + M_1 X^2(t-T_1). \tag{53}$$

It means that (49) holds when $p=1$.

Now suppose that (49) holds for a integer $p>0$. Then from (53) weo an see that

$$X_j(t-pT_1) \leqslant \sum_{i=m+1}^{n} b_{ji}^{(0)} X_i(t-(p+1)T_1)$$
$$+H_1 \int_{t-(p+1)T_1}^{t-pT_1} X(\tau) Y(\tau) d\tau + M_1 X^2(t-(p+1)T_1), \quad m+1 \leqslant j \leqslant n. \tag{54}$$

Substituting (54) into (49), we obtain

$$\begin{aligned} X_s(t) \leqslant & \sum_{j=m+1}^{n} \sum_{i=m+1}^{n} b_{sj}^{(p)} b_{ji}^{(0)} X_i(t-(p+1)T_1) \\ & + \sum_{j=m+1}^{n} b_{sj}^{(p)} H_1 \int_{t-(p+1)T_1}^{t-pT_1} X(\tau) Y(\tau) d\tau + H_p \int_{t-pT_1}^{t} X(\tau) Y(\tau) d\tau \\ & + \sum_{j=m+1}^{n} M_1 b_{sj}^{(p)} X^2(t-(p+1)T_1) + M_p X^2(t-pT_1) \\ & \leqslant \sum_{i=m+1}^{n} b_{si}^{(p+1)} X_i(t-(p+1)T_1) \\ & + H_{p+1} \int_{t-(p+1)T_1}^{t} X(\tau) Y(\tau) d\tau + M_{p+1} X^2(t-(p+1)T_1). \end{aligned}$$

From above equation, we can see that (49) holds for all positive integers p.

For such p which makes (18) hold, from (49) we have

$$X_s(t) \leqslant \sigma_0 X(t-pT_1) + H_p \int_{t-pT_1}^{t} X(\tau) Y(\tau) d\tau + M_p X^2(t-pT_1), \quad (s=m+1, \cdots, n). \tag{55}$$

Similarly, we can obtain

$$X_r(t) \leqslant \sigma_0 X(t-pT_1) + H_p \int_{t-pT_1}^{t} X(\tau) Y(\tau) d\tau + M_p X^2(t-pT_1), \quad (r=1, \cdots, m). \tag{56}$$

Combining (55) and (56), we can obtain immediately (47). In the same way we can prove the estimate (48).

Thus Lemma 1 has been proved.

Lemma 2. *For any* $\sigma \in (\sigma_0, 1)$, *set* $\delta_2 = \min\left(\delta_1, \dfrac{\sigma-\sigma_0}{M_p}\right)$. *Then when* $X(t) < \delta_2$, *we have*

$$W(t) \leqslant \sigma W(t-pT_1) + C_1 \int_{t-pT_1}^{t} W^2(\tau) d\tau, \tag{57}$$

where C_1 *is a constant.*

Proof Combining (47) and (48), it is easily seen that

$$W(t) \leqslant \sigma_0 W(t-pT_1) + H_p \int_{t-pT_1}^{t} W^2(\tau) d\tau + M_p X(t-pT_1) W(t-pT_1).$$

Taking into account $X(0) < \delta_2$ and the choice of δ_2, from above equation, we obtain immediatly (57), completing the proof.

Lemma 3. *Under the condition in Lemma 2, the following estimate holds:*

$$W(t) \leqslant \sigma^k W(0) + C_1 \sigma^{-1} \int_0^t \sigma^{\frac{t-\tau}{pT_1}} W^2(\tau) d\tau, \tag{58}$$

where $k = \left[\frac{t}{pT_1}\right]$.

Proof Using Lemma 2 successively, we have

$$\sigma^i W(t - ipT_1) \leqslant \sigma^{i+1} W(t-(i+1)pT_1) + C_1 \sigma^i \int_{t-(i+1)pT_1}^{t-ipT_1} W^2(T) d\tau \quad (i=0, 1, \cdots, k-1). \tag{59}$$

Moreover, when $\frac{t}{pT_1}$ is not an integer, from the definition of $W(t)$ we have

$$\sigma^k W(t - kpT_1) \leqslant \sigma^k W(0). \tag{60}$$

Combining the inequalities (59) and (60), we get

$$W(t) \leqslant \sigma^k W(0) + C_1 \sum_{i=0}^{k-1} \sigma^i \int_{t-(i+1)pT_1}^{t-ipT_1} W^2(\tau) d\tau. \tag{61}$$

Moreover, for any $\tau \in [t-(i+1)pT_1, t-ipT_1]$, clearly $i \geqslant \frac{t-\tau}{pT_1} - 1$. Therefore

$$W(t) \leqslant \sigma^k W(0) + C_1 \sigma^{-1} \sum_{i=0}^{k-1} \int_{t-(i+1)pT_1}^{t-ipT_1} \sigma^{\frac{t-\tau}{pT_1}} W^2(\tau) d\tau. \tag{62}$$

This meaes that (58) holds, which completes the proof.

Lemma 4. *Under the condition in Lemma 2, we have*

$$W(t) \leqslant \frac{1}{\sigma} e^{-\alpha t} W(0) + C_1 \sigma^{-1} \int_0^t e^{-\alpha(t-\tau)} W^2(\tau) d\tau, \quad \alpha = -\frac{\ln \sigma}{pT_1} > 0. \tag{63}$$

Proof From $k = \left[\frac{t}{pT_1}\right]$, it follows that $k \geqslant \frac{t}{pT_1} - 1$. Therefore $\sigma^k = e^{k \ln \sigma} \leqslant \sigma^{-1} e^{-\alpha t}$. Moreover, clearly $\sigma^{\frac{t-\tau}{pT_1}} = e^{-\alpha(t-\tau)}$. From (58) in Lemma 3, the proof of Lemma 4 follows

The proof of Theorem First, we assume that

$$W(t) < \delta_3 = \min\left(\delta_1, \frac{\sigma - \sigma_0}{M_p}, \frac{\alpha \sigma^2}{2C_1}\right). \tag{64}$$

At the moment, Lemma 4 holds. Set

$$P(t) = e^{\alpha t} W(t). \tag{65}$$

From (63) in Lemma 4, it follows that

$$P(t) \leqslant \sigma^{-1} P(0) + C_1 \sigma^{-1} \int_0^t e^{-\alpha t} P^2(\tau) d\tau. \tag{66}$$

Now, we consider $Q(t)$ which satisfies the following initial value problem for ordinary differential equation, as in [6]:

$$\begin{cases} \frac{dQ(t)}{dt} = C_1 \sigma^{-1} e^{-\alpha t} Q^2(t), \\ t = 0: Q(t) = \sigma^{-1} P(0) = \sigma^{-1} W(0). \end{cases} \tag{67}$$

The solution to initial value problem (67) is

$$Q(t)=\frac{1}{\frac{\sigma}{W(0)}-\frac{C_1}{2\sigma}(1-e^{-\alpha t})}. \tag{68}$$

Noticing the choice (64) of δ_3, from (68) we get

$$Q(t)\leqslant 2\sigma^{-1}W(0). \tag{69}$$

Moreover, from Bihari's inequality (cf. [9]), we have

$$P(t)\leqslant Q(t). \tag{70}$$

Therefore, from (65), (69) and (70), it follows that

$$W(t)\leqslant 2\sigma^{-1}W(0)e^{-\alpha t}. \tag{71}$$

The above inequality means that if (64) is satisfied, then the C^0 norms of the solutions and their first order derivatives for mixed initial-boundary value problem (8), (11), (12), (13) decay exponentially to zero, as $t\to+\infty$.

Now, we show the existence of the global smooth solution for mixed problem (8), (11), (12), (13). At first, we take t_0 such that $2\sigma^{-1}e^{-\alpha t_0}=1$. Takeng δ so small that when $|\varphi|_{C^1}<\delta$, there exists a smooth solution for mixed problem (8), (11) (12), (13) on $[0, t_0]$, and moreover

$$\max_{1\leqslant i\leqslant n}|U_i(t,x)|_{C^0}+\max_{1\leqslant i\leqslant n}|V_i(t,x)|_{C^0}<\delta_3,\ t\in[0,t_0]. \tag{72}$$

At the moment, from (71) we get $W(t)<2\sigma^{-1}\delta_3 e^{-\alpha t}$, $\forall t\in[0,t_0]$.

From (74) it is not difficult to show that there exists a global smooth solution for mixed problem (8), (11), (12), (13) on $[0,+\infty)$, and moreover, the C^0 norms of the solution and its first order derivatives decay exponentially to zero, as $t\to+\infty$. The Theorem is proved.

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