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# CIOSED-LOOP SYNTHESES FOR QUADRATIC DSFFERENTIAL GAME OF DISTRIBUTED SYSTEMS

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**Abstract**

To differential game problems of linear distributed parameter systems with quadratic criterion, closed-loop syntheses of optimal strategy are proved and solution of related operator Riccati equation is investigated.

## § 1. Introduction

Differential games of distributed parameter systems are of interest to some practical examples such as pollution control and competitive fishing in a water region[1]. The latter problem, especially, can be formulated as a game of parabolic system with respect to quadratic criterion.

In [2] (Chap. 6) various results are given on the properties and the open-loop necessary conditions of quadratic optimal strategies of linear distributed parameter systems.

In this paper we explore the closed-loop syntheses of such a class of differential games described as follows.

Let real Hilbert spaces $X$, $U$ and $V$ be the value spaces of state $x(t)$, controls $u(t)$ and $v(t)$ respectively. $t_1>0$ fixed. Denote $\mathscr{X}=L^2(0, t_1; X)$, $\mathscr{U}=L^2(0, t_1; U)$ and $\mathscr{V}=L^2(0, t_1; V)$. Consider a linear evolution system

$$x(t)=T(t)x_0+\int_0^t T(t-s)[Bu(s)+Cv(s)]ds,\ t\geqslant 0, \tag{1.1}$$

and a quadratic criterion

$$J(u, v)=\langle Qx(t_1), x(t_1)\rangle+\int_0^{t_1}[\langle Wx(t), x(t)\rangle+\langle R_1u(t), u(t)\rangle+\langle R_2v(t), v(t)\rangle]dt, \tag{1.2}$$

the game problem is to find a strategy $(u_*, v_*)\in\mathscr{U}\times\mathscr{V}$ such that

$$J(u_*, v)\leqslant J(u_*, v_*)\leqslant J(u, v_*),\ \forall u\in\mathscr{U},\ \forall v\in\mathscr{V}. \tag{1.3}$$

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Such a strategy $(u_*, v_*)$ is called optimal in the sense of saddle point. This problem will be briefly denoted by (GP).

Here we assume that $T(t)$ $(t\geqslant 0)$ is a $C_0$-semigroup of operators on $X$,

$$B\in\mathscr{L}(U;X),\ C\in\mathscr{L}(V;X),$$

$Q$ and $W$ are self-adjoint in $\mathscr{L}(X)$, $R_1\in\mathscr{L}(U)$ and $R_2\in\mathscr{L}(V)$ are coercively positive and negative self-adjoint respectively.

Define some operators:

$$\begin{aligned}&(Ku)(t)=\int_0^t T(t-s)Bu(s)ds,\ K_1u=(Ku)(t_1),\ u\in\mathscr{U};\\&(Lv)(t)=\int_0^t T(t-s)Cv(s)ds,\ L_1v=(Lv)(t_1),\ v\in\mathscr{V};\\&\Phi=R_1+K^*WK+K_1^*QK_1,\ \Psi=R_2+L^*WL+L_1^*QL_1.\end{aligned}\tag{1.4}$$

Obviously, $K\in\mathscr{L}(\mathscr{U};\mathscr{X})$, $K_1\in\mathscr{L}(\mathscr{U};X)$, $L\in\mathscr{L}(\mathscr{V};\mathscr{X})$, $L_1\in\mathscr{L}(\mathscr{V};X)$. $\Phi\in\mathscr{L}(\mathscr{U})$ and $\Psi\in\mathscr{L}(\mathscr{V})$ are self-adjoint.

## § 2. Closed-loop Result

**Hypothesis 1.** *$\Phi>0$ (coercively positive) and $\Psi<0$ (coercively negative).*

**Theorem 1.** *Under Hypothesis 1, for any given $x_0\in X$, there exists a unique optimal strategy of (GP), $(u_*, v_*)$, which satisfies the open-loop equations:*

$$\left.\begin{aligned}u_*(t)&=-R_1^{-1}B^*[T^*(t_1-t)Qx_*(t_1)+\int_t^{t_1}T^*(\sigma-t)Wx_*(\sigma)d\sigma],\\v_*(t)&=-R_2^{-1}C^*[T^*(t_1-t)Qx_*(t_1)+\int_t^{t_1}T^*(\sigma-t)Wx_*(\sigma)d\sigma],\end{aligned}\right\}\ t\in[0,t_1].\tag{2.1}$$

*Proof* $J(u,v)$ can be written as follows

$$\begin{aligned}J(u,v)=&\langle Q(K_1u+L_1v+T(t_1)x_0),\ K_1u+L_1v+T(t_1)x_0\rangle_X\\&+\langle W(Ku+Lv+T(\cdot)x_0,\ Ku+Lv+T(\cdot)x_0\rangle_{\mathscr{X}}\\&+\langle R_1u,u\rangle_{\mathscr{U}}+\langle R_2v,v\rangle_{\mathscr{V}}\\=&\langle\Phi u,u\rangle+2\langle(K_1^*QL_1+K^*WL)v,u\rangle+\langle\Psi v,v\rangle\\&+2\langle K_1^*QT(t_1)x_0+K^*WT(\cdot)x_0,u\rangle+2\langle L_1^*QT(t_1)x_0+L^*WT(\cdot)x_0,v\rangle\\&+\langle QT(t_1)x_0,T(t_1)x_0\rangle+\langle WT(\cdot)x_0,T(\cdot)x_0\rangle,\ \forall(u,v)\in\mathscr{U}\times\mathscr{V}.\end{aligned}\tag{2.2}$$

From (2.2) it can be seen that there is a $(u_*, v_*)$ which satisfies (1.3) if and only if the following system of equations admits a solution,

$$\begin{bmatrix}\Phi & K^*WL+K_1^*QL_1\\ L^*WK+L_1^*QK_1 & \Psi\end{bmatrix}\begin{bmatrix}u_*\\ v_*\end{bmatrix}=-\begin{bmatrix}K^*WT(\cdot)+K_1^*QT(t_1)\\ L^*WT(\cdot)+L_1^*QT(t_1)\end{bmatrix}x_0.\tag{2.3}$$

In view of the auxiliary lemma described later, we know that the operator

$$H=\begin{bmatrix}\Phi & K^*WL+K_1^*QL_1\\ L^*WK+L_1^*QK_1 & \Psi\end{bmatrix}\tag{2.4}$$

has a bounded inverse operator $H^{-1}$. Thus (2.3) admits a unique solution.

Taking note of

$$(K_1^*h)(t)=B^*T^*(t_1-t)h,\ (L_1^*h)(t)=C^*T^*(t_1-t)h,\ \forall h\in X;$$

$$(K^*y)(t)=\int_t^{t_1}B^*T^*(\sigma-t)y(\sigma)d\sigma,\ (L^*y)(t)=\int_t^{t_1}C^*T^*(\sigma-t)y(\sigma)d\sigma,\ \forall y\in\mathscr{X}, \tag{2.5}$$

we can verify that (2.3) is equivalent to (2.1). Q. E. D.

**Auxiliary Lemma.** *Let $H_1$ and $H_2$ be Hilbert spaces, $E\in\mathscr{L}(H_1)$ and $G\in\mathscr{L}(H_2)$ be coercively positive and negative self-adjoint respectively, and $S\in\mathscr{L}(H_2;H_1)$. Then the operator*

$$\begin{pmatrix}E & S\\ S^* & G\end{pmatrix}\in\mathscr{L}(H_1\times H_2)$$

*has a bounded inverse operator given by*

$$\begin{pmatrix}E & S\\ S^* & G\end{pmatrix}^{-1}=\begin{pmatrix}(E-SG^{-1}S^*)^{-1} & -(E-SG^{-1}S^*)^{-1}SG^{-1}\\ -(G-S^*E^{-1}S)^{-1}S^*E^{-1} & (G-S^*E^{-1}S)^{-1}\end{pmatrix}. \tag{2.6}$$

*Proof* As $E-SG^{-1}S^*\geqslant E>0$ and $G-S^*E^{-1}S\leqslant G<0$, the operator on the right side of (2.6) can be verified to be the left inverse of $\begin{pmatrix}E & S\\ S^* & G\end{pmatrix}$. Similarly, its right inverse also exists. Thus (2.6) is valid.

**Theorem 2** (Closed-loop Theorem I). *Under Hypothesis 1, $(u_*, v_*)$ is the optimal strategy of (GP) if and only if it is the linear state feedback given by*

$$\begin{aligned}u_*(t)&=-R_1^{-1}B^*P(t)x_*(t),\\ v_*(t)&=-R_2^{-1}C^*P(t)x_*(t),\end{aligned}\quad t\in[0,\ t_1],\ x_0\in X, \tag{2.7}$$

*where $x_*$ is the corresponding trajectory, and $P(t)$ $(0\leqslant t\leqslant t_1)$ is a strongly continuous and self-adjoint solution of the operator Riccati equation*

$$\begin{aligned}P(t)=T^*(t_1-t)QT(t_1-t)+\int_t^{t_1}T^*(\sigma-t)[W-P(\sigma)(BR_1^{-1}B^*\\ +CR_2^{-1}C^*)P(\sigma)]T(\sigma-t)d\sigma.\end{aligned} \tag{2.8}$$

*And the following equality holds*

$$J^*\equiv J(u_*, v_*)=\langle P(0)x_0, x_0\rangle. \tag{2.9}$$

## § 3. Proof of Theorem 2

1. Proof of the "only if" part of Theorem 2.

We exploit the optimality principle of dynamic programming in proving this. For any given $\tau\in[0,\ t_1]$, $(GP)_\tau$ is referred to the corresponding game problem for which (1.1) and (1.2) are replaced by

$$x(t)=T(t-\tau)x_{0\tau}+\int_\tau^{t_1}T(t-s)[Bu(s)+Cv(s)]ds,\ t\geqslant\tau, \tag{3.1}$$

$$J_\tau(u,v)=\langle Qx(t_1),x(t_1)\rangle+\int_\tau^{t_1}[\langle Wx(t),x(t)\rangle+\langle R_1u(t),u(t)\rangle+\langle R_2v(t),v(t)\rangle]dt, \tag{3.2}$$

where $u\in\mathscr{U}_\tau=L^2(\tau,t_1;U)$ and $v\in\mathscr{V}_\tau=L^2(\tau,t_1;V)$.

To $(GP)_\tau$ we attach the subscript $\tau$ to the counterparts of those operators shown in § 1 and § 2. $\Phi>0$ implies $\Phi_\tau>0$ and $\Psi<0$ implies $\Psi_\tau<0$ by null extension.

**Lemma 1.** *For any given $x_{0\tau}\in X$, the unique optimal strategy $(x_{*\tau},v_{*\tau})$ of $(GP)_\tau$ can be expressed by*

$$u_{*\tau}(t)=M_\tau(t)x_{0\tau},\ v_{*\tau}(t)=N_\tau(t)x_{0\tau},\ t\in[\tau,t_1], \tag{3.3}$$

*where operators $M_\tau(t)$ and $N_\tau(t)$ are strongly continuous with respect to $t$ and such that*

$$\sup_{0\le\tau\le t\le t_1}\|M_\tau(t)\|_{\mathscr{L}(X;U)}<\infty,\ \sup_{0\le\tau\le t\le t_1}\|N_\tau(t)\|_{\mathscr{L}(X;V)}<\infty. \tag{3.4}$$

*Proof* Analogous to (2.3) and (2.4), we have

$$\begin{pmatrix}u_{*\tau}\\ v_{*\tau}\end{pmatrix}=-H_\tau^{-1}\begin{pmatrix}K_\tau^*WT(\cdot-\tau)+K_{1\tau}^*QT(t_1-\tau)\\ L_\tau^*WT(\cdot-\tau)+L_{1\tau}^*QT(t_1-\tau)\end{pmatrix}x_{0\tau}, \tag{3.5}$$

in view of (2.6), $H_\tau^{-1}$ is given by

$$H_\tau^{-1}=\begin{pmatrix}\Pi_\tau^{-1} & -\Pi_\tau^{-1}(K_\tau^*WL_\tau+K_{1\tau}^*QL_{1\tau})\Psi_\tau^{-1}\\ -\Gamma_\tau^{-1}(L_\tau^*WK_\tau+L_{1\tau}^*QK_{1\tau})\Phi_\tau^{-1} & \Gamma_\tau^{-1}\end{pmatrix}, \tag{3.6}$$

where

$$\begin{aligned}\Pi_\tau&=\Phi_\tau-(K_\tau^*WL_\tau+K_{1\tau}^*QL_{1\tau})\Psi_\tau^{-1}(L_\tau^*WK_\tau+L_{1\tau}^*QK_{1\tau}),\\ \Gamma_\tau&=\Psi_\tau-(L_\tau^*WK_\tau+L_{1\tau}^*QK_{1\tau})\Phi_\tau^{-1}(K_\tau^*WL_\tau+K_{1\tau}^*QL_{1\tau}).\end{aligned} \tag{3.7}$$

By (1.4) and (2.5) it can be seen that

$$\begin{aligned}&\sup_{0\le\tau\le t_1}\|K_\tau^*WT(\cdot-\tau)+K_{1\tau}^*QT(t_1-\tau)\|_{\mathscr{L}(X;C([\tau,t_1];U))}<\infty,\\ &\sup_{0\le\tau\le t_1}\|L_\tau^*WT(\cdot-\tau)+L_{1\tau}^*QT(t_1-\tau)\|_{\mathscr{L}(X;C([\tau,t_1];V))}<\infty;\\ &\sup_{0\le\tau\le t_1}\|K_\tau^*WL_\tau+K_{1\tau}^*QL_{1\tau}\|_{\mathscr{L}(C([\tau,t_1];V);C([\tau,t_1];U))}<\infty,\\ &\sup_{0\le\tau\le t_1}\|L_\tau^*WK_\tau+L_{1\tau}^*QK_{1\tau}\|_{\mathscr{L}(C([\tau,t_1];U);C([\tau,t_1];V))}<\infty.\end{aligned} \tag{3.8}$$

As Hypothesis 1 implies $\sup_{0\le\tau\le t_1}\|\Phi_\tau^{-1}\|_{\mathscr{L}(\mathscr{U}_\tau)}<\infty$ and $\sup_{0\le\tau\le t_1}\|\Psi_\tau^{-1}\|_{\mathscr{L}(\mathscr{V}_\tau)}<\infty$, it turns out from (3.7), $\Pi_\tau\ge\Phi_\tau$ and $\Gamma_\tau\le\Psi_\tau$ that

$$\sup_{0\le\tau\le t_1}\|\Pi_\tau^{-1}\|_{\mathscr{L}(\mathscr{U}_\tau)}<\infty,\ \sup_{0\le\tau\le t_1}\|\Gamma_\tau^{-1}\|_{\mathscr{L}(\mathscr{V}_\tau)}<\infty. \tag{3.9}$$

Taking note of (1.4) and (2.5), we have the explicit expressions of $\Phi_\tau$ and $\Psi_\tau$:

$$\begin{aligned}(\Phi_\tau g)(t)&=R_1g(t)+\int_\tau^{t_1}Z_1(t,\sigma)g(\sigma)d\sigma,\ g\in\mathscr{U}_\tau,\\ (\Psi_\tau h)(t)&=R_2h(t)+\int_\tau^{t_1}Z_2(t,\sigma)h(\sigma)d\sigma,\ h\in\mathscr{V}_\tau,\end{aligned} \tag{3.10}$$

where

$$Z_1(t,\sigma)=B^*\Big[T^*(t_1-t)QT(t_1-\sigma)+\int_{\max(t,\sigma)}^{t_1}T^*(\eta-t)WT(\eta-\sigma)d\eta\Big]B,$$
$$Z_2(t,\sigma)=C^*\Big[T^*(t_1-t)QT(t_1-\sigma)+\int_{\max(t,\sigma)}^{t_1}T^*(\eta-t)WT(\eta-\sigma)d\eta\Big]C,$$
$$(t,\sigma)\in[\tau,t_1]^2. \quad (3.11)$$

Similarly to the Lemma 3 of [3], $\Phi_\tau$ and $\Psi_\tau$ are bijections on $C([\tau,t_1];U)$ and $C([\tau,t_1];V)$ respectively, and

$$\sup_{0\leq\tau\leq t_1}\|\Phi_\tau^{-1}\|_{\mathscr{L}(C([\tau,t_1];U))}<\infty,\ \sup_{0\leq\tau\leq t_1}\|\Psi_\tau^{-1}\|_{\mathscr{L}(C([\tau,t_1];V))}<\infty. \quad (3.12)$$

Hence, (3.7) implies that $\Pi_\tau$ and $\Gamma_\tau$ are bijections on $C([\tau,t_1];U)$ and $C([\tau,t_1];V)$ respectively, and

$$\sup_{0\leq\tau\leq t_1}\|\Pi_\tau^{-1}\|_{\mathscr{L}(C([\tau,t_1];U))}<\infty,\ \sup_{0\leq\tau\leq t_1}\|\Gamma_\tau^{-1}\|_{\mathscr{L}(C([\tau,t_1];V))}<\infty. \quad (3.13)$$

In fact, for example, if $\Pi_\tau g=f\in C([\tau,t_1];U)$, we have

$$\begin{aligned}\|g\|_{C([\tau,t_1];U)}\leq&\sup_{t\in[\tau,t_1]}\Big\|R_1^{-1}\int_\tau^{t_1}Z_1(t,\sigma)g(\sigma)d\sigma\Big\|\\&+\|R_1^{-1}(K_\tau^*WL_\tau+K_{1\tau}^*QL_{1\tau})\Psi_\tau^{-1}(L_\tau^*WK_\tau+L_{1\tau}^*QK_{1\tau})g\|_{C([\tau,t_1];U)}\\&+\|R_1^{-1}f\|_{C([\tau,t_1];U)}\\\leq&\text{const}\|g\|_{\mathscr{U}_\tau}+\text{const}\|f\|_{C([\tau,t_1];U)}\\\leq&\text{const}\|\Pi_\tau^{-1}\|_{\mathscr{L}(\mathscr{U}_\tau)}\|f\|_{\mathscr{U}_\tau}+\text{const}\|f\|_{C([\tau,t_1];U)}\leq\text{const}\ \|f\|_{C([\tau,t_1];U)}.\end{aligned}$$

Combining (3.5) and (3.6) with (3.8), (3.12) and (3.13), we obtain (3.4).

**Lemma 2.** *For any given $x_0\in X$, the optimal trajectory of (GP) is*

$$x_*(t)=G(t,\xi)\ x_*(\xi)=G(t,0)x_0,\ 0\leq\xi\leq t\leq t_1, \quad (3.14)$$

*where $G(t,\xi)$ $(0\leq\xi\leq t\leq t_1)$ is a family of mild evolution operators with uniformly bounded norms. The optimal strategy $(u_*,v_*)$ of (GP) is given by (2.7) where the feedback operator $P(\cdot)$ is characterized in Theorem 2.*

*Proof* The optimality principle of dynamic programming indicates

$$x_*(t;0,x_0)=x_{*\xi}(t;\xi,x_*(\xi;0,x_0)),\ 0\leq\xi\leq t\leq t_1,\ x_0\in X. \quad (3.15)$$

Let $G(t,\tau)$ be given as follows

$$G(t,\tau)=T(t-\tau)+\int_\tau^t T(t-s)[BM_\tau(s)+CN_\tau(s)]ds,\ 0\leq\tau\leq t\leq t_1, \quad (3.16)$$

where $M_\tau(\cdot)$ and $N_\tau(\cdot)$ are shown in (3.3). Hence (3.14) holds. (3.15) implies that $G(t,\tau)$ possesses the evolution property and it is strongly continuous with respect to $t\in[\tau,t_1]$. (3.4) implies that $\|G(t,\tau)\|_{\mathscr{L}(X)}$ are uniformly bounded for $0\leq\tau\leq t\leq t_1$. These two facts in turn imply its strong continuity with respect to $\tau\in[0,t]$.

Substituting (3.14) into (2.1), we obtain (2.7), where $P(\cdot)$ is given by

$$P(t)=T^*(t_1-t)QG(t_1,t)+\int_t^{t_1}T^*(\sigma-t)WG(\sigma,t)d\sigma,\ t\in[0,t_1]. \quad (3.17)$$

Similar to the proof of the Lemma 5 of [3], it can be proved that $P(\cdot)$ given by

(3.17) is a strongly continuous and self-adjoint solution of the Riccati equation (2.8). Q. E. D.

Thus we have completed the proof of the "only if" part of Theorem 2 except (2.9) which will be proved later in lemma 5.

2. Proof of the "if" part of Theorem 2.

**Lemma 3.** *Let $F$ be any self-adjoint operator in $\mathscr{L}(X)$. For any two game processes $\{x_0, u, v, x\}$ and $\{\hat{x}_0, \hat{u}, \hat{v}, \hat{x}\}$, the following identity holds:*

$$\begin{aligned}&\langle FT(\sigma-t)x(t), T(\sigma-t)\hat{x}(t)\rangle\\&=\langle Fx(\sigma), \hat{x}(\sigma)\rangle-\int_t^\sigma\langle FT(\sigma-s)x(s), T(\sigma-s)[B\hat{u}(s)+C\hat{v}(s)]\rangle ds\\&\quad-\int_t^\sigma\langle FT(\sigma-s)\hat{x}(s), T(\sigma-s)[Bu(s)+Cv(s)]\rangle ds, 0\leqslant t\leqslant\sigma.\end{aligned}\tag{3.18}$$

It can be verified directly.

**Lemma 4.** *Let $P(t)$ $(0\leqslant t\leqslant t_1)$ be a strongly continuous and self-adjoint solution of the Riccati equation (2.8). For any two game processes $\{x_0, u, v, x\}$ and $\{\hat{x}_0, \hat{u}, \hat{v}, \hat{x}\}$, the following identity holds:*

$$\begin{aligned}&\langle P(t)x(t), \hat{x}(t)\rangle\\&=\langle Qx(t_1), \hat{x}(t_1)\rangle-\int_t^{t_1}\langle x(s), P(s)[B\hat{u}(s)+C\hat{v}(s)]\rangle ds\\&\quad-\int_t^{t_1}\langle\hat{x}(s), P(s)[Bu(s)+Cv(s)]\rangle ds\\&\quad+\int_t^{t_1}\langle(W-P(s)[BR_1^{-1}B^*+CR_2^{-1}C^*]P(s))x(s), \hat{x}(s)\rangle ds, t\in[0, t_1].\end{aligned}\tag{3.19}$$

*Proof* Let $F=Q$ in (3.18), we obtain

$$\begin{aligned}&\langle QT(t_1-t)x(t), T(t_1-t)\hat{x}(t)\rangle\\&=\langle Qx(t_1), \hat{x}(t_1)\rangle-\int_t^{t_1}\langle QT(t_1-s)x(s), T(t_1-s)[B\hat{u}(s)+C\hat{v}(s)]\rangle ds\\&\quad-\int_t^{t_1}\langle QT(t_1-s)\hat{x}(s), T(t_1-s)[Bu(s)+Cv(s)]\rangle ds.\end{aligned}\tag{3.20}$$

Let $F(\sigma)=W-P(\sigma)(BR_1^{-1}B^*+CR_2^{-1}C^*)P(\sigma)$ in (3.18). Then integrate it with respect to $\sigma\in[t, t_1]$, we obtain

$$\begin{aligned}&\left\langle\int_t^{t_1}T^*(\sigma-t)(W-P(\sigma)(BR_1^{-1}B^*+CR_2^{-1}C^*)P(\sigma))T(\sigma-t)d\sigma x(t), \hat{x}(t)\right\rangle\\&=\int_t^{t_1}\langle(W-P(s)[BR_1^{-1}B^*+CR_2^{-1}C^*]P(s)x(s), \hat{x}(s)\rangle ds\\&\quad-\int_t^{t_1}\langle x(s), \left\{\int_s^{t_1}T^*(\sigma-s)(W-P(\sigma)[BR_1^{-1}B^*+CR_2^{-1}C^*]P(\sigma))T(\sigma-s)d\sigma\right\}\\&\quad\{B\hat{u}(s)+C\hat{v}(s)\}\rangle ds\\&\quad-\int_t^{t_1}\langle\hat{x}(s), \left\{\int_s^{t_1}T^*(\sigma-s)(W-P(\sigma)[BR_1^{-1}B^*+CR_2^{-1}C^*]P(\sigma))T(\sigma-s)d\sigma\right\}\\&\quad\{Bu(s)+Cv(s)\}\rangle ds.\end{aligned}\tag{3.21}$$

Summing up the two sides of (3.20) and (3.21) respectively leads immediately to (3.19).

**Lemma 5.** *Let $P(t)$ $(0 \leqslant t \leqslant t_1)$ be a strongly continuous and self-adjoint solution of the Riccati equation (2.8). For any given $x_0 \in X$, if a strategy $(\bar{u}, \bar{v})$ and its corresponding trajectory $\bar{x}$ satisfy*

$$\begin{aligned} \bar{u}(t) &= -R_1^{-1}B^*P(t)\bar{x}(t), \\ \bar{v}(t) &= -R_2^{-1}C^*P(t)\bar{x}(t), \end{aligned} \quad t \in [0, t_1], \tag{3.22}$$

*then $J(\bar{u}, \bar{v}) = \langle P(0)x_0, x_0\rangle$.*

*Proof* Let $\{\hat{x}_0, \hat{u}, \hat{v}, \hat{x}\} = \{x_0, u, v, x\}$ and $t=0$ in (3.19), we obtain

$$\langle P(0)x_0, x_0\rangle = J(u, v) - \int_0^{t_1} \langle R_1(u(t) + R_1^{-1}B^*P(t)x(t)), u(t) + R_1^{-1}B^*P(t)x(t)\rangle dt$$

$$- \int_0^{t_1} \langle R_2(v(t) + R_2^{-1}C^*P(t)x(t)), v(t) + R_2^{-1}C^*P(t)x(t)\rangle dt. \tag{3.23}$$

Hence, if $(\bar{u}, \bar{v}, \bar{x})$ satisfies (3.22), then $J(\bar{u}, \bar{v}) = \langle P(0)x_0, x_0\rangle$.

**Lemma 6.** *If $\Phi \geqslant 0$ and $\Psi \leqslant 0$, and $P(t)$ $(0 \leqslant t \leqslant t_1)$ is a strongly continuous and self-adjoint solution of the Riccati equation (2.8), then for any given $x_0 \in X$, the feedback strategy. $(\bar{u}, \bar{v})$ given by (3.22) must be the optimal strategy of (GP).*

*Proof* For any given $x_0 \in X$, the game process $\{x_0, \bar{u}, \bar{v}, \bar{x}\}$ is given by (3.22), and for any $u \in \mathscr{U}$, we have another game process $\{x_0, u, \bar{v}, x\}$. Denote $u_e = u - \bar{u}$, $x_e = x - \bar{x}$. Obviously, $x_e$ is the trajectory corresponding to the initial state zero and the strategy $(u_e, 0)$.

$$J(u, \bar{v})|_{x(0)=x_0} = J(\bar{u}, \bar{v})|_{\bar{x}(0)=x_0} + J(u_e, 0)|_{x_e(0)=0} + J_1, \ \forall u \in \mathscr{U}, \tag{3.24}$$

where

$$J_1 = 2\left\{\langle Q\bar{x}(t_1), x_e(t_1)\rangle + \int_0^{t_1} [\langle W\bar{x}(t), x_e(t)\rangle + \langle R_1\bar{u}(t), u_e(t)\rangle] dt\right\}.$$

According to Lemma 4, now for the two game processes $\{x_0, \bar{u}, \bar{v}, \bar{x}\}$ and $\{0, u_e, 0, x_e\}$, taking $t=0$ in (3.19), we obtain

$$0 = \langle P(0)x_0, 0\rangle = \langle Q\bar{x}(t_1), x_e(t_1)\rangle - \int_0^{t_1} \langle \bar{x}(s), P(s)Bu_e(s)\rangle ds$$

$$- \int_0^{t_1} \langle x_e(s), P(s)[B\bar{u}(s) + C\bar{v}(s)]\rangle ds + \int_0^{t_1} \langle W\bar{x}(s), x_e(s)\rangle ds$$

$$- \int_0^{t_1} \langle P(s)[BR_1^{-1}B^* + CR_2^{-1}C^*]P(s)\bar{x}(s), x_e(s)\rangle ds = \frac{1}{2} J_1.$$

Hence, $J_1 = 0$. According to (2.2), we have

$$J(u_e, 0)|_{x_e(0)=0} = \langle \Phi u_e, u_e\rangle \geqslant 0.$$

Thus $J(u, \bar{v})|_{x(0)=x_0} \geqslant J(\bar{u}, \bar{v})|_{\bar{x}(0)=x_0}$. Similarly we can prove that $J(\bar{u}, \bar{v})|_{\bar{x}(0)=x_0} \geqslant J(\bar{u}, v)|_{x(0)=x_0}$ for any $v \in \mathscr{V}$. Q. E. D.

Thus we have proved the "if" part of Theorem 2. The proof of this theorem is completed.

## § 4. Solution of Riccati Equation (2.8)

**Hypothesis 2.** *U and V are finite dimensional spaces.*

By introducing new equivalent norms of $U$ and $V$, we can assume

$$R_1 = I_U \text{ and } R_2 = -I_V$$

for brevity.

The open-loop equation (2.3) can be written concretely as follows:

$$\tilde{I}\begin{pmatrix} u_*(t) \\ v_*(t) \end{pmatrix} + \int_0^{t_1} E(t,\sigma)\begin{pmatrix} u_*(\sigma) \\ v_*(\sigma) \end{pmatrix} d\sigma = -\begin{pmatrix} B^* y(t;\, x_0) \\ C^* y(t;\, x_0) \end{pmatrix},\ t \in [0,\, t_1], \tag{4.1}$$

where $\tilde{I} = \begin{pmatrix} I_U & \\ & -I_V \end{pmatrix}$, and

$$E(t,\,\sigma) = \begin{pmatrix} B^* \\ C^* \end{pmatrix}\left[T^*(t_1 - t)QT(t_1 - \sigma) + \int_{\max(t,\sigma)}^{t_1} T^*(\eta - t)WT(\eta - \sigma)d\eta\right](B,\, C). \tag{4.2}$$

$$y(t;\, x_0) = T^*(t_1 - t)QT(t_1)x_0 + \int_t^{t_1} T^*(\sigma - t)WT(\sigma)d\sigma x_0.$$

For $0 \leqslant \tau \leqslant t_1$, define an operator $E_\tau \in \mathscr{L}(\mathscr{U}_\tau \times \mathscr{V}_\tau)$ by

$$\left[E_\tau \begin{pmatrix} u \\ v \end{pmatrix}\right](t) = \int_\tau^{t_1} E(t,\,\sigma)\begin{pmatrix} u(\sigma) \\ v(\sigma) \end{pmatrix} d\sigma,\ t \in [\tau,\, t_1],\ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathscr{U}_\tau \times \mathscr{V}_\tau, \tag{4.3}$$

where $E(t,\,\sigma)$ is shown by (4.2). As $E(t,\,\sigma)$ is a matrix-valued function which is continuous with respect to $(t,\,\sigma) \in [0,\, t_1]^2$, $E_\tau$ is a self-adjoint Hilbert-Schmidt operator on $\mathscr{U}_\tau \times \mathscr{V}_\tau$.

Define an open-loop resolvent operators $F_\tau \in \mathscr{L}(\mathscr{U}_\tau \times \mathscr{V}_\tau)$ by

$$F_\tau = \tilde{I}_\tau - (\tilde{I}_\tau + E_\tau)^{-1}, \tag{4.4}$$

where

$$\tilde{I}_\tau = \begin{pmatrix} I_{\mathscr{U}_\tau} & \\ & -I_{\mathscr{V}_\tau} \end{pmatrix},$$

$I_{\mathscr{U}_\tau}$ and $I_{\mathscr{V}_\tau}$ are the identity operators on $\mathscr{U}_\tau$ and $\mathscr{V}_\tau$ respectively. As $\tilde{I}_\tau + E_\tau = H_\tau$ is invertible, (4.4) is well-defined.

By a similar approach shown in [4], § 2, we can prove the following result. Its proof is omitted here.

**Lemma 7.** *Under Hypotheses 1 and 2, $F_\tau$ admits the following expression*

$$\left[F_\tau \begin{pmatrix} u \\ v \end{pmatrix}\right](t) = \int_\tau^{t_1} F_\tau(t,\,\sigma)\begin{pmatrix} u(\sigma) \\ v(\sigma) \end{pmatrix} d\sigma,\ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathscr{U}_\tau \times \mathscr{V}_\tau, \tag{4.5}$$

*where the kernel $F_\tau(t,\,\sigma)$ is a unique solution of linear matrix integral equation*

$$\begin{aligned} F_\tau(t,\,\sigma) &= \tilde{I}E(t,\,\sigma)\tilde{I} - \int_\tau^{t_1} \tilde{I}E(t,\,\eta)F_\tau(\eta,\,\sigma)d\eta \\ &= \tilde{I}E(t,\sigma)\tilde{I} - \int_\tau^{t_1} F_\tau(t,\,\eta)E(\eta,\,\sigma)\tilde{I}d\eta,\ (t,\,\sigma) \in [\tau,\, t_1]^2, \end{aligned} \tag{4.6}$$

*and* $F_\tau(t, \sigma) = F_\tau^*(\sigma, t)$ *is continuous with respect to* $(\tau, t, \sigma)$ *on*

$$\Delta = \{(\tau, t, \sigma) \mid 0 \leqslant \tau \leqslant t_1,\ \tau \leqslant t \leqslant t_1,\ \tau \leqslant \sigma \leqslant t_1\}.$$

The following result indicates that the unique solution of the Riccati equation (2.8) can be expressed by means of the resolvent kernel $F_\tau(t, \sigma)$.

**Theorem 3.** *Under Hypotheses* 1 *and* 2, *the unique strongly continuous and self-adjoint solution of the Riccati equation* (2.8) *is given by*

$$\begin{aligned}
P(t) &= T^*(t_1-t)QT(t_1-t) + \int_t^{t_1} T^*(\sigma-t)WT(\sigma-t)d\sigma \\
&\quad - \int_t^{t_1} T^*(t_1-t)QT(t_1-\xi)(BB^*-CC^*)T^*(t_1-\xi)QT(t_1-t)d\xi \\
&\quad - \int_t^{t_1}\int_t^{\sigma} T^*(\sigma-t)WT(\sigma-\xi)(BB^*-CC^*)T^*(t_1-\xi)QT(t_1-t)d\xi\, d\sigma \\
&\quad - \int_t^{t_1}\int_t^{\sigma} T^*(t_1-t)QT(t_1-\xi)(BB^*-CC^*)T^*(\sigma-\xi)WT(\sigma-t)d\xi\, d\sigma \\
&\quad + \int_t^{t_1}\int_t^{t_1} T^*(t_1-t)QT(t_1-\xi)(B,\ C)\ F_t(\xi, \eta)\begin{pmatrix} B^* \\ C^* \end{pmatrix} T^*(t_1-\eta)QT(t_1-t)d\eta\, d\xi \\
&\quad - \int_t^{t_1}\int_t^{\sigma}\int_\xi^{t_1} T^*(\sigma-t)WT(\sigma-\xi)(BB^*-CC^*)T^*(\eta-\xi)WT(\eta-t)d\eta\, d\xi\, d\sigma \\
&\quad + \int_t^{t_1}\int_t^{t_1}\int_t^{\sigma} T^*(\sigma-t)WT(\sigma-\xi)(B,\ C)F_t(\xi, \eta) \\
&\qquad \times \begin{pmatrix} B^* \\ C^* \end{pmatrix} T^*(t_1-\eta)QT(t_1-t)d\xi\, d\eta\, d\sigma \\
&\quad + \int_t^{t_1}\int_t^{t_1}\int_t^{\sigma} T^*(t_1-t)QT(t_1-\eta)(B,\ C)F_t(\eta, \xi) \\
&\qquad \times \begin{pmatrix} B^* \\ C^* \end{pmatrix} T^*(\sigma-\xi)WT(\sigma-t)d\xi\, d\eta\, d\sigma \\
&\quad + \int_t^{t_1}\int_t^{\sigma}\int_t^{t_1}\int_t^{\eta} T^*(\sigma-t)WT(\sigma-\xi)(B,\ C)\ F_t\ (\xi,\rho) \\
&\qquad \times \begin{pmatrix} B^* \\ C^* \end{pmatrix} T^*(\eta-\rho)WT(\eta-t)d\rho\, d\eta\, d\xi\, d\sigma,\quad t \in [0,\ t_1],
\end{aligned} \tag{4.7}$$

*where* $F_\tau(t, \sigma)$ *is the unique continuous solution of the equation* (4.6).

*Proof* From (3.3), (3.5), (4.1), (4.4) and (4.5) we obtain

$$\begin{aligned}
\begin{pmatrix} M_\tau(t) \\ N_\tau(t) \end{pmatrix} &= -\tilde{I}_\tau \begin{pmatrix} B^* \\ C^* \end{pmatrix} \left(T^*(t_1-t)QT(t_1-\tau) + \int_t^{t_1} T^*(\sigma-t)WT(\sigma-\tau)d\sigma\right) \\
&\quad + \int_\tau^{t_1} F_\tau(t, \sigma) \begin{pmatrix} B^* \\ C^* \end{pmatrix} \Big(T^*(t_1-\sigma)QT(t_1-\tau) \\
&\quad + \int_\sigma^{t_1} T^*(\eta-\sigma)WT(\eta\tau)d\eta\Big)d\sigma.
\end{aligned} \tag{4.8}$$

Substituting (4.8) into (3.16) we have

$$G(t, \tau) = T(t-\tau) + \int_\tau^t T(t-s)(B,\ C)\begin{pmatrix} M_\tau(s) \\ N_\tau(s) \end{pmatrix} ds$$

$$=T(t-\tau)-\int_\tau^t T(t-s)(B,\ C)\begin{pmatrix} B^* \\ -C^* \end{pmatrix}(T^*(t_1-s)QT(t_1-\tau)$$

$$+\int_s^{t_1} T^*(\sigma-s)WT(\sigma-\tau)d\sigma)ds$$

$$+\int_\tau^t T(t-s)(B,\ C)\int_\tau^{t_1} F_\tau(s,\ \sigma)\begin{pmatrix} B^* \\ C^* \end{pmatrix}(T^*(t_1-\sigma)QT(t_1-\tau)$$

$$+\int_\sigma^{t_1} T^*(\eta-\sigma)WT(\eta-\tau)d\eta)d\sigma\, ds. \tag{4.9}$$

Then substituting (4.9) into (3.17), after rearrangement, we obtain finally (4.7).

**Theorem 4.** (Closed-loop Theorem II) *Under Hypotheses* 1 *and* 2, $(u_*,\ v_*)$ *is the optimal strategy of (GP) if and only if it is the linear state feedback given by*

$$\begin{pmatrix} u_*(t) \\ v_*(t) \end{pmatrix}=\left[-\Lambda(t,\ t)+\tilde{I}\int_t^{t_1} F_t(t,\ \xi)\Lambda(\xi,\ t)\ d\xi\right]x_*(t),\ t\in[0,\ t_1], \tag{4.10}$$

*where* $\tilde{I}=\begin{pmatrix} I_U & \\ & -I_V \end{pmatrix}$, *and*

$$\Lambda(t,\ \sigma)=\begin{pmatrix} B^* \\ C^* \end{pmatrix}\left[T^*(t_1-t)QT(t_1-\sigma)+\int_{\max(t,\sigma)}^{t_1} T^*(\xi-t)WT(\xi-\sigma)d\xi\right],$$

$$(t,\ \sigma)\in[0,\ t_1]^2. \tag{4.11}$$

*Proof* According to Theorems 2 and 3, it remains only to show that the result of left multiplication of (4.7) by $-\begin{pmatrix} B^* \\ C^* \end{pmatrix}$ is no other than the feedback operator shown in (4.10). In fact, such a result contains ten terms where

i) the sum of terms 1 and 2 is equal to $-\Lambda(t,\ t)$;

ii) the sum of terms 3, 4, 6 and 8 is equal to

$$\tilde{I}\int_t^{t_1} F_t(t,\ \xi)\begin{pmatrix} B^* \\ C^* \end{pmatrix}T^*(t_1-\xi)QT(t_1-t)d\xi;$$

and iii) the sum of terms 5, 7, 9 and 10 is equal to

$$\tilde{I}\int_t^{t_1} F_t(t,\ \xi)\begin{pmatrix} B^* \\ C^* \end{pmatrix}\int_\xi^{t_1} T^*(\rho-\xi)WT(\rho-t)d\rho\, d\xi.$$

Thus we have proved the conclusion of this Theorem.

## References


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