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ON COHOMOLOGY OF INFINITESIMAL NEIGHBOURHOODS OF COMPLEX MANIFOLDS

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Abstract

In this paper, we introduce the concept of the (k, l)-th C^∞ infinitesimal neighbourhoods of complex manifold M and difine defferential modules $\mathscr{A}^{p,q}_{M,k-p,l-q}$ and $\mathscr{A}^{r}_{M,k,l}$ for the (k,l)-th C^∞ infinitesimal neighbourhoods. We prove some isomorphism theorems of cohomology and hyper cohomology concerning $\widetilde{\Omega}^p_{M,k-p}$ and $\Omega^p_{M,k-p}$, as follows

$$H^p(M, \widetilde{\Omega}^r_{M,k-r}) \approx H^p_{\bar\partial}(M, \mathscr{A}^{r,*}_{M,k-r,l-*}),$$
$$H^p(M, \mathscr{A}^*_{M,k,l}) \approx H^p_{DR}(M, \mathbf{C})$$

and for hyper cohomology

$$\mathbf{H}^p(M, \widetilde{\Omega}^*_{M,k-*}) \approx H^p(M, \mathbf{C}),$$
$$\mathbf{H}^p(M, \Omega^*_{M,k-*}) \approx H^p(M, \mathcal{O}_M).$$

The concept of the k-th infinitesimal neighbourhoods was introduced by A. Grothendieck[2], B. Malgrange[3] used the k-th infinitesimal neighbourhoods of C^∞ manifolds to Lie equation theory. In [4] we introduced two kinds of differential modules $\widetilde{\Omega}^p_{M,k-p}$ and $\Omega^p_{M,k-p}$ for the k-t h infinitesimal neighbourhood. In this paper we prove some isomorphism theorems of cohomology and hypercohomology concerning $\widetilde{\Omega}^p_{M,k-p}$ and $\Omega^p_{M,k-p}$ for complex manifold M. We introduce the concept of the ((k, l)-th C^∞ infinitesimal neighbourhood of complex manifold M and define the differential module of (p, q)-type $\widetilde{\mathscr{A}}^{p,q}_{M,k-p,l-q}$ for the (k, l)-th C^∞ infinitesimal neighbourhood. The main results are as follows

$$H^p(M, \widetilde{\Omega}^r_{M,k-r}) \approx H^p_{\bar\partial}(M, \widetilde{\mathscr{A}}^{r,*}_{M,k-r,1-*}).$$

This is similar to Dolbeault isomorphism theorem.

If $\widetilde{\mathscr{A}}^r_{M,k,l} = \bigoplus_{p+q=r} \widetilde{\mathscr{A}}^{p,q}_{M,k-p,l-q}$, the cohomology groups of the complex $\widetilde{\mathscr{A}}^*_{M,k,l}$ are

$$H^p(M, \widetilde{\mathscr{A}}^*_{M,k,l}) \approx H^p_{DR}(M, \mathbf{C}),$$

where $H^p_{DR}(M, \mathbf{C})$ is the p-th complex valued C^∞ DeRham cohomology group of M.

On hypercohomology, there are isomorphisms

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$$\mathbf{H}^p(M, \widetilde{\Omega}^*_{M,k-*}) \approx H^p(M, \mathbf{C}),$$
$$\mathbf{H}^p(M, \Omega^*_{M,k-*}) \approx H^p(M, \mathscr{O}_M).$$

All symbols and concepts concerning complex manifolds we use here follow those in [1].

Suppose M is an m-dimensional complex manifold. C^∞_M is the sheaf of germs of complex-valued C^∞ functions on M. $\alpha^{p,q}_M$ is the sheaf of germs of C^∞ forms of type (p, q) on M. $\mathscr{O}_M$ is the sheaf of germs of holomorphic functions on M. $\overline{\mathscr{O}}_M$ is the sheaf of germs of antiholomorphic functions on M. Ω^p_M is the sheaf of germs of usual holomorphic p-forms on M (it is denoted by $\overset{*}{\Omega}{}^p_M$ in [4]). $M^{(k)}$ is the holomorphic k-th infinitesimal neighbourhood of M. $\mathscr{O}_{M^{(k)}}$ is the strucure sheaf of $M^{(k)}$. $\widetilde{\Omega}^p_{M,k-p}$ and $\Omega^p_{M,k-p}$ are sheaves of germs of holomorphic p-forms defined in (4) for $M^{(k)}$.

At first we describe the holomorphic differential modules $\widetilde{\Omega}^p_{M,k-p}$ and $\Omega^p_{M,k-p}$ by somewhat different way so as to match the content of this paper. Let Δ: $M \to M\times M$ be the diagonal map and π_1 (or π_2): $M\times M \to M$ be the projection to the first (or second) factor of $M\times M$. π_1^* (or π_2^*): $\mathscr{O}_M \to \mathscr{O}_{M\times M}$ is the pullback by π_1 (or π_2). Let x's denote the points of the first factor of $M\times M$ and y's denote the points of the second factor of $M\times M$. If $y_i - x_i$, $i=1, \cdots, m$, in $M\times M$ correspond to the formal coordinates ξ_i, $i=1, \cdots, m$ of $J_{M,k}$ (see [4]), dx_i and $d(y_i - x_i)$, $i=1, \cdots, m$, in $M\times M$ correspond to $\tilde{d}x_i$ and $\tilde{d}\xi_i$, $i=1, \cdots, m$, in $\widetilde{\Omega}_{M,k-1}$ respectively, the projection p: $\mathscr{O}_{M\times M} \to \mathscr{O}_{M^{(k)}}$ induces the following commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{M\times M} & \xrightarrow{p} & \mathscr{O}_{M^{(k)}} \\ d\downarrow & & \downarrow d \\ \Omega_{M\times M} & \longrightarrow & \widetilde{\Omega}_{M,k-1}. \end{array}$$

Therefore $\widetilde{\Omega}_{M,k-1} = \Omega_{M\times M} \otimes_{\mathscr{O}_{M\times M}} \mathscr{O}_{M^{(k-1)}}|_{\Delta(M)}$. Similarly $\widetilde{\Omega}^p_{M,k-p} = \Omega^p_{M\times M} \otimes_{\mathscr{O}_{M\times M}} \mathscr{O}_{M^{(k-p)}}|_{\Delta(M)}$. We can consider that $\tilde{d}$ is induced by d. Similarly, let ${}_{\text{II}}d$ be the partial differential with respect to y on $M\times M$ and ${}_{\text{II}}\Omega_{M\times M}$ be the module of partial differentials with respect to y on M, i.e. ${}_{\text{II}}\Omega_{M\times M} = \sum_{i=1}^{m} \mathscr{O}_{M\times M} dy_i$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{M\times M} & \xrightarrow{p} & \mathscr{O}_{M^{(k)}} \\ d\downarrow & & \downarrow D \\ \Omega_{M\times M} & \longrightarrow & \Omega_{M,k-1}, \end{array}$$

$\Omega^p_{M,k-p} = {}_{\text{II}}\Omega^p_{M\times M} \otimes_{\mathscr{O}_{M\times M}} \mathscr{O}_{M^{(k-p)}}|_{\Delta(M)}$ and D is induced by d.

Let $\mathscr{I}'$ be the idea sheaf of $C^\infty_{M\times M}$ generated dy $\pi_1^* f - \pi_2^* f$, $f\in \mathscr{O}_M$ and $\mathscr{I}''$ be the idea sheaf of $C^\infty_{M\times M}$ generated by $\pi_1^* g - \pi_2^* g$, $g \in \overline{\mathscr{O}}_M$. We define

$$\mathscr{I}^{k+1,l+1} = (\mathscr{I}')^{k+1} + (\mathscr{I}'')^{l+1}$$

to be the idea sheaf of $C^\infty_{M\times M}$.

Definition 1. *The (k, l)-th C^∞ infinitesimal neighbourhood of the complex manifold M is the ringed space $M^{(k,l)}=(M, C^\infty_{M^{(k,l)}})$, whose underlying space is M and structure sheaf is*

$$C^\infty_{M^{(k,l)}}=C^\infty_{M\times M}/\mathscr{I}^{k+1,\,l+1}\big|_{\Delta(M)}.$$

If U is an open coordinate neighbourhood on M, $f(x, \bar{x}, y, \bar{y})\in C^\infty_{M\times M}(U\times U)$,

$$f(x, \bar{x}, y, \bar{y})\equiv\sum_{|\alpha|\leq k\ |\beta|\leq 1}\frac{1}{\alpha!\beta!}\frac{\partial^{|\alpha+\beta|}f(x, \bar{x}, y, \bar{y})}{\partial y^\alpha\partial\bar{y}^\beta}(y-x)^\alpha(\bar{y}-\bar{x})^\beta \bmod \mathscr{I}^{k+1,\,l+1}.$$

Hence

$$C^\infty_{M^{(k,l)}}(U)=\{\sum_{|\alpha|\leq k,\ |\beta|\leq 1}f_{\alpha\beta}(x, \bar{x})\,(y-x)^\alpha(\bar{y}-\bar{x})^\beta \bmod \mathscr{I}^{k+1,\,l+1}\}.$$

Definition 2. *Let*

$$\widetilde{\mathscr{A}}^{p,q}_{M,k,l}=a^{p,q}_{M\times M}\otimes_{\mathscr{O}_{M\times M}}C^\infty_{M^{(k,l)}}\big|_{\Delta(M)}.$$

$\widetilde{\mathscr{A}}^{p,q}_{M,k,l}$ is a $C^\infty_{M^{(k,l)}}$-module and

$$\widetilde{\mathscr{A}}^{p,q}_{M,k,l}(U)=\sum_{|I+J|=p,\ |K+L|=q}C^\infty_{M^{(k,l)}}(U)\,dx^{\wedge I}\wedge dy^{\wedge J}\wedge d\bar{x}^{\wedge K}\wedge d\bar{y}^{\wedge L}.$$

By $\partial: a^{p,q}_{M\times M}\to a^{p+1,q}_{M\times M}$ and $\bar{\partial}: a^{p,q}_{M\times M}\to a^{p,q+1}_{M\times M}$, it is clear that

$$\partial(\mathscr{I}^{k+1,\,l+1}a^{p,q}_{M\times M})\subset\mathscr{I}^{k,\,l+1}a^{p+1,q}_{M\times M}$$

and

$$\bar{\partial}(\mathscr{I}^{k+1,\,l+1}a^{p,q}_{M\times M})\subset\mathscr{I}^{k+1,\,l}a^{p,q+1}_{M\times M}.$$

Hence $\partial: a^{p,q}_{M\times M}\to a^{p+1,q}_{M\times M}$ and $\bar{\partial}: a^{p,q}_{M\times M}\to a^{p,q+1}_{M\times M}$ induce $\partial:\widetilde{\mathscr{A}}^{p,q}_{M,k,l}\to\widetilde{\mathscr{A}}^{p+1,q}_{M,k-1,l}$ and $\bar{\partial}:\widetilde{\mathscr{A}}^{p,q}_{M,k,l}\to\widetilde{\mathscr{A}}^{p,q+1}_{M,k,l-1}$ respectively. $d=\partial+\bar{\partial}: a^{p,q}_{M\times M}\to a^{p+1,q}_{M\times M}\oplus a^{p,q+1}_{M\times M}$ induces

$$d=\bar{\partial}+\partial:\widetilde{\mathscr{A}}^{p,q}_{M,k,l}\to\widetilde{\mathscr{A}}^{p+1,q}_{M,k-1,l}\oplus\widetilde{\mathscr{A}}^{p,q+1}_{M,k,l+1}.$$

The following diagrams are commutative

$$\begin{array}{ccc}
a^{p,q}_{M\times M} & \xrightarrow{\ \partial\ } & a^{p+1,q}_{M\times M}\\
\text{projection}\downarrow & & \downarrow\text{projection}\\
\widetilde{\mathscr{A}}^{p,q}_{M,k,l} & \xrightarrow{\ \partial\ } & \widetilde{\mathscr{A}}^{p+1,q}_{M,k-1,l},
\end{array}$$

$$\begin{array}{ccc}
a^{p,q}_{M\times M} & \xrightarrow{\ \bar{\partial}\ } & a^{p+q,1}_{M\times M}\\
\text{projection}\downarrow & & \downarrow\text{projection}\\
\widetilde{\mathscr{A}}^{p,q}_{M,k,l} & \xrightarrow{\ \bar{\partial}\ } & \widetilde{\mathscr{A}}^{p,q+1}_{M,k,l-1}
\end{array}$$

and

$$\begin{array}{ccc}
a^{p,q}_{M\times M} & \xrightarrow{\ d\ } & a^{p+1,q}_{M\times M}\oplus a^{p,q+1}_{M\times M}\\
\text{projection}\downarrow & & \downarrow\text{projection}\\
\widetilde{\mathscr{A}}^{p,q}_{M,k,l} & \xrightarrow{\ d\ } & \widetilde{\mathscr{A}}^{p+1,q}_{M,k-1,l}\oplus\widetilde{\mathscr{A}}^{p,q+1}_{M,k,l-1},
\end{array}$$

and the composition

$$\widetilde{\mathscr{A}}^{p,q}_{M,k,l}\xrightarrow{\ d\ }\widetilde{\mathscr{A}}^{p+1,q}_{M,k-1,l}\oplus\widetilde{\mathscr{A}}^{p,q+1}_{M,k,l-1}\xrightarrow{\ d\ }\widetilde{\mathscr{A}}^{p+2,q}_{M,k-2,l}\oplus\widetilde{\mathscr{A}}^{p+1,q+1}_{M,k-1,l-1}\oplus\widetilde{\mathscr{A}}^{p,q+2}_{M,k,l-2}$$

is zero.

We have a sheaf complex

$$\widetilde{\mathscr{A}}^{r,*}_{M,k-r,l-*}:\widetilde{\mathscr{A}}^{r,0}_{M,k-r,l}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\widetilde{\mathscr{A}}^{r,p}_{M,k-r,l-p}\xrightarrow{\bar\partial}\widetilde{\mathscr{A}}^{r,p+1}_{M,k-r,l-p-1}\xrightarrow{\bar\partial}\cdots.$$

Theorem 1. *There are isomorphisms of cohomology groups*

$$H^p(M, \tilde{\Omega}^r_{M,k-r})\approx H^p_{\bar\partial}(M, \widetilde{\mathscr{A}}^{r,*}_{M,k-r,l-*}),\ p=0, 1, \cdots.$$

In order to prove the theorem we introduce the following sheaves and concepts.

Let $T'_{(x,y)}(M\times M)=\mathbf{C}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right)_{i,j=1,\cdots,m}$ be the holomorphic tangent space to $M\times M$ at (x, y) (see [1]) and

$$T''_{(x,y)}(M\times M)=\mathbf{C}\left(\frac{\partial}{\partial \bar{x}_i}, \frac{\partial}{\partial \bar{y}_j}\right)_{i,j=1,\cdots,m}$$

be the antiholomorphic tangent space to $M\times M$ at (x, y). $T'(M\times M)$ is the holomorphic tangent bundle on $M\times M$ and $T''(M\times M)$ is the antiholomorphic tangent bundle on $M\times M$. We consider the vector bundle $\Lambda^pT'(M\times M)\otimes\Lambda^qT''(M\times M)$ on $M\times M$. Let $\mathrm{Der}^{p,q}_{\mathscr{O}}(M\times M)$ be the sheaf of germs of C^∞ cross sections of $\Lambda^pT'(M\times M)\otimes\Lambda^qT''(M\times M)$. If U is an open coordinate neighbourhood on M,

$$\mathrm{Der}^{p,q}_{\mathscr{O}}(M\times M)(U\times U)=\sum_{\substack{|I+J|=p\\|K+L|=q}} C^\infty_{M\times M}(U)\left(\frac{\partial}{\partial x}\right)^{\wedge I}\wedge\left(\frac{\partial}{\partial y}\right)^{\wedge J}\wedge\left(\frac{\partial}{\partial \bar{x}}\right)^{\wedge K}\wedge\left(\frac{\partial}{\partial \bar{y}}\right)^{\wedge L}.$$

Definition 3. *Let*

$$\mathrm{Der}^{p,q}_{\mathscr{O},k,l}(M)=\mathrm{Der}^{p,q}_{\mathscr{O}}(M\times M)\otimes_{\mathscr{O}_{M\times M}}C^\infty_{M^{(k,l)}}|_{\Delta(M)}.$$

If $\langle\ ,\ \rangle:\mathrm{Der}^{p,q}_{\mathscr{O}}(M\times M)\times a^{p,q}_{M\times M}\to C^\infty_{M\times M}$ is the natural pairing,

$$\langle\mathrm{Der}^{p,q}_{\mathscr{O}}(M\times M), \mathscr{I}^{k+1,l+1}a^{p,q}_{M\times M}\rangle\subset\mathscr{I}^{k+1,l+1}$$

and

$$\langle\mathscr{I}^{k+1,l+1}\mathrm{Der}^{p,q}_{\mathscr{O}}(M\times M), a^{p,q}_{M\times M}\rangle\subset\mathscr{I}^{k+1,l+1}.$$

The pairing $\langle\ ,\ \rangle:\mathrm{Der}^{p,q}_{\mathscr{O}}(M\times M)\times a^{p,q}_{M\times M}\to C^\infty_{M\times M}$ induces the pairing

$$\langle\ ,\ \rangle:\mathrm{Der}^{p,q}_{\mathscr{O},k,l}(M)\times\widetilde{\mathscr{A}}^{p,q}_{M,k,l}\to C^\infty_{M^{(k,l)}}.$$

By the pairing $\widetilde{\mathscr{A}}^{p,q}_{M,k,l}$ is locally free $C^\infty_{M^{(k,l)}}$-module with locally free basis $dx^{\wedge I}\wedge dy^{\wedge J}\wedge d\bar{x}^{\wedge K}\wedge d\bar{y}^{\wedge L}$, $|I+J|=p$, $|K+L|=q$.

We consider the linear subspace ${}_{\mathrm{II}}\overset{*}{T}{}'_{(x,y)}(M\times M)=\mathbf{C}\{dy_i\}_{i=1,\cdots,m}$ of the holomorphic cotangent space to $M\times M$ at (x, y),

$$\overset{*}{T}{}'_{(x,y)}(M\times M)=C\{dx_i, dy_j\}_{i,j=1,\cdots,m}$$

and the linear subspace

$${}_{\mathrm{II}}\overset{*}{T}{}''_{(x,y)}(M\times M)=C\{d\bar{y}_i\}_{i=1,\cdots,m}$$

of the antiholomorphic cotangent space to $M\times M$ at (x, y)

$$\overset{*}{T}{}''_{(x,y)}(M\times M)=C\{d\bar{x}_i, d\bar{y}_j\}_{i,j=1,\cdots,m}.$$

They are invariant under locally holomorphic coordinate transformations on M. We get vector bundles

$${}_{\mathrm{II}}\overset{*}{T}{}'(M\times M)=\bigcup_{(x,y)\in M\times M}{}_{\mathrm{II}}\overset{*}{T}{}'_{(x,y)}(M\times M)$$

and

$${}_{\mathrm{II}}\overset{*}{T}{}''(M\times M)=\bigcup_{(x,y)\in(M\times M)}{}_{\mathrm{II}}\overset{*}{T}{}''_{(x,y)}(M\times M).$$

Let $\mathscr{B}^{p,q}_{M\times M}$ be the sheaf of germs of C^∞ cross sections of $\Lambda^p{}_{\mathrm{II}}\overset{*}{T}'(M\times M)\otimes\Lambda^q{}_{\mathrm{II}}\overset{*}{T}''(M\times M)$. Locally

$$\mathscr{B}^{p,q}_{M\times M}(U\times U)=\sum_{|I|=p,\ |J|=q}C^\infty_{M\times M}dy^{\wedge I}\wedge d\bar{y}^{\wedge J}.$$

The partial differential ${}_{\mathrm{II}}d$ with respect to y can be decomposed into ${}_{\mathrm{II}}d={}_{\mathrm{II}}\partial+{}_{\mathrm{II}}\bar{\partial}$, where

$${}_{\mathrm{II}}\partial f(x,\bar{x},y,\bar{y})=\sum_{i=1}^{m}\frac{\partial f}{\partial y_i}dy_i$$

and

$${}_{\mathrm{II}}\bar{\partial} f(x,\bar{x},y,\bar{y})=\sum_{i=1}^{m}\frac{\partial f}{\partial \bar{y}_i}d\bar{y}_i,$$

where $f(x,\bar{x},y,\bar{y})\in C^\infty_{M\times M}$. They are independent of holomorphic coordinate transformations on M.

Definition 4. *Let*

$$\mathscr{A}^{p,q}_{M,k,l}=\mathscr{B}^{p,q}_{M\times M}\otimes_{\mathscr{O}_{M\times M}}C^\infty_{M^{(k,l)}}\big|_{\Delta(M)}.$$

Locally

$$\mathscr{A}^{p,q}_{M,k,l}(U)=\sum_{|I|=p,\ |J|=q}C^\infty_{M^{(k,l)}}(U)Dy^{\wedge I}\wedge D\bar{y}^{\wedge J},$$

where $Dy=dy\otimes1\in\mathscr{A}^{1,0}_{M,k,l}(U)$ and $D\bar{y}=d\bar{y}\otimes1\in\mathscr{A}^{0,1}_{M,k,l}(U)$. By the following inclusion relations

$${}_{\mathrm{II}}\partial(\mathscr{I}^{k+1,l+1}\mathscr{B}^{p,q}_{M\times M})\subset\mathscr{I}^{k,l+1}\mathscr{B}^{p+1,q}_{M\times M}$$

and

$${}_{\mathrm{II}}\bar{\partial}(\mathscr{I}^{k+1,l+1}\mathscr{B}^{p,q}_{M\times M})\subset\mathscr{I}^{k+1,l+1}\mathscr{B}^{p,q+1}_{M\times M},$$

the partial differential ${}_{\mathrm{II}}d$: $\mathscr{B}^{p,q}_{M\times M}\to\mathscr{B}^{p+1,q}_{M\times M}\oplus\mathscr{B}^{p,q+1}_{M\times M}$ induces differential D: $\mathscr{A}^{p,q}_{M,k,l}\to\mathscr{A}^{p+1,q}_{M,k-1,l}\oplus\mathscr{A}^{p,q+1}_{M,k,l-1}$ and we have the following commutative diagram

$$\begin{array}{ccc}\mathscr{B}^{p,q}_{M\times M} & \xrightarrow{\ d\ } & \mathscr{B}^{p+1,q}_{M\times M}\oplus\mathscr{B}^{p,q+1}_{M\times M}\\ \text{projection}\downarrow & & \downarrow\text{projection}\\ \mathscr{A}^{p,q}_{M,k,l} & \xrightarrow{\ D\ } & \mathscr{A}^{p+1,q}_{M,k-1,l}\oplus\mathscr{A}^{p,q+1}_{M,k,l-1}.\end{array}$$

It is clear that $DD=0$.

Now we consider the linear subspace

$${}_{\mathrm{II}}T'_{(x,y)}(M\times M)=\mathbf{C}\left\{\frac{\partial}{\partial y_i}\right\}_{i=1,\cdots,m}$$

of the holomorphic tangent space to $M\times M$ at (x,y) $T'_{(x,y)}(M\times M)$ and the linear subspace

$${}_{\mathrm{II}}T''_{(x,y)}(M\times M)=\mathbf{C}\left\{\frac{\partial}{\partial \bar{y}_i}\right\}_{i=1,\cdots,m}$$

of the antiholomorphic tangent space to $M\times M$ at $(x,y)T''_{(x,y)}(M\times M)$. They are invariant under the local holomorphic coordinate transformations on M. We have vector bundles

$${}_{\mathrm{II}}T'(M\times M)=\bigcup_{(x,y)\in M\times M}{}_{\mathrm{II}}T'_{(x,y)}(M\times M)$$

and

$$ {}_{\mathrm{II}}T''(M\times M)=\bigcup_{(x,y)}{}_{\mathrm{II}}T''_{(x,y)}(M\times M). $$

Let $\mathrm{Der}^{p,q}_{C^\infty_M}(M\times M)$ be the sheaf of germs of C^∞ cross sections of $\Lambda^p{}_{\mathrm{II}}T'(M\times M)\otimes \Lambda^q{}_{\mathrm{II}}T''(M\times M)$. It is a $C^\infty_{M\times M}$-module and locally

$$ \mathrm{Der}^{p,q}_{C^\infty_M}(M\times M)(U\times U)=\sum_{|I|=p,|J|=q}C^\infty_{M\times M}(U\times U)\Big(\frac{\partial}{\partial y}\Big)^{\wedge I}\wedge\Big(\frac{\partial}{\partial \bar y}\Big)^{\wedge J}. $$

Definition 5. *Let*

$$ \mathrm{Der}^{p,q}_{C^\infty_M,k,l}(M)=\mathrm{Der}^{p,q}_{C^\infty_M}(M\times M)\otimes_{\mathcal{O}_{M\times N}}C^\infty_M(k,l)\,|_{\Delta(M)}. $$

There is a natural pairing $\langle\ ,\ \rangle$: $\mathrm{Der}^{p,q}_{C^\infty_M}(M\times M)\times\mathscr{B}^{p,q}_{M\times M}\to C^\infty_{M\times M}$ *and*

$$ \langle \mathrm{Der}^{p,q}_{C^\infty_M}(M\times M),\ \mathscr{I}^{k+1,l+1}\mathscr{B}^{p,q}_{M\times M}\rangle\subset\mathscr{I}^{k+1,l+1}, $$

$$ \langle \mathscr{I}^{k+1,l+1}\,\mathrm{Der}^{p,q}_{C^\infty_M}(M\times M),\ \mathscr{B}^{p,q}_{M\times M}\rangle\subset\mathscr{I}^{k+1,l+1}. $$

Therefore the pairing $\langle\ ,\ \rangle$: $\mathrm{Der}^{p,o}_{C^\infty_M}(M\times M)\times\mathscr{B}^{p,p}_{M\times M}\to C^\infty_{M\times M}$ induces the pairing

$$ \langle Der^{p,q}_{C^\infty_{M,k,l}}(M),\ \mathscr{A}^{p,q}_{M,k,l}\rangle\to C^\infty_{M(k,l)} $$

and $\mathscr{A}^{p,q}_{M,k,l}$ is a locally free $C^\infty_{M(k,l)}$-module with locally free basis $Dy^{\wedge I}\wedge D\bar y^{\wedge J}$, $|I|=p$, $|J|=q$.

Now we define the following filtration for the complex $(\widetilde{\mathscr{A}}^{r,*}_{M,k-r,l-*},\bar\partial)$.

$$ F_p(\widetilde{\mathscr{A}}^{r,q}_{M,k-r,l-q})=\mathrm{Ann}((\mathrm{Der}^{0,q-p+1}_{C^\infty_M,k-r,l-q}(M)\otimes\mathrm{Der}^{0,p-1}_{C,k-r,l-q}(M)), $$

where Ann means annihilator sheaf. Locally

$$ F_p(\widetilde{\mathscr{A}}^{r,q}_{M,k-q,l-r}(U))=\sum_{|K+L|=q,|r|\geq p}\widetilde{\mathscr{A}}^{r,0}_{M,k-r,l-q}(U)d\bar x^{\wedge k}\wedge d\bar y^{\wedge L}. $$

By the filtration we can compute the spectral sequence

$$ E_0^{p,q}=F_p\widetilde{\mathscr{A}}^{r,p+q}_{M,k-r,l-p-q}/F_{p+1}\widetilde{\mathscr{A}}^{r,p+q}_{M,k-r,l-p-p}. $$

Locally

$$ E_0^{p,q}(U)=\sum_{|I|=p,|J|=q}\widetilde{\mathscr{A}}^{r,0}_{M,k-r,l-p-q}(U)d\bar y^{\wedge I}\wedge d\bar x^{\wedge J} $$

and $d_0^{p,q}$: $E_0^{p,q}\to E_0^{p,q+1}$ is equal to ${}_{\mathrm{II}}\bar\partial$. We have

$$ E_1^{p,q}(U)=\begin{cases}\sum_{|I|=p}\widetilde{\mathscr{A}}^{r,0}_{M,k-r,0}(U)d\bar x^{\wedge I}, & q=0,\\ 0 & q\geq 1,\end{cases} $$

and $d_1^{p,q}$: $E_1^{p,q}\to E_1^{p+1,q}$ is equal to ${}_{\mathrm{I}}\bar\partial$, where

$$ {}_{\mathrm{I}}\bar\partial f(x,\bar x,y,\bar y)=\sum_{i=1}^m\frac{\partial f}{\partial\bar x_i}d\bar x_i. $$

We get sheaf exact sequence

$$ 0\to\widetilde\Omega^r_{M,k-r}\to E_1^{0,0}\xrightarrow{d_1^{0,0}}E_1^{1,0}\to\cdots. $$

Hence globally we get

$$ H^p(M,\widetilde\Omega^r_{M,k-r})=E_2^{p,0}(M)=E_\infty^{p,0}(M)=H^p_{\bar\partial}(M,\widetilde{\mathscr{A}}^{r,*}_{M,k-r,l-*}). $$

Theorem 1 has been proved.

We define sheaves

$$\widetilde{\mathscr{A}}^r_{M,k,l}=\bigoplus_{p+q=r}\widetilde{\mathscr{A}}^{p,q}_{M,k-p,q-l}.$$

We get a complex

$$\widetilde{\mathscr{A}}^*_{M,k,l}:\ \widetilde{\mathscr{A}}^0_{M,k,l}\to\cdots\to\widetilde{\mathscr{A}}^r_{M,k,l}\xrightarrow{d}\widetilde{\mathscr{A}}^{r+1}_{M,k,l}\to\cdots.$$

Theorem 2. *There are isomorphisms of cohomology groups*

$$H^p(M,\widetilde{\mathscr{A}}^*_{M,k,l})=H^p_{DR}(M,\mathbf{C}),\ p=0,1,\cdots,$$

where $H^p_{DR}(M,\mathbf{C})$ is the p-th complex-valued DeRham cohomology group of M.

We define a filtration for the complex $(\widetilde{\mathscr{A}}^*_{M,k,l},d)$

$$F_p(\widetilde{\mathscr{A}}^r_{M,k,l})=\mathrm{Ann}\Big(\bigoplus_{\substack{a+c=i\\ b+d=j\\ a+b=r-q+1\\ c+d=p-1}}\mathrm{Der}^{a,b}_{C^\infty_M,k-i,l-j}(M)\mathrm{Der}^{c,d}_{\mathrm{C},k-i,l-j}(M)\Big).$$

Locally

$$E^{p,q}_0=\Big\{\sum_{\substack{I+J+K+L=p+q\\ I+K=p}}\eta_{IJKL}(x,\bar{x},y,\bar{y})dx^{\wedge I}\wedge dy^{\wedge J}\wedge d\bar{x}^{\wedge K}\wedge d\bar{y}^{\wedge L}\Big\},$$

where $\eta_{IJKL}(x,\bar{x},y,\bar{y})\in C^\infty_{M(k-|I+J|,l-|K+L|)}$. The differential $d^{p,q}_0$: $E^{p,q}_0\to E^{p,q+1}_0$ is equal to ${}_{II}d={}_{II}\partial+{}_{II}\bar{\partial}$.

We have

$$E^{p,q}_1=\begin{cases}\sum_{a+b=p}a^{a,b}_M,\ q=0,\\ 0,\ q\geqslant 1,\end{cases}$$

and $d^{p,q}_1$: $E^{p,q}_1\to E^{p+1,q}_1$ is equal to d: $\sum_{a+b=p}a^{a,b}_M\to\sum_{c+d=p+1}a^{c,d}_M$. Globally we have

$$H^p(M,\widetilde{\mathscr{A}}^*_{M,k,l})=E^{p,q}_2(M)=E^{p,q}_\infty(M)=H^p_{DR}(M,\mathbf{C}).$$

Theorem 2 has been proved.

Corollary. *Poincare lemma holds for the complex $\widetilde{\mathscr{A}}^*_{M,k,l}$.*

Theorem 3. *The hypercohomology groups of the complex $\widetilde{\Omega}^*_{M,k-*}$ and $\widetilde{\Omega}^*_{M,k-*}$ are as follows:*

$$\mathbf{H}^*(M,\widetilde{\Omega}^*_{M,k-*})=\mathbf{H}^*(M,\Omega^*_M)=H^*(M,\mathbf{C}),$$

where $H^*(M,\mathbf{C})$ are cohomology groups with complex coefficients.

$$\mathbf{H}^*(M,\Omega^*_{M,k-*})=H^*(M,\mathscr{O}_M).$$

Let i be the composition $\Omega^p_M\xrightarrow{\pi_1^*}\Omega^p_{M\times M}\xrightarrow{\text{projection}}\Omega^p_{M,k-*}$. By (4) i's induce quasi-isomorphisms of complexes

$$\begin{array}{ccccccccc}\widetilde{\Omega}^*_{M,k-*}: & \mathscr{O}_{M(k)} & \to\cdots\to & \widetilde{\Omega}^p_{M,k-p} & \to & \widetilde{\Omega}^{p+1}_{M,k-p-1} & \to\cdots\\ \uparrow & \uparrow & & \uparrow & & \uparrow & \\ \Omega^*_M\ : & \mathscr{O}_M & \to\cdots\to & \Omega^p_M & \to & \Omega^{p+1}_M & \to\cdots\\ \uparrow & \uparrow & & \uparrow & & \uparrow & \\ \mathbf{C}^*\ : & \mathbf{C} & \to\cdots\to & 0 & \to & 0 & \to\cdots.\end{array}$$

Hence $\mathbf{H}^*(M,\Omega^*_{M,k-*})=\mathbf{H}^*(M,\Omega^*_M)=H^*(M,\mathbf{C})$.

i: $\mathscr{O}_M\to\mathscr{O}_{M(k)}$ induces quasi-isomorphism of complexes

$$\begin{array}{ccccccccc}\Omega^*_{M,k-*}: & \mathscr{O}_{M(k)} & \to\cdots\to & \Omega^p_{M,k-p} & \to & \Omega^{p+1}_{M,k-p-1} & \to\cdots\\ \uparrow & \uparrow & & \uparrow & & \uparrow & \\ \mathscr{O}^*_M\ : & \mathscr{O}_M & \to 0\to\cdots\to & 0 & \to & 0 & \to\cdots.\end{array}$$

Hence $\mathbf{H}^*(M,\Omega^*_{M,k-*})=H^*(M,\mathscr{O}_M)$.

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