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WEIGHTED NORM INEQUALITIES ON MIXED HOMOGENEITY SPACE

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Abstract

In this paper, the properties of the weight functions in mixed homogeneity spaces are discussed and the corresponding weighted norm inequalities for maximal functions, Macinkiewicz integral and singular integrals are given.

§ 1. Introduction

Suppose that $R^n=R^1\times\cdots\times R^1$, $R^1=(-\infty,+\infty)$ and $\|X\|^2=(x,x)$, where (x, y) is the inner product of vectors x and y in R^n. Let $A_t=\begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n}\end{pmatrix}$, $t>0$, $\lambda_j\geqslant 1$ $(j=1, 2, \cdots, n)$. Obviously, $A_t\cdot A_s=A_{t.s}$. Let

$$F(x, \rho)=\left(\sum_{j=1}^n\left(\frac{x_j}{\rho\lambda_j}\right)^2\right)^{\frac{1}{2}}\quad(\rho>0).$$

For any fixed $x(\neq 0)$, $F(x, \rho)$ is a strictly decreasing function of variable ρ. So there exists a unique t such that $F(x, t)=1$ for any $x\neq 0$. Thus we obtain a function $\rho(x)=t$ for $x\neq 0$. Now set $\rho(0)=0$. It is easy to prove that $\rho(x)\geqslant 0$, $\rho(x+y)\leqslant\rho(x)+\rho(y)$ and that $\rho(x)=0$ if $x=0$. Hence $\rho(x)$ may be considered as a metric on R^n. The space (R^n, ρ) is a special case of the parabolic space defined in [1].

We have the following properties:

1) $\rho(A_tx)=t\rho(x)$;

2) If $k\geqslant 1$, then $\rho(kx)\geqslant\rho(x)$;

3) $\int_{\rho(x,y)<\sigma}dy=C\sigma^\gamma$, where $\sigma\geqslant 0$, $\gamma=\sum_{j=1}^n\lambda_j$ and $\rho(x, y)=\rho(x-y)$;

4) (R^n, ρ, μ) is a homogeneous space, where ρ is the metric defined above and μ is the Lebesgue measure on R^n. The definition of the homogeneous space is given in [2].

5) Denote by $Q(x, a)$ the parallelotope with center at x and of measure $2a^{\lambda_1}\times 2a^{\lambda_2}\times\cdots\times 2a^{\lambda_n}$, whose sides are parallel to the axes. Then there exist balls $B(x, a)$ and $B(x, \beta a)$ such that

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a) $B(x, a)\subseteq Q(x, a)\subseteq B(x,\beta a)$,

b) $C_1|B(x, a)|=|Q(x, a)|=C_2|B(x, \beta a)|$,

where β, C_1, C_2 are constants independent of x and a, $B(x, t)=\{y\in\mathbf{R}^n\ \rho(x, y)<t\}$, and $|E|$ denotes the Lebesgue measure of a measurable set E.

It is easy to verify 1) and 2) by the definition of $\rho(x)$. For 3) a generierialized polar coordinates transformation formula is needed. 4) follows directely from 3). Using 2) and the following figure, we have 5)a) with $\beta=\rho(x)$ and $\bar{x}=(1, 1, 1, \cdots, 1)$. 5)b) follows from 3) and the fact that $|Q(x, t)|=2^n t^\gamma$.

The definitions of A_p conditions of weight functions in the mixed homogeneity spaces are the same as that on Euclidean space except that the averrages are now taken in balls $B(x, t)$.

If we use the parallelotope $Q(x, t)$ instead of the ball $B(x, t)$ in our definition, the corresponding A_p conditions will be denoted by A_p^*.

We shall establish the following theorems:

Theorem 1. (*Weighted norm inequality for maximal functions*)
Let

$$Mf(x)=f^*(x)=\sup_{t<0}\frac{1}{|B(x, t)|}\int_{B(x,t)}|f(y)|dy.$$

a) *If* $\omega(x)\in A_1$, *then* $|\{x\in\mathbf{R}^n: f^*(x)>\lambda\}|_\omega\leqslant\frac{C}{\lambda}\|f\|_{1.\omega}$;

b) *If* $\omega(x)\in A_p(1<p<\infty)$, *then* $\|f^*\|_{p.\omega}\leqslant C_{p.\omega}\|f\|_{p.\omega}$, *where*

$$\|f\|_{p.\omega}=\left(\int_{\mathbf{R}^n}|f(x)|^p\omega(x)dx\right)^{\frac{1}{p}}.$$

Theorem 2. (*Weighted norm inequality for maximal vector-valued functions*)
Suppose that $f(x)=(f_1(x), f_2(x), \cdots)$, $f^*(x)=(f_1^*(x), f_2^*(x), \cdots)$ *and*

$$|f(x)|_r=\left(\sum_{i=1}^{\infty}|f_i(x)|^r\right)^{\frac{1}{r}}\quad(r\geqslant 1).$$

a) *If* $\omega(x)\in A_1$, *then* $|\{x\in\mathbf{R}^n: |f^*(x)|_r>\lambda\}|\leqslant\frac{C}{\lambda}\||f(x)|_r\|_{1,\omega}$;

b) *If* $\omega(x)\in A_p(1<p<\infty)$, *then* $\||f^*(x)|_r\|_{p,\omega}\leqslant C_{p,\omega}\||f(x)|_r\|_{p,\omega}$.

Theorem 3. (*Weighted norm inequality for Marcinkiewicz integrals*)
Suppose that P is a closed set and $\delta(y)=\inf_{x\in p}P(x, y)$. *Define*

$$J_\alpha f(x)=\int_{\mathbf{R}^n}\frac{f(y)\delta^\alpha(y)}{P^{\alpha+\gamma}(x-y)+\delta^{\alpha+\gamma}(y)}dy,$$

where $\gamma=\sum_{j=1}^{\infty}\lambda_j$ *and* $\alpha>0$. *If* $\omega(x)\in A_p$, $1\leqslant p<\infty$, *then*

$$\|J_\alpha(f)\|_{p,\omega}\leqslant C_{p,\omega}\|f\|_{p,\omega}.$$

Theorem 4. (*Weighted norm inequality for singular integrals*)
Assume that $K(x)=\frac{\Omega(x)}{\rho^\gamma(x)}$, *where* $\gamma=\sum_{j=1}^{\infty}\lambda_j$ *and* $\Omega(x)$ *satisfies the following*

conditions:

(1) $\Omega(A_t x)=\Omega(x)$;

(2) $|\Omega(x)|\leqslant B$;

(3) $\int_{\Sigma_{n-1}}\Omega(x)H(\sigma)d\sigma=0$, *where* $dx=r^{\gamma-1}H(\sigma)d\sigma\, dr$ *and* $\sum_{n-1}=\{x\in\mathbf{R}^n,\ \rho(x)=1\}$;

(4) $|\Omega(x'-y)-\Omega(x')|\leqslant B\rho^{\cdot}(y)$, *where* $\rho(x')=1$ *and* $0<\alpha\min\limits_j{}^{-1}\{\lambda_j\}<1$.

Let
$$T_{r,R}f(x)=K_{r,R}*f(x),$$
where
$$K_{r,R}(x)=\begin{cases}K(x) & (\text{if } r\leqslant\rho(x)\leqslant R),\\ 0 & (\text{otherwise}),\end{cases}$$
and $T^*f(x)=\sup\limits_{r/R<0}|T_{r,R}f(x)|$.

Suppose that $\omega(x)\in A_p$, $(1<p<\infty)$, *then we have* $\|T^*f\|_{p,\omega}\leqslant C\|f\|_{p,\omega}$. *In case* $p=1$, i. e, $\omega\in A_1$, *we obtain*
$$|\{x\in\mathbf{R}^n:\ T^*f(x)>\lambda\}|_\omega\leqslant\frac{C}{\lambda}\|f\|_{1,\omega}.$$

Theorem 5. *If we weaken the condition of Theorem* 4 *in replacing* (4) *by*

(4′) $\int_0^{\frac{1}{2}}\frac{w(t)}{t}dt<\infty$, $w(t)=\sup\limits_{x'\in\Sigma_{n-1},(\rho(y)<t)}\{|\Omega(x'-y)-\Omega(x')|\}$

and keep the rest of the conditions unchanged, then the results of Theorem 4 *remain true.*

§ 2. Preliminaries

Lemma 1. $w(x)\in A_p$, *iff* $w(x)\in A_p^*$.

This follows directly from property 5 of $\rho(x)$.

Lemma 2. (*Calderon–Zygmund decomposition on mixed homogeneity space*)

Suppose $f\in L^p(R^n)$, $(1\leqslant p<\infty)$. *Then for any* $\alpha>0$, *there is a decomposition of* R^n: $R^n=\Omega\cup\Omega^c$, *with* $\Omega=\bigcup\limits_k I_k$ *satisfying:*

1) $I_k^0\cap I_j^0=\phi$, $(K\neq j)$ (I_k^0 *is the interior of the parallelotope* I_k);

2) *For any* $I_k\subseteq\Omega$, *there exists a parallelotope* $Q_k=Q\ (x,\ a_k)$, *defined above, satisfying* $I_k\supseteq I_k$ *and* $|Q_k|\leqslant\beta|I_n|$, *where* β *is independent of* I_k;

3) $|I_k|\to 0$, *when* $k\to+\infty$;

4) *For any* $I_k\subseteq\Omega$,
$$\alpha<\frac{1}{|I_k|}\int_{I_k}|f(x)dx\leqslant B\alpha,$$
where B *is a constant independent of* I_k;

5) *If* $x\in\Omega^c$, *then* $|f(x)|\leqslant\alpha$ $(a,\ e)$.

Proof Let
$$R^n=R_1\times R_2\times\cdots\times R_n,\ R_j=(-\infty,\ +\infty)\quad(j=1,\ 2,\ \cdots,\ n).$$

Divide R_j into intervals$(Kt_0^{\lambda_j},\ (K+1)t_0^{\lambda_j})$, $(K=\cdots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \cdots)$ with the same length $t_0^{\lambda_j}$. Their product sets are parallelotopes $\{I_0'\}$ of R^n with the same measure $t_0^{\lambda_1}\times t_0^{\lambda_2}\times\cdots\times t_0^{\lambda_n}$ (We denote every such parallelotope by I_0' without making distinction). We can select t_0 large enough so that

$$\frac{1}{|I_0'|}\int_{I_0'}|f(x)|\,dx\leqslant\alpha$$

for every I_0'.

Denote by $K_1^j=[2^{\lambda_j}]$ the integer part of 2^{λ_j}. Divide the j^{-th} side of every I_0' into K_1^j subintervals with the length $(K_1^j)^{-1}\,t_0^{\lambda_j}$. Their product sets are sub-parallelotopes of I_0'. We denote every one of them by I_1' without making distinction. Now two different cases occur:

The first case:

$$\frac{1}{|I_1'|}\int_{I_1'}|f(x)|\,dx>\alpha.$$

The second case:

$$\frac{1}{|I_1'|}\int_{I_1'}|f(x)|\,dx\leqslant\alpha.$$

Select those I_1' which satisfy the first case and put them into a set, say Ω. We denote them by $\{I_1\}$. Now we subdivide those of I_1' which satisfy the second case into $\{I_2'\}$ as before and repeat the above process until we are forced into the first case. In the ν^{th} step we subdivide $I_{\nu-1}'$ into $\prod\limits_{j=1}^{n}K_\nu^j$ sub-parallelotopes I_ν' with the measure $\prod\limits_{j=1}^{n}(K_1^jK_2^j\cdots K_\nu^j)^{-1}t_0^{\lambda_j}$, where

$$K_\nu^j=[(K_1^jK_2^j\cdots K_{\nu-1}^j)^{-1}t_0^{\lambda_j}/(2^{-\nu}t_0)^{\lambda_j}]=[(K_1^jK_2^j\cdots K_{\nu-1}^j)^{-1}\cdot2^{\nu\lambda_j}].$$

According to our subdivision, we obtain the following relations:

(1) $(K_1^jK_2^j\cdots K_\nu^j)^{-1}\cdot2^{(\nu+1)\lambda_j}=(K_\nu^j)^{-1}\{(K_1^jK_2^j\cdots K_{\nu-1}^j)^{-1}\cdot2^{\nu\lambda_j}\}\cdot\{2^{\lambda_j}\}\geqslant2^{\lambda_j}\geqslant2.$

(2) $(K_1^jK_2^j\cdots K_\nu^j)^{-1}\cdot2^{\nu\lambda_j}=(K_\nu^j)^{-1}\{K_1^jK_2^j\cdots K_{\nu-1}^j)^{-1}\cdot2^{\nu\lambda_j}\}\leqslant(K_\nu^j)^{-1}\{K_\nu^j+1\}\leqslant2.$

From (1) and (2), we obtain $(K_1^jK_2^j\cdots K_\nu^j)^{-1}t_0^{\lambda_j}\geqslant2^{-\nu\lambda_j}t_0^{\lambda_j}$, and $(K_1^jK_2^j\cdots K_\nu^j)^{-1}t_0^{\lambda_j}\leqslant2\cdot(2^{-\nu\lambda_j}t_0^{\lambda_j})$ respectively.

Therefore there is a parallelotope $Q_\nu=Q(x,\ 2^{-\nu}t_0)$ with the same center x as I_ν such that $I_\nu\subseteq Q_\nu$. Moreover we know that there exists a ball $B(x,\ \beta2^{-\nu}t_0)\supseteq I_\nu$. Obviously, $|B(x,\ \beta2^{-\nu}t_0)|\leqslant C|I_\nu|$. 1) and 2) are proved.

Since $K_\nu^j\geqslant[2^{\lambda_j}]\geqslant2$ for any ν, 3) follows directly.

Since

$$K_{\nu+1}^j\leqslant(K_1^jK_2^j\cdots K_\nu^j)^{-1}(2^{(\nu+1)\lambda_j})=\{(K_1^jK_2^j\cdots K_\nu^j)^{-1}\cdot2^{\nu\lambda_j}\}\cdot2^{\lambda_j}\leqslant2\cdot2^{\lambda_j}=2^{\lambda_j+1}$$

for any $I_k\subseteq\Omega$, we have

$$\alpha<\frac{1}{|I_k|}\int_{I_k}|f(x)|\,dx<B\alpha,$$

where $B=\prod\limits_{j=1}^{n}2^{\lambda_j+1}$. 4) is proved.

As to 5), since $x\in\Omega^c$, there is a sequnce of parallelotopes I'_k which satisfy the second case and 2), 3). If we apply the theorem about maximal function on homogeneous space [2], the proof of 5) is similar to the case on Euclidean space[10].

Remark 1. Lemma 2 is an extension of the Lemma 3 in [3].

Remark 2. The other decomposition of Calderon–Zygmund on mixed homogeneity space has been given in [4]. But the proof of a sublemma in [4] is not correct. However the Lemma 2 here can be used as a substitute for it.

Because of Lemma 2, A_p condition in mixed homogeneity space then plays the same important role as in Euclidean space.

Corollary. a) *If $w(x)\in A_p$ $(p>1)$, then there exist $C>0$ and $\delta>0$ such that*

$$\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}w(y)^{1+\delta}dy\right)^{\frac{1}{1+\delta}}\leqslant C\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}w(y)\,dy\right).$$

b) *If $w(x)\in A_p$, then $w(x)\in A_\infty$.*

c) *If $w(x)\in A_p(p>1)$, then there is an $\varepsilon>0$ such that $w(x)\in A_{p-\varepsilon}$.*

Proof The proof is similar to that of Lemma 2 in [5]. But we should point out

(1) By Lemma 1, we can deal with A_p^* condition instead of A_p condition.

(2) Since $Q(x,t)$ consists of 2 congruent I_k, we can apply Lemma 2 for $Q(x,t)$, although Lemma 2 is the decomposition on whole space.

(3) By 2) of Lemma 2, for $I_k\subseteq\Omega$, we have

$$\left(\frac{1}{|I_k|}\int_{I_k}w(x)dx\right)\left(\frac{1}{(I_k)}\int_{I_k}w(x)^{-\frac{1}{p-1}}dx\right)^{p-1}\leqslant C.$$

As in Euclidean spaces, A_p condition on mixed homogeneity spaces has the following elementary properties:

a) $w(x)\in A_p$, if $w(x)^{-\frac{1}{p-1}}\in A'_p$, where $p^{-1}+(p')^{-1}=1$.

b) If $w(x)\in A_p(p>1)$ and E is a measurable subset of $B(x,t)$, then

$$\frac{|E|_\omega}{|B(x,t)|_\omega}\geqslant C\left(\frac{|E|}{|B(x,t)|}\right)^p.$$

Lemma 3. *Let $\psi(x)=\text{esssup}\ |\varphi(y)|$ and $\varphi_t(x)=t^{-\gamma}\varphi(A_tx)$, where $\gamma=\sum_{j=1}^n\lambda_j$. If $\psi(x)\in L^1(\mathbf{R}^n)$, then $\sup\limits_{t<0}|f*\psi_t(x)|\leqslant Cf^*(x)$.*

Proof *Let $Tf=f*\varphi$*, then

$$(A_t^{-1}TA_t)f(x)=\int_{R^n}f(A_ty)\varphi(A_t^{-1}x-y)dy=\int_{R^n}f(z)t^{-\gamma}\varphi(A_t^{-1}(x-z))dz=f*\varphi_t$$

$$A_tf^*(x)=\sup_{\sigma<0}\frac{1}{|B(A_t^x,\sigma)|}\int_{B(A_tx,\sigma)}|f(y)|dy=\sup_{\sigma>0}\frac{t^\gamma}{|B(A_tx,\sigma)|}\int_{B(x,\frac{\sigma}{t})}|f(A_tz)|dz$$

$$=\sup_{\sigma>0}\frac{1}{\left|B\left(x,\frac{\sigma}{t}\right)\right|}\int_{B(x,\frac{\sigma}{t})}|f(A_tz)|dz=M(A_tf)(x).$$

Let

$$A_x(r)=\int_0^r t^{\gamma-1}\,dt\int_{\Sigma_{n-1}} H(\sigma)\,|f(x-\sigma t)|\,d\sigma=C\int_{\rho(x,y)<r}|f(y)|\,dy=O(r^\gamma).$$

Let $\Psi(t)=\psi(x)$, if $t=\rho(x)$. Since $\Psi(t)$ is a decreasing function and $\psi(x)\in L^1(\mathbf{R}^n)$, then

$$\int_{\frac{r}{2}<\rho(x)<r}\psi(x)\,dx\geqslant C\Psi\left(\frac{r}{2}\right)\cdot r^\gamma.$$

Therefore $\Psi(r)\cdot r^\gamma\to 0$. If $r\to 0$ or $r\to+\infty$,

$$\begin{aligned}\left|\int_{\mathbf{R}^n} f(y)\varphi(x-y)\,dy\right|&\leqslant\int_{\mathbf{R}^n}|f(y)|\,\psi(x-y)\,dy=\int_0^{+\infty}\Psi(t)t^{\gamma-1}\,dt\int_{\Sigma_{n-1}}H(\sigma)\,|f(x-\sigma t)|\,d\sigma\\
&=\int_0^{+\infty}\Psi(t)\,dA_x(t)=A_x(t)\Psi(t)\Big|_0^{+\infty}-\int_0^{\infty}A_x(t)\,d\Psi(t)\\
&=Cf^*(x)\int_0^{+\infty}t^\gamma\,d\Psi(t)\\
&=-Cf^*(x)t^\gamma\Psi(t)\Big|_0^{+\infty}+Cf^*(x)\int_0^{+\infty}\Psi(t)t^{\gamma-1}\,dt\leqslant Cf^*(x)\quad(C>0).\end{aligned}$$

§ 3. Proof of the theorems

Observing that

$$\frac{|B(x,t)|_\omega}{|B(x,2t)|_\omega}\geqslant C\left(\frac{|B(x,t)|}{|B(x,2t)|}\right)^p=\frac{C}{2^{\gamma p}},$$

where $\omega(x)\in A_p(1<p<\infty)$ and $\gamma=\sum_{j=1}^{n}\lambda_j$, we know that the space $(\mathbf{R}^n,\ \rho,\ 15)$ with $dv=\omega(x)dx$ is still a homogeneous space. The proof of Theorems 1 and 2 is similar to the proof given in [5] and [8] by using Lemma 2 instead of Calderon Zygmund decomposition in Euclidean space [10].

For the proof of Theorem 3, one can refer to [6] noticing that

$$\varphi(x)=1/(1+\rho(x))^{\gamma+\alpha}\in L^1(\mathbf{R}^n)$$

for $\alpha>0$.

Now we come to the proof of Theorem 4. From the definition of $\rho(x)$ we obtain

$$\rho(x)^{\min_j\{\lambda_j\}}\leqslant\|x\|\leqslant\rho^{\max_j\{\lambda_j\}},\ \text{when}\ \rho(x)\geqslant 1;$$

$$\rho(x)^{\max_j\{\lambda_j\}}\leqslant\|x\|\leqslant\rho^{\min_j\{\lambda_j\}},\ \text{when}\ \rho(x)\leqslant 1.$$

Since $0<\alpha\max\limits_j{}^{-1}\{\lambda_j\}\leqslant\alpha\min{}^{-1}\{\lambda_j\}<1$, we know that $\Omega(x)$ is not a constant.

First of all we introduce two lemmas.

Lemma 4. *Suppose that $(X,\ \rho,\ \mu)$ is a homogeneous space, and that $U\subseteq X$ is a bounded open set, then there exists a sequence of balls $B(x_j,\ t_j)$ such that*

(i) $U=\bigcup_{j=1}^{\infty}B_j(x_j,\ t_j)$;

(ii) *if $x\in U$, then x belongs to at most M balls, where M is a constant;*

(iii) *if $\bar{\bar{B}}_j=B(x_j\ 3Kt_j)$, then $\bar{\bar{B}}_j\cap\{X-U\}\neq\phi$, where K is the constant appearing in the definition of homogeneous space* ([2], Theorem 3.2).

Lemma 5. *Suppose that*

(i) $K(x) \in L^1(\Sigma_{n-1})$ *and* $\int_{\Sigma_{n-1}} K(x)H(\sigma)d\sigma = 0$,

(ii) $K(A_t x) = t^{-\gamma}K(xt^{-1})$, (iii) $\int_{\rho(x)<2\rho(y)} |K(x-y)-K(x)|dx \leqslant C$.

Then a) $|\{T^*f(x)>\lambda\}| \leqslant \frac{C}{\lambda}\|f\|_1$, b) $\|T^*f\|_p \leqslant C_p\|f\|_p$,

where Tf is defined as the maximal operator in Theorem 4 ([7], Theorem 7.1).

Now, we are going to verify that $K(x) = \Omega(x)/\rho(x)$ satisfies the conditions required in Lemma 5.

Since $|\Omega(x)| < B$ and $\int_{\Sigma_{n-1}} \Omega(x)H(\sigma)d\sigma = 0$, (i) follows directly.

Observing that $\Omega(A_t x) = \Omega(x)$ and $\rho(A_t x) = t\rho(x)$, we get the equality (ii).

For (iii), when $\rho(x) \geqslant 2\rho(y)$,

$$\left|\frac{\Omega(x-y)}{\rho^\gamma(x-y)} - \frac{\Omega(x)}{\rho^\gamma(x)}\right| \leqslant B\left|\frac{1}{\rho^\gamma(x-y)} - \frac{1}{\rho^\gamma(x)}\right| + B\frac{|\Omega(x-y)-\Omega(x)|}{\rho^\gamma(x)} = I_1 + I_2.$$

Since $\rho(A^{-1}_{\rho(x)}x) = 1$,

$$|\Omega(x-y)-\Omega(x)| = |\Omega(A^{-1}_{\rho(x)}(x-y)-\Omega(A^{-1}_{\rho(x)}x)| \leqslant B\rho^\alpha(A^{-1}_{\rho(x)}y) \leqslant B\left(\frac{\rho(y)}{\rho(x)}\right)^\alpha.$$

So $I_2 \leqslant B\rho^\alpha(y)/\rho^{\gamma+\alpha}(x)$,

$$I_1 \leqslant \frac{C}{\rho^{2\gamma}(x)}[\rho^\gamma(x-y)-\rho^\gamma(x)] = \frac{C}{\rho^\gamma(x)}[\rho^\gamma(A^{-1}_{\rho(x)}(x-y)-\rho^\gamma(A^{-1}_{\rho(x)}x)]$$

$$= \frac{C}{\rho^\gamma(x)}\left|\nabla\rho^\gamma(\xi)\cdot A_{\frac{\rho(y)}{\rho(x)}}y''\right| \leqslant \frac{C}{\rho^\gamma(x)}\sum_{j=1}^n\left|\frac{\rho(y)}{\rho(x)}\right|^{\lambda_j} \leqslant C\frac{\rho(y)}{\rho(x)^{\gamma+1}} \quad (\lambda_j \geqslant 1),$$

where $\frac{1}{2} \leqslant \rho(\xi) \leqslant \frac{3}{2}$ and $y'' = A^{-1}_{\rho(y)}y\,(\rho(y'')=1)$. Here, we have used $\rho(x) \in C^\infty(\mathbf{R}^n - 0)$.

Hence

$$\int_{\rho(x)>2\rho(y)} |K(x-y)-K(x)|dx \leqslant C\int_{2\rho(y)}^{+\infty} \frac{\rho(y)}{\rho(x)^{\gamma+1}}dx + C\int_{2\rho(y)}^{+\infty}\frac{\rho^\alpha(y)}{\rho^{\gamma+\alpha}(x)}dx \leqslant C.$$

the conclusions of Lemma 5 are valid for the operator defined in the theorem.

When $p>1$, we make the decomposition $\{x\in\mathbf{R}^n: T^*f>\lambda\} = \bigcup_{j=1}^\infty B_j(x_j, t_j)$ by using Lemma 4. All we need to prove is that

$$(*)\ |\{x\in B_j(x_j,t_j): T^*f(x)>\beta\lambda,\ f^*(x)\leqslant h\lambda\}| \leqslant C\frac{h}{\beta}|B_j(x_j,t_j)| \quad (\beta>1,\ h<1),$$

then $|\{T^*f(x)>\beta\lambda, f^*(x)\leqslant h\lambda\}| = |\bigcup_{j=1}^\infty\{x\in B_j(x_j,t_j): T^*f(x)>\beta\lambda, f^*(x)<h\lambda\}|$

$$\leqslant C\frac{h}{\beta}\sum_{j=1}^\infty|B_j(x_j,t_j)| \leqslant CM\frac{h}{\beta}|\{T^*f(x)>\lambda\}|,$$

where M is a constant defined in Lemma 4.

Therefore $\qquad |\{T^*f>\beta\lambda, f^*<h\lambda\}|_\omega \leqslant \frac{CMh}{\beta}|T^*f>\lambda\}|_\omega$,

and $\qquad \|T^*f\|_{p,\omega} \leqslant C_{p,\omega}\|f\|_{p,\omega}, \quad (p>1)$.

However, the proof of (*) is standard[5]. In the proof, Lemma 5 and Theorem

3.5 in [2] are used.

In case $p=1$, using Lemma 4, we make the decomposition

$$\Omega=\{x\in\mathbf{R}^n: f^*(x)>\lambda\}=\bigcup_{j=1}^{\infty}B_j(y_j,\ t_j).$$

Let $f=g+b$, where

$$g(x)=\begin{cases} f(x), \text{ when } x\in\mathbf{R}^n-\Omega, \\ \sum_{j=1}^{\infty} m_j(\chi_j f)\cdot\chi_{B_j}(x), \text{ when } x\in\Omega. \end{cases}$$

Here $\chi_j(x)$ is the characteristic function of ball $B_j\ (y_j,\ t_j)$,

$$\eta_j=\frac{\chi_j(x)}{\sum_{j=1}^{\infty}\chi_j(x)} \quad\text{and}\quad m_j(g)=\frac{1}{|B_j(y_j,\ t_j)|}\int_{B_j(y_j,t_j)}|g(y)|dy.$$

By using Lemma 4, we obtain $\sum_{j=1}^{\infty}\chi_j(x)\leqslant M$, $\sum_{j=1}^{\infty}\int_{B_j}\omega\leqslant C\cdot M|\Omega|_\omega$ and

$$\sum_{j=1}^{\infty}\int_{B_j}|f(x)|\omega(x)dx\leqslant M\int_{R^n}|f(x)|\omega(x)dx.$$

The rest of the proof of Theorem 4 b) is standard[9].

The proof of Theorem 5 is similar to what we have just done.

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