Chin. Ann. of Math.
6B (3) 1985

THE ESSENTIAL MAXIMALITY AND SOME OTHER PROPERTIES OF A RIEMANN SURFACE OF O_{AD}^0

QIU SHUXI (邱曙熙)*

Abstract

The essential maximality of a Riemann surface of O_{AD}^0 is verified. Actually we have proved something a little stronger, i. e. a modified Stoilow principle and its converse for a functiou meromorphic on a non-compact subregion on a Riemann surface of O_{AD}^0. The significance of these assertions lies in that they fail to hold for the wider and more familiar class O_{AD}.

§ 1. Introduction and Preliminaries

In the present paper we shall at first establish two theorems on the behavior of a meromorphic function near the ideal boundary of a Riemann surface of O_{AD}^0, one of which asserts that a modified Stoilow principle is valid in this case. As a consequence the essential maximality of a Riemann surface of O_{AD}^0 will be verified. It seems worthwhile to pointout here that this is not a property possessed by every Riemann surface of the wider and more familiar class O_{AD}, though every Riemann surface of finite genus of O_{AD} should be essentially maximal[1].

Let us recall briefly some notations, terminologies and preliminaries.

An AD-(resp. AB-, ABD-) function means an analytic function with finite Dirichlet integral (resp. bounded, bounded and with finite Dirichlet integral). A Riemann surface on which every AD-(resp. AB-, ABD-) function reduces to a constant will be said to be of the class O_{AD} (resp. O_{AB}, O_{ABD}).

M. Sakai proved in 1979 the following important theorem which we shall refer to as (ST)[2]:

$$(ST) \qquad O_{AD}=O_{ABD}.$$

Manuscript received June 16, 1983.

* Department of Mathematics, Xiamen University, Fujian, China.

** The author would like to express his heartfelt thanks to his tutor, professor Zhang Mingyong, from whomthe author got considerable benefit inthe accomplishment of this thesis.

An AD-(resp. AB-, ABD-) function on a subregion Δ whose real part vanishes continuously on the boundary γ of Δ is called' an AD^0-(resp. AB^0-, ABD^0-) function. A subregion on which every AD^0-(resp. AB^0-, ABD^0-) function reduces to a constant is said to be of the class SO_{AD} (resp. SO_{AB}, SO_{ABD}). A Riemann surface on which every subregion is of SO_{AD} (resp. SO_{AB}, SO_{ABD}) is said to be of the class O^0_{AD} (resp. O^0_{AB}, O^0_{ABD}).

The following lemma (KL) is due to Kuroda[3].

(KL) Let Δ be a subregion and $\hat{\Delta}$ the double of Δ along its boundary. Then $\hat{\Delta}\in O_{AD}$ (resp. O_{AB}, O_{ABD}) if and only if $\Delta\in SO_{AD}$ (resp. SO_{AB}, SO_{ABD}).

The proof is straight forw ard and will be omitted.

From (ST) and (KL) follows immediately

Corollary 1. $SO_{AD}=SO_{ABD}$, $O^0_{AD}=O^0_{ABD}$.

§ 2. Stoilow Principle and its Converse

A Riemann surface R is said to have (L_D)-(resp. (L_B)-, (L_{BD})-) property if for any subregion Δ with boundary γ on R and for any AD-(resp. AB-, ABD-) function $f(z)$ on $\Delta\cup\gamma$, $|f(z)|=M=\text{const.}$ on γ implies $|f(z)|\leqslant M$ in Δ. We have

Lemma 1. *The following statements are equivalent*[4]:

(i) $R\in O^0_{AD}$.

(ii) $R\in O^0_{ABD}$.

(iii) *R has (L_D)-property.*

(iv) *R has (L_{BD})-property.*

Proof For (i)→(iii) and (ii)↔(iv) see [4]. That (iii)→(iv) is evident and that (i)↔(ii) have been shown in Corollary 1.

Let R be a Riemann surface spread over the complex sphere and G a domain on the complex sphere. A component of the set (assumed non-empty) of all points on R whose projections lie in G is called a *pennisula* over G. A pennisula of R over G is *lacunary* if its projection is not dense everywhere in G.

Theorem 1. *A Riemann surface $R\in O^0_{AD}$ if and only if for every subregion Δ with boundaty γ on R, for every meromorphic function $f(p)$ on $\Delta\cup\gamma$ and for every circular disk G such that $f(\gamma)\cap G=\emptyset$ and $f(\Delta)\cap G\neq\emptyset$, every lacunary pennisula over G of the Riemann surface $\widetilde{f(\Delta)}$ produced by f from Δ has infinite spherical area.*

Proof Suppose that for a Riemann surface R the condition stated in Theorem 1 fails to hold, then there exists a subregion Δ with boundary γ on R, a meromorphic function f on $\Delta\cup\gamma$ and a circular disk G as discribed above such that there is a lacunary pennisula Φ_G of $\widetilde{f(\Delta)}$ over G with finite sphreical area. Since Φ_G is lacunary, there exists a point $w_0\in G$ and two positive numbers η and ε, such that the two circular disk G_η and G_ε of common center w_0 and radii η and ε respectively are included in G and such that $G_\eta \cap f(\Delta) \neq \emptyset$ and $G_\varepsilon \cap f(\Delta) = \emptyset$. Hence $\widetilde{f(\Delta)}$ has a lacunary pennisula Φ_{G_η} over G_η with finite spherical area. Denote the inverse image of Φ_{G_η} on Δ by Δ_η which is again a subregion with boundary γ_η on R. Define

$$g(p) = \frac{1}{f(p) - w_0}, \quad p \in \Delta_\eta \cup \gamma_\eta,$$

then

$$\frac{1}{\eta} \leqslant |g(p)| \leqslant \frac{1}{\varepsilon}, \quad p \in \Delta_\eta \cup \gamma_\eta,$$

$$g(p) = \frac{1}{\eta}, \qquad p \in \gamma_\eta, \tag{1}$$

and there exists at least one point $p_0 \in \Delta_\eta$ such that

$$|g(p_0)| > \frac{1}{\varepsilon}. \tag{2}$$

Moreover, $g(p)$ is an AD-function on Δ_η. In fact, denote by $D_\Omega(u)$ and $D^*_\Omega(u)$ the Dirichlet integral and the spherical Dirichlet integral respectively of a function u on a domain Ω, then

$$\begin{aligned} D_{\Delta_\eta}(g) &= \iint_{\Delta_\eta} \frac{|f'(z)|^2}{|f(z) - w_0|^4}\, dx\, dy \leqslant \frac{1}{\varepsilon^4} \iint_{\Delta_\eta} |f'(z)|^2 dx\, dy \\ &\leqslant \frac{(1+\eta^2)^2}{\varepsilon^4} \iint_{\Delta_\eta} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\, dx\, dy \leqslant \frac{(1+\eta^2)^2}{\varepsilon^4} D^*_{\Delta_\eta}(f) \\ &\leqslant \frac{(1+\eta^2)^2}{\varepsilon^4} D^*_\Delta(f) < +\infty, \end{aligned} \tag{3}$$

where $z = x + iy$ denotes the local coordinate of the variable point p in Δ.

It follows from (1), (2) and (3) that R has not (L_D)-property, hence by Lemma 1 $R \overline{\in} O^0_{AD}$. The necessity of the condition is proved.

In order to prove the sufficiency of the condition we suppose on the contrary that $R \overline{\in} O^0_{AD}$. Then by Lemma 1 R fails to have (L_{BD})-property, i. e. there exist a subgregion Δ with boundary γ on R, an ABD-function $f(p)$ on Δ whose modulus continuously equals to a constant M on γ and a point $p_1 \in \Delta$ such that $|f(p_1)| > M$. Hence

$$M < \sup_{p \in \Delta} |f(p)| = M' < +\infty$$

since $f(p)$ is bounded. Consequently there exists a point $w_0 \in \overline{f(\Delta)}$ and $|w_0| = M'$. Let G be a circular disk of center w_0 and radius $\frac{M'-M}{2}$, then $G \cap f(\Delta) \neq \emptyset$, $G \cap f(\gamma) = \emptyset$ and there is at least a lacunary pennisula Φ over G of the Riemann surface $\widetilde{f(\Delta)}$ produced by f from Δ. Since f is an AD-function, Φ has finite area and hence finite spherical area. This shows that R does not satisfy the condition stated in Theorem 1. The sufficiency is proved.

A totally disconnected closed set E on a Riemann surface is called AD-(resp.

AB–, ABD–) removable if, for every neighbourhood G of E, every AD–(resp. AB–, ABD–) function on $G\backslash E$ can be extended analytically to be an AD–(resp. AB–, ABD–) function on G.

Lemma 2. *Let $R\in O^0_{AD}$, Ω be any subregion with boundary α on R and $w=f(p)$ be a meromorphic function on $\Omega\cup\alpha$. Let G be a simply connected bounded subregion on the complex w-plane such that $G\cap f(\alpha)=\emptyset$ and $G\cap f(\Omega)\neq\emptyset$, and let Δ be a component of the set $\{p|f(p)\in G\}$. If $G\backslash f(\Delta)$ is non–AD–removable, then*

$$\sup_{w\in G} n_{f_\Delta}(w)=+\infty,$$

where $n_{f_\Delta}(w)$ for a given w is the cardinal number of the set $f_\Delta^{-1}(w)$, and $f_\Delta(p)$ is the restriction of $f(p)$ to Δ.

Proof Since $E=G\backslash f(\Delta)$ is non–AD–removable, the double $f(\Delta)$ of $f(\Delta)$ along the boundary σ of G does not belong to O_{AD}. By (KL) we have $f(\Delta)\overline{\in} SO_{AD}$ and there exists a nonconstant AD–function $\varphi(w)$ on $f(\Delta)$ whose real part vanishes continuously on σ. Evidently Δ is a subregion of R. We denote its boundary by γ. Then $\varphi\circ f(p)$ is a non–constant analytic function on $\Delta\cup\gamma$, whose real part vanishes continuously on γ. If

$$\sup_{w\in G} n_{f_\Delta}(w)=n<+\infty,$$

then

$$D_\Delta(\varphi\circ f)\leqslant nD_G(\varphi)<+\infty,$$

so that $R\overline{\in} O^0_{AD}$. This contradiction shows

$$\sup_{w\in G} n_{f_\Delta}(w)=+\infty.$$

By means of Lemma 2 we are able to establish the following modified Stoilow principle for meromorphic function in a subregion on a Riemann surface of O^0_{AD}.

Theorem 2. *Let Ω be a non–compact subregion of $R\in O^0_{AD}$ with compact boundary α and $w=f(p)$ be a non–constart meromorphic function on $\Omega\cup\alpha$. Then the supremum of the valence function $n_f(w)=\mathrm{card}\{p\in\Omega|f(p)=w\}$ is bounded is and only if the cluster set $C_\Omega(f,\beta)$ of f at the ideal boundary β of Ω is AD–removable.*

Proof To prove the necessity of the condition stated in Theorem 2, we show that the non–AD–removablllity of $C_\Omega(f,\beta)$ implies the unboundedness of $n_f(w)$. Suppose that $C_\Omega(f,\beta)$ is non–AD–removable. Denote by A the subset of $C_\Omega(f,\beta)$ consisting of all the points each of which possesses no neighbourhood intersecting $C_\Omega(f,\beta)$ in an AD–removable set. Then it is not difficult to see that A is non–empty and perfect, and every point of A possesses no neighbourhood intersecting A in an AD–removable set. We denote by A_0 the intersection of A with an arbitrary closed circular disk centered at a point $w_0\in A$. There are two possible cases: $n_f(w)=0$ for all $w\in A_0$ or $n_f(w_1)>0$ for some $w_1\in A_0$.

If it happens to be the first case, then we may construct a non–compact subregion Ω_0 with compact boundary α_0 such that $\Omega_0 \cup \alpha_0 \subset \Omega$ and $f(\alpha_0) \cap A_0 = \emptyset$. Since $f(\alpha_0)$ is compact, there exists a circular disk Δ centered at a point $w' \in A_0$ such that $\Delta \cap f(\alpha_0) = \emptyset$. Hence by Lemma 2, the valence function of the restriction f_0 of f to Ω_0 and consequently $n_f(w)$ should be unbounded. This is what we want to prove.

Consider now the second case, i. e.

$$n_f(w_1) = n_1 > 0 \quad \text{for some } w_1 \in A_0.$$

We shall show that the assumption $\sup n_f(w) < \infty$ would lead to a contradiction. For then we should have $0 < n_1 < \infty$ and may assume without loss of generality that w_1 is not the projection of any branch point of the n_1 sheets of the Riemann surface $\widetilde{f(\Omega)}$ produced by f from Ω that cover w_1. These n_1 sheet should cover schlichtly a common closed circular disk Δ_1 centered at w_1. Denote $A_1 = \Delta_1 \cap A_0$. Removing the inverse image of these n_1 schlicht islands from Ω we get a new non-compact subregion Ω_1. Denote f_1 the restriction of f to Ω_1, then

$$\sup_{w \in A_1} n_{f_1}(w) \leqslant \sup_{w \in A_1} n_f(w) - n_1 \leqslant \sup n_f(w) - n_1.$$

If it happens to be that $\sup\limits_{w \in A_1} n_{f_1}(w) = 0$, then by the same arguments as in the preceding paragraph, $n_{f_1}(w)$ and consequently $n_f(w)$ would be unbounded, and we arrive at a contradiction. If otherwise for some $w_2 \in A_1$, $n_{f_1}(w_2) = n_2 > 0$, then the similar procedure may be repeated so as to obtain a non-compact subregion Ω_2 and the restriction f_2 of f_1 to Ω_2 such that

$$\sup_{w \in A_1} n_{f_2}(w) \leqslant \sup n_f(w) - n_1 - n_2.$$

Since $\sup n_f(w)$ is assumed finite, by repeating the same procedure at most finite times we shall get ultimately a positive integer k such that

$$\sup_{w \in A_k} n_{f_k}(w) = 0,$$

where $A_1, A_2, \cdots, A_k$ and $f_1, f_2, \cdots, f_k$ are understood obviously. Therefore $n_{f_k}(w)$ and consequently $n_f(w)$ should be unbounded and a contradiction is obtained. The proof of the necessity is completed.

Next, suppose that the cluster set $C_\Omega(f, \beta)$ is AD-removable, then $w = f(p)$ has localizable Iversen's property[5]. By Stoilow principle, we have

$$\sup n_f(w) < +\infty$$

and the sufficiency is tenable.

§ 3. The Essential Maximality of a Riemann Surface of O^0_{AD}

Theorem 3. *If $R \in O^0_{AD}$, then the following statements are true:*

(i) *R is essentially maximal,*

(ii) *If φ is a conformal homeomorphism of R into a Riemann surface $\tilde{R}$, then $\tilde{R}\backslash\varphi(R)$ is AD-removable.*

(iii) *Any two maximal prolongations of R are conformally homeomorphic to each other.*

Proof To prove (i) we suppose that $\tilde{R}$ is a prolongation of R and φ is a conformal homeomorphism of R into $\tilde{R}$. It suffices to show that $\tilde{R}\backslash\varphi(R)$ is totally disconnected. If, on the contrary $\tilde{R}\backslash\varphi(R)$, the complement of $\varphi(R)$ relative to $\tilde{R}$, contained a continuum, then the boundary of $\varphi(R)$ relative to $\tilde{R}$ would contain a continuum C. Take $w_0 \in C$ and a coordinate neighbourhood G of w_0. We may imagine G as a closed circular disk on the complex plane with center w_0. $\varphi(R)\cap G$ would have at least a component Φ which would be an one-sheeted pennisula of $\varphi(R)$ over G, and $G\backslash\Phi$ contained a non-AD-removable set. This contradicts Lemma 2. Therefore (i) is true.

We now prove (ii). From the above we have already observed that $E=\tilde{R}\backslash\varphi(R)$ is the boundary of $\varphi(R)$ relative to $\tilde{R}$. By similar arguments it follows that every point $w_0\in E$ should possess a neighbourhood G with $G\cap E$ AD-removable. (ii) is proved.

To prove (iii) we suppose that $\tilde{R}_1$ and $\tilde{R}_2$ are two maximal prolongations of a Riemann surface R of O^0_{AD} and φ_1 and φ_2 are the corresponding conformal homeomorphism of R into $\tilde{R}_1$ and $\tilde{R}_2$ respectively. Take $w_1\in\tilde{R}_1\backslash\varphi_1(R)$ and a sufficiently small neighbourhood G_1 of w_1 whose boundary is an analytic Jordan curve lying entirely in $\varphi_1(R)$[5]. As usual we may imagine G_1 as subregion lying within a circular disk. Then $\varphi_2\circ\varphi_1^{-1}$ maps $\varphi_1(R)\cap G_1$ onto a subregion G_2 on $\varphi_2(R)$. Since $\tilde{R}_2\backslash\varphi_2(R)$ is AD-removable, the inverse schlicht mapping $\varphi\circ\varphi_1^{-1}$ can be extended to be a conformal homeomorphism of the closure $\bar{G}_2$ of G_2 relative to $\bar{R}_2$ into G_1. This must be an onto mapping since otherwise $\tilde{R}_2$ would be prolongable. Therefore $\varphi_2\circ\varphi_1^{-1}$ can be extended to a conformal homeomorphism of G_1 into $\tilde{R}_2$. Letting w_1 range over the whole $\tilde{R}_1\backslash\varphi_1(R)$, we assert that $\varphi_2\circ\varphi_1^{-1}$ can be extended to be a conformal homeomorphism of $\tilde{R}_1$ into $\bar{R}_2$. Since $\bar{R}_1$ is maximal, this is an onto conformal homeomorphism. (iii) is proved.

Corollary 2. *If $R\in O^0_{AB}$, then the following statements are true:*

(i) *R is essentially maximal.*

(ii) *If φ is a conformal homeomorphism of R. into a Riemann surface $\tilde{R}$, then $\tilde{R}\backslash\varphi(R)$ is AB-removable.*

(iii) *Any two maximal prolongations of R are conformally homeomorphic to each other.*

Proof (i) follows immediately from the fact that $O^0_{AB}\subset O^0_{AD}$. (ii) and (iii) may be proved in a similar way as for Theorem 3.

We note that there exists a maximal Riemann surface $F \in O_{AD}$[6], Let E be an AD-removable set on F, then $F' = F \setminus E \in O_{AD}$. So $F' \in O^0_{AD}$. Therefore, (i) and (ii) of Theorem 3 are not sufficient conditions for a Riemann surface to be of O^0_{AD}.

It is known that a Riemann surface R of finite genus belonging to O_{AD} is essentially maximal and all closed extensions of R are conformally equivalent[1]. However Myrberg's example [1] shows that there is a Riemann surface of infinite genus belonging to O_{AD} but not essentially maximal.

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