Chin. Ann. of Math.
6B (3) 1985

ON HYPONORMAL WEIGHTED SHIFT (II)

SUN SHUNHUA (孙顺华) **

Abstract

It is shown in this paper that a hyponormal weighted shift of norm one is unitarily equivalent to a Toeplitz operator if and only if its weights $\{a_n\}_0^\infty$ satisfy $1-|a_n|^2=(1-|a_0|^2)^{n+1}$ $\forall n\geqslant 0$. In particular, this answers the Abrahamse's Problem 2. As a consequence, the (surprising) answer, obtained first by C. C. Cowen in an explicit form, to Halmos' Question 5 is recaptured.

This is a sequel of [1]. It is shown in this paper that the answer to the Abrahamse's Problem 2 (cf. [2]) is that the only class of the hyponormal weighted shift that is unitarily equivalent to a Toeplitz operator is the multiple of those weighted shifts whose weights $\{a_n\}_0^\infty$ satisfy $1-a_n^2=(1-a_0^2)^{n+1}$, $\forall n\geqslant 0$. As a consequence, we obtain the (surprising) answer to the Question 5 of Halmos given in [3] (cf. also [4]): there is a Toeplitz operator which is a subnormal weighted shift, but is neither normal nor analytic. C. C. Cowen is the first one to give the answer to the above Halmos' question in such a way[5].

The main result of this paper is as follows.

Theorem 1. *Let T be a hyponormal weighted (unilateral) shift on some Hilbert space $\mathscr{H}$ with weights $\{a_n\}_0^\infty$. Then T is unitarily equivalent to a Toeplitz operator if and only if $1-|a_n|^2=(1-|a_0|^2)^{n+1}$ $\forall n\geqslant 0$, where we assume*

$$\lim_{n\to\infty}|a_n|=1.$$

Proof By Theorem 3 of [1], it only remains to prove the "if" part.

For the weights $\{a_n\}_0^\infty$ satisfying the assumption of the theorem, without loss of generality we assume $0<a_n<1$ $\forall n\geqslant 0$ (cf. [1]). We define an operator Q on some Hilbert space $\mathscr{K}\supset\mathscr{H}$ with the following matrix representation (cf. [1] p. 107):
where $\hat{T}$ is the matrix representation of T with respect to T——shifted basis $\{e_n\}_0^\infty\subset\mathscr{H}$. The associated orthonormal basis in $\mathscr{K}$ for the representation (1) is denoted by $\{e_n\}_{-\infty}^\infty$ where span $\{e_n: n\leqslant -1\}=\mathscr{K}\ominus\mathscr{H}$. We identify $\mathscr{K}$ with $L^2(\Delta)$ and $\mathscr{H}$ with the Hardy space $H^2(\subset L^2(\Delta))$ (cf. [1] for definitions) in what follows, where Δ is the unit circle in the complex plane.

Manuscript received August 31, 1983.
*) This research was supported in part by a Foundation from Academy of Science of China.
** Department of Mathematics, Sichuan University, Sichuan, China.

$$Q\sim\begin{array}{cl}
\begin{array}{cccccccc} -(n+2) & \cdots & -2 & -1 & 0 & \cdots & n & \cdots\end{array} & \\
\left(\begin{array}{cccccccc}
a_n & \cdots & \cdots & 0 & & (1-a_n^2)^{1/2} & \cdots & \\
0 & & a_0 & \cdots & (1-a_0^2)^{1/2} & 0 & & \\
 & & & -1 & & & & \\
 & & -(1-a_0^2)^{1/2} & & a_0 & & & \\
-(1-a_n^2)^{1/2} & & & 0 & & & a_n\cdots &
\end{array}\right) & \begin{array}{l} -(n+1)\\ -1\\ 0\\ 1\\ n+1\end{array}
\end{array}$$

$$\equiv\begin{array}{cl}
\begin{array}{cc} -1 & 0\end{array} & \\
\left(\begin{array}{cc}
\hat{T}^* & -(1-a_0^2)^{1/2}\xi \\
\xi^* & \hat{T}
\end{array}\right) & \begin{array}{l} -1\\ 0\end{array}
\end{array} \tag{1}$$

Obviously, Q is unitary. Let

$$Q=\int_{|\zeta|=1}\zeta\,dE(\zeta)$$

be the spectral resolution of Q. Consider the curve Ω in the complex plane

$$\Omega\equiv\left\{\zeta+(1-a_0^2)^{1/2}\frac{1}{\zeta}:\ |\zeta|=1\right\}. \tag{2}$$

Ω is an ellipse. Let $\mathring{\Omega}$ be the interior domain bounded by Ω. Also, let F be the Riemann conformal mapping of $\mathring{\Omega}$ onto the unit disc with $F|_{z=0}=0$. Thus

$$|F\circ(\zeta+(1-a_0^2)^{1/2}\bar{\zeta})|=1,\ \forall|\zeta|=1.$$

By the Riesz functional calculus, we can define the operator

$$S\equiv F(Q_\rho)=\int_{|\zeta|=1}F\circ(\zeta+(1-a_0^2)^{1/2}\bar{\zeta})\,dE(\zeta), \tag{3}$$

where $Q_\rho=Q+(1-a_0^2)^{1/2}Q^*$. It is easy to verify that S is unitary.

It follows from (1) that $Q_\rho H^2\subset H^2$. Since F is analytic on $\mathring{\Omega}$, we have

$$SH^2=F(Q_\rho)H^2\subset H^2.$$

Moreover, it is easy to check that $\hat{S}\equiv S|_{H^2}$ is a completely non-unitary isometry with $\dim\{H^2\ominus\hat{S}H^2\}=1$. Therefore $\hat{S}\cong M_z$ (cf. [6], p. 30), the multiplication by z on H^2. Noticing that both S and Q_ρ have lower triangular-block matrix representation with respect to the decomposition $H^{2\perp}\oplus H^2=L^2(\Delta)$ and that S commutes with Q_ρ, we obtain immediately $\hat{S}\hat{Q}_\rho=\hat{Q}_\rho\hat{S}$, where $\hat{Q}_\rho=Q_\rho|_{H^2}$. This leads to

$\hat{Q}_\rho \cong M_{f(z)}$, the multiplication by $f(z)$ on H^2, where $f \in H^\infty$ (cf. [7], p. 73).

On the other hand, it follows directly from the definition of Q_ρ and the expression (1) that

$$\hat{Q}_\rho - (1-a_0^2)^{1/2}\hat{Q}_\rho^* = a_0^2 T. \tag{4}$$

As is just proved above, the left hand side of (4) is unitarily equivalent to the Toeplitz operator $T_f - (1-a_0^2)^{1/2}T_f^* = T_{f-(1-a_0^2)^{1/2}\bar{f}}$. Hence T is unitarily equivalent to a Toeplitz operator T_ψ, where

$$\psi = \frac{1}{a_0^2}(f - (1-a_0^2)^{1/2}\bar{f}).$$

This ocmpletes the proof of Thorem 1.

As a consequence we obtain the (surprising) answer to Question 5 of Halmos in [3].

Corollary 2[5]. *There is a Toeplitz operator which is a subnormal weighted shift, but is neither normal nor analytic.*

Proof Let $a_n^2 = 1 - \alpha^{n+1} (n \geqslant 0)$, where α is a constant satisfying $0 < \alpha < 1$. It is not difficult to verify that the sequence $\{\beta_n\}_0^\infty$ is a moment sequence of some probability measure on [0, 1], where

$$\beta_n = \prod_{l=0}^{n-1} a_l \quad (n > 0)$$

and $\beta_0 = 1$ (cf. [8], p. 107 for the solvability of the moment problem). Thus, by the Proposition 25 of [9], the weighted shift T with weights $\{a_n\}_0^\infty$ is subnormal. But Theorem 1 concludes that $T \cong T_\psi$, where $\psi = (f - (1-a_0^2)^{1/2}\bar{f})$ with $f \in H^\infty$ and $a_0^2 = 1 - \alpha$. Obviously, T_ψ is neither normal nor analytic (cf. [10]). This ends the proof.

References

[1] Sun Shunhua, On hyponormal weighted shift, *Chinese Annals of Mathematics*, **5B**: 1 (1984), 101—108.

[2] Abrahamse, M. B., Subnormal Toeplitz operators and function of bounded type, *Duke Mathematical J.*, **43** (1976), 597—604.

[3] Halmos P. R., Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76** (1970), 887—933.

[4] Halmos P.R., Ten years in Hilbert space, *Integral Equations and Operator Theory*, **2**: 4 (1979), 529—564.

[5] Cowen C. C., Personal communication, Aug., 1983.

[6] Никольский. Н. К., Лекции об операторе сдвига, Москва 1980.

[7] Halmos P. R., A Hilbert space problem book, Springer-Verlag, 1974.

[8] Yosida K., Functional analysis, Springer-Verlag, 1965.

[9] Shields A. L., Weighted shift operators and analytic function theory, Topics in Operotor Theory (edited by C. Pearcy) 1974, 49—128.

[10] Brown A., Halmos P. R., Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.*, **231** (1963), 89—102.