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COMPLETE SUBMANIFOLDS IN E^{n+p} WITH PARALLEL MEAN CURVATURE*

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Abstract

This paper gives a pinching condition by which complete submanifolds in a Euclidean space can be classified completely. In particular, a characterization for a complete submanifold in a Euclidean space to be totally umbilical is shown.

§ 1. Introduction

Let E^{n+p} denote a Euclidean $(n+p)$-space. Okumura, M.[1-2] characterized under certain conditions a totally umbilical submanifold of E^{n+p} by an inequality between the length of the second fundamental form and the mean curvature of the submanifold. Later, by Yau's maximum principle, Goldberg, S. I.[3] improved the condition of [1]. Recently, Hasanis, Th. in [4] extended the results of [3].

In this paper, by generalizing Theorem 2 of [4] to a submanifold of any codimension, we improve the pinching condition of [2]. Our main results are as follows.

Theorem 1. *Let M be an $n(\geqslant 3)$-dimensional complete connected submanifold in E^{n+p} with parallel mean curvature. If the second fundamental form σ of M satisfies*

$$\|\sigma\|^2 \leqslant \|\mathrm{trace}\ \sigma\|^2/(n-1), \tag{1}$$

then M is an n-plane, an n-sphere, or a circular cylinder $S^{n-1}\times E^1$.

Theorem 2. *Let M^2 be a complete surface in E^{2+p} with parallel mean curvature. If the ineqality (1) is satisfied, then M^2 is a plane, a sphere, a circular cylinder $S^1\times E^1$ or a product of circles $S^1(r_1)\times S^1(r_2)$, where $\|\mathrm{trace}\ \sigma\|^2=(1/r_1)^2+(1/r_2)^2$.*

From Theorem 1 and Theoaem 2 we have the following

Corollary. *Let M be an $n(\geqslant 2)$-dimensional complete connected submanifold in E^{n+p} with parallel mean curvature. If the second fundamental form σ of M satisfies*

$$\|\sigma\|^2 < \|\mathrm{trace}\ \sigma\|^2/(n-1), \tag{2}$$

then M is a totally umbilical n-sphere.

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Remark. If $p=1$, Theorem 1 has been proved by Hasanis, Theorem[4]. The corollary may be viewed as a generalization of the classical theorem of Liebmann, H. and was obtained by Okumura, M.[2] under the additional conditions that M is compact and the connection of the normal bundle over M is flat.

§ 2. Formulas and Lemmas

Let M be an n-dimensional submanifold immersed in E^{n+p}. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}$ in E^{n+p} such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the field of dual frames and the connection 1-forms of E^{n+p}, respectively*. Restricting these forms to M, we have (cf. [5])

$$\omega_{\alpha i}=\sum_j h_{ij}^\alpha\omega_j,\ h_{ij}^\alpha=h_{ji}^\alpha,\ H^\alpha=(h_{ij}^\alpha), \tag{3}$$

$$d\omega_i=-\sum_j \omega_{ij}\wedge\omega_j,\ \omega_{ij}+\omega_{ji}=0,$$

$$d\omega_{ij}=-\sum_k \omega_{ik}\wedge\omega_{kj}+\frac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l, \tag{4}$$

$$R_{ijkl}=\sum_\alpha(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha), \tag{5}$$

$$d\omega_{\alpha\beta}=-\sum_\gamma\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}+\frac{1}{2}\sum_{j,k}R_{\alpha\beta jk}\omega_j\wedge\omega_k,$$

$$R_{\alpha\beta jk}=\sum_i(h_{ij}^\alpha h_{ik}^\beta-h_{ik}^\alpha h_{ij}^\beta). \tag{6}$$

The second fundamental form σ and the mean curvature ξ of M are

$$\sigma=\sum_{i,j,\alpha}h_{ij}^\alpha\omega_i\otimes\omega_j\otimes e_\alpha \tag{7}$$

and

$$\xi=(\text{trace }\sigma)/n=\frac{1}{n}\sum_\alpha(\text{tr }H^\alpha)e_\alpha, \tag{8}$$

respectively. From (5), (7) and (8) the scalar curvature R of M can be written as

$$R=n^2H^2-\|\sigma\|^2, \tag{9}$$

where

$$H^2=\|\text{trace }\sigma\|^2/n^2=\|\xi\|^2,\quad \|\sigma\|^2=\sum_\alpha\text{tr}(H^\alpha)^2. \tag{10}$$

If $\xi\neq0$, we can choose e_{n+1} in such a way that its direction coincides with that of ξ. Then

$$\text{tr }H^{n+1}=nH,\ \text{tr }H^\beta=0\ (\beta\neq n+1). \tag{11}$$

Putting

$$\mu=\sum_{i,j}(h_{ij}^{n+1}-H\delta_{ij})\omega_i\otimes\omega_j\otimes e_{n+1},\ \tau=\sum_{\substack{i,j\\ \beta\neq n+1}}h_{ij}^\beta\omega_i\otimes\omega_j\otimes e_\beta, \tag{12}$$

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* We use the following convention on the range of indices:
$1\leqslant A, B, \cdots\leqslant n+p;\ 1\leqslant i, j, k, \cdots\leqslant n;\ n+1\leqslant\alpha, \beta, \cdots\leqslant n+p.$

we have

$$\text{trace}\ \mu=0,\ \text{trace}\ \tau=0, \tag{13}$$

$$\|\mu\|^2=\text{tr}\ (H^{n+1})^2-nH^2,\ \|\tau\|^2=\sum_{\beta\neq n+1}\text{tr}\ (H^\beta)^2, \tag{14}$$

$$\|\sigma\|^2=\|\tau\|^2+\|\mu\|^2+nH^2, \tag{15}$$

from which it may be seen theat $\|\tau\|^2$ as well as $\|\mu\|^2$ is independent of the choice of the frame fields and is a function globally defined on M.

A submanifold M is said to be pseudo-umbilical if it is umbilical with respect to the direction of the mean curvature ξ, i. e., $h_{ij}^{n+1}=H\delta_{ij}$. From (11),(13) and (14) one can easily see that M is pseudo-umbilical iff $\|\mu\|^2=0$ and M is totally umbilical iff it is pseudo-umbilical and $\|\tau\|^2=0$.

Now assume that the mean curvature $\xi=He_{n+1}$ of M is parallel, i. e.,

$$\omega_{\beta,n+1}=0,\ H=\text{constant}. \tag{16}$$

As has been calculated in [5], we have

$$\frac{1}{2}\Delta(\|\mu\|^2)=\|D\mu\|^2+\sum_{i,j,k,l}h_{ij}^{n+1}(h_{kl}^{n+1}R_{lijk}+h_{li}^{n+1}R_{lkjk}), \tag{17}$$

and

$$\frac{1}{2}\Delta(\|\tau\|^2)=\|D\tau\|^2+\sum_{\substack{i,j,k,l\\ \beta\neq n+1}}h_{ij}^\beta(h_{kl}^\beta R_{lijk}+h_{li}^\beta R_{lkjk})+\sum_{\substack{i,j,k\\ \beta,\gamma\neq n+1}}h_{ij}^\beta h_{ik}^\gamma R_{\gamma\beta jk}, \tag{18}$$

where D denotes the generalized covariant differentiation and Δ the Laplacian.

The following lemma can be found in [6].

Lemma 1. *Let M be an n-dimensional submanifold in E^{n+p}. If*

$$(n-1)\|\sigma\|^2\leqslant\|\text{trace}\ \sigma\|^2\quad(\text{resp.}\ <),$$

then the sectional curvatures of M are $\geqslant 0$ (resp. >0).

The following generalized maximum principle is due to Yau, S. T.-Cheng, S. Y.-Motomiya, M.[7]

Lemma 2. *Let M be a complete connected Riemannian manifold with Ricci curvature bounded below, and f be a C^2-function bounded above on M and haev no maximum. Then for any $\varepsilon>0$, there exists a point $P\in M$ such that at P*

(i) $\sup f-\varepsilon<f(P)<\sup f-\varepsilon/2$,

(ii) $|\text{grad}\ f|(P)<\varepsilon$,

(iii) $\Delta f(P)<\varepsilon$.

We now establish our main lemma.

Lemma 3. *Let M be an $n(\geqslant 3)$-dimensional complete connected submanifold in E^{n+p} with nonzero parallel mean curvature. If the second fundamental form σ of M satisfies* (1), *then either M is pseudo-umbilical or $\|\mu\|^2=nH^2/(n-1)$ on M everywhere.*

Proof From the assumption and (16), it is easy to see that

$$H^{n+1}H^\beta=H^\beta H^{n+1}. \tag{19}$$

Substituting (5) into (17) and making use of (11) and (19), we have

$$\frac{1}{2}\Delta(\|\mu\|^2)=\|D\mu\|^2-[\mathrm{tr}(H^{n+1})^2]^2+(nH)\mathrm{tr}(H^{n+1})^3-\sum_{\beta\neq n+1}[\mathrm{tr}(H^\beta H^{n+1})]^2. \quad (20)$$

By Schwarz ineqality, from (11) and (14) it follows that

$$\sum_{\beta\neq n+1}[\mathrm{tr}(H^\beta H^{n+1})]^2=\sum_{\beta\neq n+1}[\sum_{i,j}h^\beta_{ij}(h^{n+1}_{ij}-H\delta_{ij})]^2\leqslant\|\mu\|^2\|\tau\|^2. \quad (21)$$

we repeat the same calculations as in [1] and from (20), (21), (15) and (1) we get

$$\frac{1}{2}\Delta(\|\mu\|^2)\geqslant\|D\mu\|^2+\|\mu\|^2\left(nH^2-\frac{n(n-2)}{\sqrt{n(n-1)}}|H|\cdot\|\mu\|-\|\mu\|^2-\|\tau\|^2\right)$$
$$\geqslant(n-2)\sqrt{nH^2/(n-1)}\|\mu\|^2\{\sqrt{nH^2/(n-1)}-\|\mu\|\}. \quad (22)$$

Since condition (1) implies

$$\|\mu\|^2\leqslant\|\sigma\|^2-nH^2\leqslant nH^2/(n-1) \quad (23)$$

and, by Lemma 1, the sectional curvatures of M are bounded below from O, we can apply Omori–Yau's maximum principle (cf. [3]), and from (22) and (23) conclude that either $\|\mu\|^2=0$, i. e., M is pseudo-umbilical, or

$$\sup\|\mu\|^2=nH^2/(n-1). \quad (24)$$

If $\|\mu\|^2$ attains its maximum on M, then by using Hopf's well-known theorem we see from (22) and (23) that $\|\mu\|^2=$constant and thus $\|\mu\|^2=nH^2/(n-1)$ on M everywhere.

Now assume that $\|\mu\|^2$ has no maximum on M. We prove that it is impossible. In fact, by Lemma 2, we have that, for any natural number ν, there exists a point $P_\nu\in M$ such that, by (24) and (22),

$$\frac{nH^2}{n-1}-\frac{1}{\nu}<\|\mu\|^2(P_\nu)<\frac{nH^2}{n-1}-\frac{1}{2\nu}$$

and

$$(n-2)\sqrt{nH^2/(n-1)}\|\mu\|^2(P_\nu)\{\sqrt{nH^2/(n-1)}-\|\mu\|(P_\nu)\}<\frac{1}{2\nu},$$

from which together with (24) we can find (cf. the proof of Theorem 1 in [4])

$$nH^2\leqslant 2(n-1)/(n-2). \quad (25)$$

Consider a homothetic transformation $\mathscr{A}$ in E^{n+p} which is defined by

$$\bar\omega_A=\rho\omega_A, \quad (26)$$

where ρ is a positive real number. Then, by the structure equations of E^{n+p}, it follows from (26) that

$$\bar\omega_{AB}=\omega_{AB}.$$

Thus, it is easy to see that the image $\bar M=\mathscr{A}(M)$ satisfies the same conditions as M and $n\bar H^2=nH^2/\rho^2$, where $\bar H$ is the corresponding quantity for $\bar M$. Then we must have, as (25) above,

$$nH^2=\rho^2n\bar H^2\leqslant 2\rho^2(n-1)/(n-2),$$

which is impossible for $\rho<\{n(n-2)H^2/2(n-1)\}^{1/2}$. This completes the proof of Lemma 3.

§ 3. Proofs of Theorems

The proof of Theorem 1 First of all, from (10) and (16) we see that

$$\|\text{trace }\sigma\|^2=n^2\|\xi\|^2=\text{constant}.$$

Thus, if $\xi=0$, then the inequality (1) implies $\|\sigma\|^2=0$ on M, i. e., M is a totally geodesic n-plane in E^{n+p}. So the theorem holds.

Now assume $\xi\neq 0$ on M. By Lemma 3 we separate two cases.

Case I. $\|\mu\|^2=0$ everywhere, i. e., M is pseudo-umbilical. Substituting (5) and (6) into (18) and noting that $h_{ij}^{n+1}=H\delta_{ij}$, we have

$$\frac{1}{2}\Delta(\|\tau\|^2)=\|D\tau\|^2+\sum_{\beta,\gamma\neq n+1}\{\text{tr}(H^\beta H^\gamma-H^\gamma H^\beta)^2-[\text{tr}(H^\beta H^\gamma)]^2\}+nH^2\|\tau\|^2$$
$$\geqslant\left(2-\frac{1}{p-1}\right)\|\tau\|^2\left(\frac{p-1}{2p-3}nH^2-\|\tau\|^2\right), \tag{27}$$

where the last inequality is from the following estimation (cf. [8])

$$\sum_{\beta,\gamma\neq n+1}\{\text{tr}(H^\beta H^\gamma-H^\gamma H^\beta)^2-[\text{tr}(H^\beta H^\gamma)]^2\}\geqslant-\left(2-\frac{1}{p-1}\right)\|\tau\|^4.$$

Condition (1) implies that $\|\tau\|^2(\leqslant\|\sigma\|^2)$ is bounded above and the sectional curvatures of M are bounded below (Lemma 1). Applying Omori-Yau's maximum principle[8], (27) gives rise to either $\|\tau\|^2=0$ or

$$\sup\|\tau\|^2\geqslant(p-1)nH^2/(2p-3). \tag{28}$$

On the other hand, by virtue of (15) and the fact that $\|\mu\|^2=0$, it follows from (1) that $\|\tau\|^2\leqslant nH^2/(n-1)$, which coutradicts (28) for $n\geqslant 3$. Hence, $\|\tau\|^2\equiv 0$ and M is a totally umbilical n-sphere in E^{n+p}.

Case II. $\|\mu\|^2=nH^2/(n-1)$ everywhere. In this case, from (15) and (1) we get

$$\|\tau\|^2+nH^2/(n-1)=\|\sigma\|^2-nH^2\leqslant nH^2/(n-1),$$

which implies $\|\tau\|^2=0$ on M everywhere. Hence, M is totally geodesic with respect to the subbundle $e_{n+2}\otimes\cdots\otimes e_{n+p}$ of the normal bundle of M. Since, from the assumption of the theorem, the subbundle $e_{n+2}\otimes\cdots\otimes e_{n+p}$ is parallel in the normal bundle, then, by Theorem 1 of [5], we conclude that M lies in a totally geodesic $(n+1)$-plane E^{n+1} of E^{n+p}, i. e., $M\hookrightarrow E^{n+1}\hookrightarrow E^{n+p}$, and e_{n+1} is just the normal vector to M in E^{n+1}. Thus, our theorem can be obtained from Theorem 2 of [4].

Therefore, Theorem 1 is proved completely.

The proof of Theorem 2 By Lemma 1, the ineqality (1) guarantees that the Gauss curvature of M^2 is nonnegative, so that it does not change sign. Thus, for $p=1$, our theorem is the direct consequence of the result of Klotz, T. and Osserman, R.[9]

We now assume $p>1$. As has been pointed out in the proof of Theorem 4 of [5],

M^2 lies in a totally geodesic 4-plane E^4 of E^{2+p}. Moreover, M^2 immersed in E^4 satisfies the same properties as in E^{2+p}. If M^2 is minimal, then condition (1) implies M^2 is totally geodesic. Hence, Theorem 2 can be shown immediately from Theorem 2 of [10].

Finally, the corollary is evident if once we note that, by Lemma 1, the ineqality (2) implies the sectional curvatures of M are positive.

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