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THE AUTOMORPHISMS OF NON-DEFECTIVE ORTHOGONAL GROUPS IN CHARACTERISTIC 2*

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Abstract

Let V be a non-defective n-dimensional quadratic space over a field F of characteristic 2. In this paper we prove that, when $n\geqslant 6$ with $n\neq 8$ and $F\neq \mathbf{F}_2$, any automorphism of $\Omega(V)$ or $O'(V)$ has the standard type Φ_g, sending σ to $g\sigma g^{-1}$, where g is a semilinear automorphism of V which preserves the quadratic structure. Therefore the automorphism groups Aut $\Omega(V)$ and Aut $O'(V)$ are isomorphic to $P\Gamma O(V)$. As a corollary, Aut $O(V)$ and Aut $O^+(V)$ are isomorphic to $P\Gamma O(V)$ as well.

Let V be a non-defective n-dimensional quadratic space over a field F of characteristic 2 with the quadratic form Q: $V\rightarrow F$ and associated symplectic form $(\ ,\ ): V\times V\rightarrow F$, $(x, y)=Q(x+y)+Q(x)+Q(y)$. The dimension of V is an even integer since V is non-defective. $O(V)$, $O^+(V)$, $O'(V)$ and $\Omega(V)$ are, respectively, the orthogonal group, the rotation group, the spinor subgroup and the commutator subgroup of the orthogonal group on V. E. A. Connors[4] showed that all automorphisms of these groups are of standard type Φ_g provided $n\geqslant 10$ and $F\neq\mathbf{F}_2$. In this paper we shall prove the result is also true when $n=6$, and provide a uniform proof for all $n\geqslant 6$ with $n\neq 8$ and $F\neq \mathbf{F}_2$.

I. Preliminaries

Through-out this paper, $F\neq\mathbf{F}_2$. We use Δ to denote either $\Omega(V)$ or $O'(V)$.

We assume familiarity with the theory of quadratic forms and orthogonal groups as treated in [1, 2, 3, 5, 8, 10]. We also assume familiarity with the residual space method as treated in [9].

Definition 1. *Let v be a non-zero vector in V, and let U be a non-zero subspace of V. We call v a singular vector (non-singular vector, resp.) if $Q(v)=0$ ($Q(v)\neq 0$, resp.). We call U non-defective (defective, totally defective, resp.) if $U\cap U^*=0$ ($U\cap U^*\neq 0$, $U\subseteq U^*$, resp.), where $U^*=\{x\in V\mid (x, U)=0\}$. U is degenerate if U is defective and there is a singular vector in $U\cap U^*$.*

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Convention Let $\sigma\in O(V)$. The letters P and R will always denote the fixed and residual spaces of σ, respectively. If Λ is an automorphism of Δ and $\sigma\in\Delta$, σ' will be used for $\Lambda(\sigma)$, and the fixed and residual spaces of σ' will be denoted by P' and R', respectively. Similarly, P_i and R_i refer to σ_i, P'_i and R'_i to σ'_i.

When R has some geometric property, we say that σ has the same property (e. g., non-defective, defective, totally defective, degenerate, etc.).

Suppose X is a subset of Δ. The centralizer of X in Δ will be denoted by $C(X)$.

The following results (Lemma 1.1—Lemma 1.7) are well-known. we omit the proof.

Lemma 1.1. *Suppose $n=2$. Then $\sigma\in O^+(V)$ is an involution if and only if $\sigma=1$. Let $\sigma\in O^+(V)$ and $\tau\in O^-(V)$. Then $\tau\sigma\tau^{-1}=\sigma^{-1}$. In particular, $O^+(V)$ is an Abelian group.*

Lemma 1.2. *Let $\sigma\in O(V)$ and $\sigma\neq 1$. Then $\sigma^2=1$ if and only if σ is totally defective. In particular, a plane rotation σ is non-defective if and only if $\sigma^2\neq 1$.*

Lemma 1.3. *Let R be a non-defective plane. The set of plane rotations with residual space R is $O^+(R)\perp l_p$ (exclude l). The set of plane rotations in Δ with residual space R is $\Omega(R)\perp l_p$ (exclude l).*

Lemma 1.4. *Every plane rotation in Δ is either non-defective or degenerate. If the Witt index $\nu=0$, every plane rotation in Δ is non-defective.*

Lemma 1.5. *Let R be a degenerate plane in V with $R=Fi\perp Fw$, $Q(i)=0$. Then the set of plane rotations with residual space R is $\{E_{i,\lambda w}\,|\,0\neq\lambda\in F\}$, where $E_{i,w}$ is the Eichler transformation, given by $F_{i,w}(x)=x+(x,w)i+(x,i)w+Q(w)(x,i)i$. Each $E_{i,w}$ is in $\Omega(V)$.*

Lemma 1.6. *If σ_1 and σ_2 are in $O(V)$ with σ_1 non-defective, then*

(a) *$\sigma_1\sigma_2=\sigma_2\sigma_1\Leftrightarrow\sigma_2R_1=R_1$ and $\sigma_2|_{R_1}$ permutes with $\sigma_1|_{R_1}$,*

(b) *$(R_1, R_2)=0\Rightarrow\sigma_1\sigma_2=\sigma_2\sigma_1$,*

(c) *$\sigma_1\sigma_2=\sigma_2\sigma_1$ and $R_1\cap R_2=0\Rightarrow(R_1, R_2)=0$.*

Lemma 1.7. *Let σ_1 and σ_2 be plane rotations. Then $\sigma_1\sigma_2$ is a plane rotation if and only if $R_1\cap R_2\neq 0$ and $\sigma_1\sigma_2\neq 1$.*

Lemma 1.8. *Suppose $n\geqslant 6$. Let $\sigma\in\Delta$ be a non-defective plane rotation. Then*

(a) *$C(\sigma)=(O^+(R)\perp O^+(P))\cap\Delta$,*

(b) *$CC(\sigma)=C\langle C(\sigma)^2\rangle=\Omega(R)\perp l$.*

Proof (a) Obviously $(O^+(R)\perp O^+(P))\cap\Delta\subseteq C(\sigma)\subseteq(O(R)\perp O(P))\cap\Delta$. For any $\rho\in C(\sigma)$, $\rho|_R$ permutes with $\sigma|_R$. But $\sigma|_R\in O^+(R)$, so $\rho|_R\in O^+(R)$ by Lemma 1.1. Hence $\rho|_P\in O^+(P)$, and so $C(\sigma)=(O^+(R)\perp O^+(P))\cap\Delta$.

(b) It follows from (a) that

$$O^+(R)\perp O^+(P)\supseteq C(\sigma)\supseteq\langle C(\sigma)^2\rangle\supseteq\langle\Omega(R)^2\rangle\perp\langle\Omega(P)^2\rangle.$$

Hence

$$\Omega(R)\perp l=(O^+(R)\perp l)\cap\Delta\supseteq C\langle C(\sigma)^2\rangle\supseteq CC(\sigma)\supseteq(O^+(R)\perp l)\cap\Delta=\Omega(R)\perp l,$$

since the centralizer of $\langle\Omega(P)^2\rangle$ in $O(P)$ is $\{l_P\}$.

Lemma 1.9.[4] *Suppose $n\geqslant 6$. $CC(E_{i,w})=\{E_{i,\lambda w}\mid\lambda\in F\}$.*

Lemma 1.10. *Suppose $n\geqslant 6$. Let σ_1 and σ_2 be plane rotations in Δ. Then*

$$R_1=R_2\Leftrightarrow CC(\sigma_1)=CC(\sigma_2)\Leftrightarrow C(\sigma_1)=C(\sigma_2).$$

Proof From Lemmas 1.8 and 1.9 we easily see that $R_1=R_2$ if and only if $CC(\sigma_1)=CC(\sigma_2)$. Since $CCC(X)=C(X)$ for any subset X in Δ, we immediately see that $CC(\sigma_1)=CC(\sigma_2)$ if and only if $C(\sigma_1)=C(\sigma_2)$.

Lemma 1.11. *Suppose $n\geqslant 6$. Let σ_1 and σ_2 be non-defective plane rotations in Δ. Then $(R_1, R_2)=0$ if and only if $\sigma_1\sigma_2=\sigma_2\sigma_1$ and $C(\sigma_1)\neq C(\sigma_2)$.*

Proof If $(R_1, R_2)=0$, then $\sigma_1\sigma_2=\sigma_2\sigma_1$ by Lemma 1.6, and $C(\sigma_1)\neq C(\sigma_2)$ by Lemma 1.10. Conversely, if $\sigma_1\sigma_2=\sigma_2\sigma_1$ and $C(\sigma_1)\neq C(\sigma_2)$, then $\sigma_2R_1=R_1$, $\sigma_2|_{R_1}\in O^+(R_1)$, and $R_1\neq R_2$ by Lemmas 1.6, 1.8, and 1.10. We have $(\sigma_2+1)R_1\subseteq R_1\cap R_2$ and $\dim R_1\cap R_2<\dim R_1$. So there is a non-zero x in R_1 such that $(\sigma_2+1)x=0$, which implies $\sigma_2|_{R_1}=1$. Hence $(R_1, R_2)=0$.

II. Action of Λ on the Non-Defective Plane Rotations

Definition 2. *Suppose Λ is an automorphism of Δ, and R is a plane in V. We say R behaves under Λ if there is a plane rotation $\sigma\in\Delta$ with the residual space R such that $\sigma'=\Lambda(\sigma)$ is also a plane rotation.*

The purpose of this section is to prove that all non-defective planes in V behave under Λ. The steps of proof are as follows. Suppose $\sigma\in\Delta$ is any non-defective plane rotation. First of all, we determine some restriction in res σ' and prove σ' is non-defective. Then we show if one non-defective plane behaves under Λ then all non-defective planes behave under Λ. Finally, we prove at least one non-defective plane behaves under Λ.

Proposition 2.1. *Suppose $n\geqslant 6$. Let R be a non-defective plane which behaves under Λ. Then $\Lambda(\Omega(R)\perp l)=\Omega(R')\perp l$, where R' is as shown in the above convention.*

Proof By definition there is a plane rotation $\sigma\in\Delta$ with the residual space R such that $\sigma'=\Lambda(\sigma)$ is also a plane rotation with the residual space R'. Since σ is non-defective, $\sigma^2\neq 1$, and so $\sigma'^2\neq 1$. Hence σ' is non-defective by Lemma 1.4. Using Lemma 1.8, we obtain $\Omega(R')\perp l=CC(\sigma')=\Lambda(CC(\sigma))=\Lambda(\Omega(R)\perp l)$.

Proposition 2.2. *Suppose $n\geqslant 6$. Let $\sigma\in\Delta$ be a non-defective plane rotation. If σ' is non-defective, then* res $\sigma'=2$ *or n.*

Proof We can prove res $\sigma'=2$, $n-2$ or n by an analogous procedure as in 2.14 of [9]. We now show res $\sigma'\neq n-2$. Otherwise suppose $\dim P'=2$.

By Lemma 1.8, $CC(\sigma)=C\langle C(\sigma)^2\rangle$. Applying Λ, we have $CC(\sigma')=C\langle C(\sigma')^2\rangle$.

Now $\langle C(\sigma')^2\rangle \subseteq \Omega(R')\perp\Omega(P')$ since $C(\sigma')\subseteq O(R')\perp O(P')$. So

$$CC(\sigma')=C\langle C(\sigma')^2\rangle \supseteq l\perp\Omega(P')$$

since $\dim P'=2$ and $\dim R'>2$.

Take $l\neq\rho'\in l\perp\Omega(P')$ and put $\rho=\Lambda^{-1}(\rho')$. Then $\rho'\in CC(\sigma')$ and $\rho\in CC(\sigma)=\Omega(R)\perp l$ by Lemmal. 8. Therefore $C(\rho)=C(\sigma)$, and so $C(\rho')=C(\sigma')$. Thus $\sigma'\in CC(\sigma')=CC(\rho')=l\perp\Omega(P')$, which implies $\sigma'=1$. This is a contradiction.

Proposition 2.3. *Suppose $n\geqslant 6$. Let $\sigma\in\Delta$ be a non-defective plane rotation Then $\sigma'=\Lambda(\sigma)$ is a non-defective rotation with residual index 2 or n.*

Proof It suffices to prove σ' is non-defective.

Suppose σ' is defective. Then res $\sigma'<n$, $R'\cap P'\neq 0$, and we have a splitting $R'=U\perp(R'\cap P')$ where U is a non-defective subspace. So $(\sigma'^2+1)V=(\sigma'+1)U$, and res $\sigma'^2<$res σ'. If σ'^2 is also defective, then res $\sigma'^4<$res σ'^2. Because dim V is finite and $\sigma'^{2^k}\neq 1$ for all integers $k\geqslant 0$, there is an integer j such that σ'^{2^j} is non-defective and res $\sigma'^{2^j}<n$. But σ^{2^j} is a non-defective plane rotation with the residual space R', and $\sigma'^{2^j}=\Lambda(\sigma^{2^j})$. So res $\sigma'^{2^j}=2$ by Proposition 2.2, and R behaves under Λ. Applying Proposition 2.1, both σ'^{2^j} and σ' are non-defective plane rotations. Hence σ' must not be defective.

Proposition 2.4. *Suppose $n\geqslant 6$. If there is a non-defective plane R which behaves under Λ, then all non-defective planes behave under Λ.*

Proof Take a non-defective plane rotation σ in Δ with the residual space R. Then $\sigma'=\Lambda(\sigma)$ is also a non-defective plane rotation with the residual space R'. Let π be any non-defective plane in V.

1) First suppose $(R,\pi)=0$. Write $V=R\perp P=R\perp\pi\perp U$ for some non-defective subspace U, and $V=R'\perp P'$. Take $\tau\in\Omega(\pi)\perp l$. Then $\tau\in C(\sigma)$, and $\tau'=\Lambda(\tau)\in C(\sigma')=(O^+(R')\perp O^+(P'))\cap\Delta$. Denote $\tau'=\tau_1'\perp\tau_2'$, where $\tau_1'\in O^+(R')$ and $\tau_2'\in O^+(P')$.

Take a non-singular vector x in R'. Define Φ: $\rho\mapsto\tau_x\rho\tau_x^{-1}$ for all ρ in Δ, where τ_x is the symmetry determined by x. Put $\Psi=\Lambda^{-1}\Phi\Lambda$. Clearly, both Φ and Ψ are automorphisms of Δ.

$$\begin{aligned}\Psi(\tau)&=\Lambda^{-1}\Phi\Lambda(\tau)=\Lambda^{-1}\Phi(\tau')=\Lambda^{-1}(\tau_x(\tau_1'\perp\tau_2')\tau_x^{-1})=\Lambda^{-1}(\tau_1'^{-1}\perp\tau_2')\\&=\Lambda^{-1}((\tau_1'^{-2}\perp l)\tau')=\Lambda^{-1}(\tau_1'^{-2}\perp l)\tau.\end{aligned}$$

Since $\tau_1'^{-2}\perp l\in\Omega(R')\perp l=CC(\sigma')$, we get $\Lambda^{-1}(\tau_1'^{-2}\perp l)\in CC(\sigma)=\Omega(R)\perp l$. Denote $\Lambda^{-1}(\tau_1'^{-2}\perp l)$ by $\theta\perp l$, where $\theta\in\Omega(R)$. So $\Psi(\tau)=(\theta\perp l_p)(\tau|_\pi\perp l_{\pi^*})=\theta\perp\tau|_\pi\perp l_U$. We see res $\Psi(\tau)\leqslant 4<n$. Applying Proposition 2.3 to the automorphism Ψ, we see $\Psi(\tau)$ is a non-defective plane rotation, which implies $\theta=1$, and so $\tau_1'^{-2}=1$. Hence $\tau_1'=1$ by Lemma 1.1. Thus $\tau'=l\perp\tau_2'$, and so res $\tau'<n$. Applying Proposition 2.3 again, we prove that τ' is a non-defective plane rotation. Therefore π behaves under Λ.

2) Now suppose $(R,\pi)\neq 0$. Write $\pi=Fx+Fy$. Without loss of generality we can assume $(R,x)\neq 0$. It is easy to see that there is a non-defective quaternary

subspace $T\supset R+Fx$, Since $n\geqslant 6$, dim $T^*\geqslant 2$. So, applying the result of step 1) twice, any non-defective plane in T behaves under Λ. In particular, we take $z\in R$ with $(z, x)\neq 0$, then $\pi_1=Fz+Fx$ is a non-defective plane in T, behaving under Λ. Choose a non-defective quaternary subspace $T_1\supset\pi_1+Fy$. As above, any non-defective plane in T_1 behaves under Λ. In particular, π behaves under Λ.

Proposition 2.5. *Suppose $n\geqslant 3$ with $n\neq 8$ and $\nu>0$. Then at least one non-defective plane behaves under Λ.*

Proof Since $\nu>0$, we can take a splitting $V=(Fi+Fj)\perp W$, where $Q(i)=Q(j)=0$ and $(i, j)=1$. So $i+j$ is a non-singular vector. Take $w\in W$ with $Q((w)\neq 0$. Put $\sigma=\tau_{i+j}\, E_{i,w}\tau_{i+j}^{-1}E_{i,w}^{-1}\in\Delta$, where τ_{i+i} is the symmetry determined by $i+j$. Now we have $E_{i,w}(i+j)=(1+Q(w))i+j+w$ and $(i+j,\ E_{i,w}(i+j))=Q(w)\neq 0$. Hence $\sigma=\tau_{i+j}\tau_{E_{i,w}}(i+j)$ is a non-defective plane rotation.

$E_{i,w}$ is an involution, so is $\Lambda(E_{i,w})$. Write $\rho=\Lambda(E_{i,w})$, and denote the residual space of ρ by π. Then $\pi\subseteq\pi^*$ and dim $\pi\leqslant\frac{n}{2}$.

1) If dim $\pi<\frac{n}{2}$, then $\sigma'=\Lambda(\sigma)=\Lambda(\tau_{i+j}E_{i,w}\tau_{i+j}^{-1}E_{i,w}^{-1})=\Lambda(E_{\sigma_{i+j}(i),\tau_{i+j}(w)}E_{i,w})=\Lambda(E_{\tau_{i+j}(i),\tau_{i+j}(w)})\rho$, and res $\sigma'<n$. Thus, res $\sigma'=2$ by proposition 2.3, and the residual plane of σ behaves under Λ.

2) We now assume dim $\pi=\frac{n}{2}$. Then $\pi=\pi^*$ and $\frac{n}{2}$ is an even integer, and so we can assume $n\geqslant 12$ Take any non-singular vector x in V, and put $y=x+\rho x\in\pi$. If $Q(y)\neq 0$, then $(x, \rho x)\neq 0$. put $\varphi=\tau_x\rho\tau_x^{-1}\rho^{-1}$ where τ_x is the symmetry determined by x. Then $\varphi=\tau_x\rho\tau_x^{-1}\rho^{-1}$ where τ_x is the symmetry determined by x. Then $\varphi=\tau_x\tau_{\rho x}$ is a non-defective plane rotation. But $\Lambda^{-1}\varphi=\Lambda^{-1}(\tau_x\rho\tau_x^{-1})E_{i,w}$, and res $\Lambda^{-1}\varphi\leqslant\frac{n}{2}+2<n$. By prorosition 2.3, $\Lambda^{-1}\varphi$ is a non-defective plane rotation, and the residual plane of $\Lambda^{-1}\varphi$ behaves under Λ.

If $Q(y)=0$, we can take $v\in(Fy)^*$ such that $v+\rho v\notin Fy$. Put $\psi=E_{y,v}\rho E_{y,v}^{-1}\rho^{-1}$. Then $\psi=E_{y,v}E_{\rho y,\rho v}=E_{y,v}E_{y,\rho v}=E_{y,v+\rho v}$, which is a degenerate plane rotation. We have res $\Lambda^{-1}\psi\leqslant 4<\frac{n}{2}$. Repeating step 1), we get through.

In order toprove at least one non-defective plane behaves under Λ in the case $\nu=0$, we need the concept of Cayley rotations.

Definition 3. *$\sigma\in O(V)$ is called a Cayley rotation on V if its minimal polynomial on V has the form $\lambda^2+\beta\lambda+1$ with $\beta\neq 0$. We eall β the residual trace of the Cayley rotation σ, and denote $\beta=$*res tr(σ).

We list some simple properties (Lemma 2.6—Lemma 2.9) of Cayley rotations. (The proof can be proceeded as in section 1B of [6] or 1D of [7].)

Lemma 2.6. *Suppose $n=2$. Then the set of Cayley rotations is $O^+(V)$ excluding*

1. If σ is a Cayley rotation on V and x is a non-singular vector, then $\{x, \sigma x\}$ is a basis of V. Given any non-zero β in F, there are at most two Cayley rotations in $O(V)$ with residual trace β. If σ is one of them, then σ^{-1} is the other.

Lemma 2.7. *Let $\sigma\in O(V)$ be a Cayley rotation on V. Suppose U is a σ-invariant non-defective subspace of V. Then $\sigma|_U$ is a Cuyley rotation on U, and*

$$\text{res tr } (\sigma|_U)=\text{res tr } (\sigma).$$

Lemma 2.8. *Suppose $\nu=0$. Then $\sigma\in O(V)$ is a Cayley rotation on V if and only if $Fx+F\sigma x$ is a σ-invariant non-defective plane for any non-zero vector x in V.*

Lemma 2.9. *Suppose $\nu=0$. Then $\sigma\in O(V)$ is a Cayley rotation on V if and only if*

(a) res $\sigma=n$,

(b) *there is a splitting $V=\pi_1\perp\cdots\perp\pi_{\frac{n}{2}}$ into non-defective planes π_i, each invariant under σ,*

(c) *$\sigma|_{\pi_i}$ is a Cayley rotation on π_i for each i,*

(d) res tr $(\sigma|_{\pi_i})=$ res tr $(\sigma|_{\pi_j})$ *for all i, j.*

In the case $\nu=0$, a plane need not be non-defective, but any plane rotation in Δ is non-defective by Lemma 1.4.

Proposition 2.10. *Suppose $n\geqslant 6$ and $\nu=0$.* Let *$\sigma\in\Delta$ be a plane rotation. If $\sigma'=\Lambda(\sigma)$ is not a Cayley rotation on V, then the non-defective plane R behaves under Λ.*

Proof By Proposition 2.3, res $\sigma'=2$ or n. If res $\sigma'=2$, then R behaves under Λ. If res $\sigma'=n$, then for any non-zero vector x in V $Fx+F\sigma'x$ is a plane since $\nu=0$. We have $0\neq Q(x+\sigma'x)=(x, \sigma'x)$, which implies the plane $Fx+F\sigma'x$ is non-defective. Since σ' is not a Cayley rotation on V, by Lemma 2.8, there is a non-zero vector y in V such that $Fy+F\sigma'y$ is not σ'-invariant. Take a plane rotation ρ in Δ with residual space $Fy+F\sigma'y$. Clearly $\rho\sigma'\neq\sigma'\rho$. The residual space of $\rho\sigma'\rho^{-1}\sigma'^{-1}$ lies in the ternary subspace $Fy+F\sigma'y+F\sigma'^2y$, and so $\rho\sigma'\rho^{-1}\sigma'^{-1}$ is a plane rotation. But $\Lambda^{-1}(\rho\sigma'\rho^{-1}\sigma'^{-1})=((\Lambda^{-1}\rho)\sigma(\Lambda^{-1}\rho)^{-1})\sigma^{-1}$, being the product of two plane rotations, has residual index at most 4. Thus, $\Lambda^{-1}(\rho\sigma'\rho^{-1}\sigma'^{-1})$ is also a plane rotation by Proposition 2.3, and the residual Plane of $\Lambda^{-1}(\rho\sigma'\rho^{-1}\sigma'^{-1})$ behaves under Λ. Therefore R behaves under Λ by proposition 2.4.

Now using the technique in 6.4 oj [7], we prove

Proposition 2.11. *Suppose $n\geqslant 6$ with $n\neq 8$ and $\nu=0$. Then at least one non-defective plane behaves under Λ.*

Proof Suppose, if possible, any non-defective plane does not behave under Λ. Then any non-defective plane does not behave unber Λ^{-1} either.

Take a plane rotation σ in Δ. By Proposition 2.10, σ' is a Cayley rotation on V. Using Lemma 2.9, we get a splitting $V=\pi_1\perp\cdots\perp\pi_k$ and $\sigma'=\Sigma_1\perp\cdots\perp\Sigma_k$, where each π_i is a σ'-invariant non-defective plane, $\Sigma_i=\sigma'|_{\pi_i}$, and $k=\frac{n}{2}$.

For each i, take a symmetry τ_i in $O(\pi_i)\perp l$. Define Φ_i: $\rho\mapsto\tau_i\rho\tau_i^{-1}$ for all ρ in Δ. Put $\Psi_i=\Lambda^{-1}\Phi_i\Lambda$. Then the Φ_i and Ψ_i are automorphisms of Δ.

Now $\Phi_i(\sigma')=\tau_i(\Sigma_1\perp\cdots\perp\Sigma_i\perp\cdots\perp\Sigma_k)\tau_i^{-1}=\Sigma_1\perp\cdots\perp\Sigma_i^{-1}\perp\cdots\perp\Sigma_k=(\Sigma_i^{-2}\perp l)\sigma'$. Denote $\Lambda^{-1}(\Sigma_i^{-2}\perp l)$ by T_i. Then T_i, T_i^2 and T_i^4 are Cayley rotations on V by the Hypothesis and Proposition 2.10. T_i permutes with σ since $\Sigma_i^{-2}\perp l$ permutes with σ'. So $T_i R=R$ for each i

Consider $\Psi_i(\sigma)=\Lambda^{-1}\Phi_i\Lambda(\sigma)=\Lambda^{-1}\Phi_i(\sigma')=\Lambda^{-1}((\Sigma_i^{-2}\perp l)\sigma')=T_i\sigma$. By Lemma 2.7, $T_i|_P$ is a Cayley rotation on P, and res $(T_i|_P)=n-2$. Applying Proposition 2.10 to the automorphism Ψ_i, we see $T_i\sigma$ is a Cayley rotation. By Lemma 2.7, res tr $(T_i)=$ res tr $(T_i|_P)=$ res tr $((T_i\sigma)|_P)=$ res tr$(T_i\sigma)$. Using Lemma 2.7 again, res tr $(T_i|_R)=$ res tr $(T_i)=$ res tr $(T_i\sigma)=$ res tr $((T_i\sigma)|_R)$, By Lemma 2.6, $(T_i\sigma)|_R=(T_i|_R)^{\pm1}$. Since $\sigma|_R\neq 1$, we obtain $(T_i\sigma)|_R=(T_i|_R)^{-1}$, and $T_i^2|_R=\sigma^{-1}|_R$ for each i. Hence the T_i^2 have the same residual trace.

Now consider the automorphism $\Psi_i\Psi_j$ for $1\leqslant i<j\leqslant k$.

$\Psi_i\Psi_j(\sigma^2)=\Lambda^{-1}\Phi_i\Phi_j(\sigma'^2)=\Lambda^{-1}((\Sigma_i^{-4}\perp l)(\Sigma_j^{-4}\perp l)\sigma'^2)=T_i^2T_j^2\sigma^2$. But $(T_i^2T_j^2\sigma^2)|_R=l$, and so $T_i^2T_j^2\sigma^2$ is a plane rotation by proposition 2.3. Let R_{ij} be the residual space of $\Psi_i\Psi_j(\sigma^2)$.

Take $R_1=R$ and write $V=R_1\perp U_1$. Since R is invariant under each T_i^2 and $T_i^2|_R=\sigma^{-1}|_R$, we can write

$$T_i^2=T_{11}\perp T_i^2|_{U_1}\quad \text{for } i=1,\cdots,k.$$

Consider $\Psi_1\Psi_2(\sigma^2)=T_1^2T_2^2\sigma^2$. Its residual plane $R_{12}\subseteq R^*=U_1$. Put $R_2=R_{12}$ and write $V=R_1\perp R_2\perp U_2$. T_i^2 permutes with $T_1^2T_2^2\sigma^2$ since $\Sigma_i^{-4}\perp l$ permutes with $\Sigma_1^{-4}\perp l$, $\Sigma_2^{-4}\perp l$ and σ'^2. So R_2 is invariant under each T_i^2. With respect to the splitting $V=R_1\perp R_2\perp U_2$, write $T_i^2=T_{11}\perp T_{i2}\perp T_i^2|_{U_2}$, where $T_{i2}=T_i^2|_{R_2}$, for each i.

Since res tr $(T_{i2})=$ res tr $(T_i^2)=$ res tr $(T_1^2)=$ res tr (T_{12}), we get $T_{i2}=T_{12}^{\pm1}$ by Lemma 2.6. In particular, since $T_1^2T_2^2\sigma^2$ has the residual space R_2, we must have $T_{22}=T_{12}$ and $T_2^2|_{U_2}=T_1^{-2}|_{U_2}$. For $i\geqslant3$, we can prove $T_{i2}=T_{12}^{-1}$. Now we write

$$T_1^2=T_{11}\perp T_{12}\perp T_1^2|_{U_2},$$
$$T_2^2=T_{11}\perp T_{12}\perp T_1^{-2}|_{U_2},$$
$$T_i^2=T_{11}\perp T_{12}^{-1}\perp T_i^2|_{U_2}\quad \text{for } i\geqslant3.$$

We now consider the plane rotation $T_1^2T_3^2\sigma^2$ and repeat the above process. Clearly, $(T_1^2T_3^2\sigma^2)|_{R_1\perp R_2}=l$, and so $R_{13}\subseteq U_2$. Put $R_3=R_{13}$ and write $V=R_1\perp R_2\perp R_3\perp U_3$ As above, R_3 is invariant under each T_i^2. With respect to this splitting of V, we have

$$T_1^2=T_{11}\perp T_{12}\perp T_{13}\perp T_1^2|_{U_3},$$
$$T_2^2=T_{11}\perp T_{12}\perp T_{13}^{-1}\perp T_1^{-2}|_{U_3},$$

$$T_3^2 = T_{11} \perp T_{12}^{-1} \perp T_{33} \perp T_3^2|_{U_3},$$
$$T_i^2 = T_{11} \perp T_{12}^{-1} \perp T_{i3} \perp T_i^2|_{U_3} \quad \text{for } i \geqslant 4.$$

As above, we can prove $T_{33} = T_{13}$ and $T_3^2|_{U_3} = T_1^{-2}|_{U_3}$.

If $n=6$, we have

$$T_2^2 = T_{11} \perp T_{12} \perp T_{13}^{-1},$$
$$T_3^2 = T_{11} \perp T_{12}^{-1} \perp T_{13},$$

with respect to the splitting $V = R_1 \perp R_2 \perp R_3$. Then $T_2^2 T_3^2 \sigma^2 = 1$. This contradicts the fact that $T_2^2 T_3^2 \sigma^2$ is a plane rotation.

If $n \geqslant 10$, then $\dim U_3 \geqslant 4$. We have

$$T_2^2 = T_{11} \perp T_{12} \perp T_{13}^{-1} \perp T_1^{-2}|_{U_3},$$
$$T_3^2 = T_{11} \perp T_{12}^{-1} \perp T_{13} \perp T_1^{-2}|_{U_3},$$

with respect to the splitting $V = R_1 \perp R_2 \perp R_3 \perp U_3$. Then $T_2^2 T_3^2 \sigma^2 = 1_{R_1 \perp R_2 \perp R_3} \perp T_1^{-4}|_{U_3}$. Since T_1^{-4} is a Cayley rotation on V, $T_1^{-4}|_{U_3}$ is a Cayley rotation on U_3, and so $\operatorname{res}(T_2^2 T_3^2 \sigma^2) = \operatorname{res}(T_1^{-4}|_{U_3}) \geqslant 4$. This is a contradiction.

Therefore, at least one non-defective plane behaves under Λ.

Theorem 2.12. *Suppose $n \geqslant 6$ with $n \neq 8$. All non-defective planes behave under Λ.*

Proof Apply propositions 2.4, 2.5 and 2.11.

III. The Automorphisms

In this section we prove any automorphism Λ of Δ has the standard type Φ_g provided $n \geqslant 6$ with $n \neq 8$, The steps of proof are as follow. First of all, we prove that Λ induces in a natural way a one-one correspondence of the set of the non-defective planes in V onto itself. Then Λ naturally induces a one-one correspondence of the set of lines in V onto itself. We show that the mapping preserves incidence between a line and a plane, and preserves the quadratic structure. Finally, using the Fundamental Theorem of Projective Geometry, we obtain the main theorem of this paper. As a corollary, we easily prove that any automorphism of $O(V)$ or $O^+(V)$ has the same type.

Suppose $n \geqslant 6$ with $n \neq 8$. Let R be a non-defective plane in V, and let σ in Δ be chosen with residual space R. Then R' is a non-defective plane by Theorem 2.12 and Proposition 2.3. Let R correspond to R'. Denote $R' = \Lambda R$. This map does not depend on the choice of σ by Lemma 1.10. It is injective by Lemma 1.10 again. It is surjective since Λ^{-1} is also an automorphism of Δ. So Λ is a bijection of the set of the non-defective planes in V onto itself.

Proposition 3.1. *Suppose $n \geqslant 6$ with $n \neq 8$. Let R_1 and R_2 be two non-defective planes in V. Then $(R_1, R_2) = 0 \Leftrightarrow (R_1', R_2') = 0$.*

Proof Apply Lemma 1.11.

Proposition 3.2. *Suppose $n\geqslant 6$ with $n\neq 8$. Let R_1 and R_2 be two non-defective planes in V. Then*

$$\dim(R_1\cap R_2)=1\Leftrightarrow\dim(R'_1\cap R'_2)=1.$$

Proof See 5.7 of [4]

Proposition 3.3. *Suppose $n\geqslant 6$ with $n\neq 8$. Let $\{R_\mu\}_{\mu\in I}$ be a family of non-defective planes in V. Then*

$$\dim\,(\bigcap_I R_\mu)=1\Leftrightarrow\dim\,(\bigcup_I R'_\mu)=1.$$

Proof It suffices to show that $\dim\,(\bigcap_I R_\mu(=1\Rightarrow\dim\,(\bigcap_I R'_\mu)=1$.

Without loss of generality we can assume card $I=3$. Let $R_1\cap R_2\cap R_3=Fx$ with R_1, R_2 and R_3 distinct.

1) First suppose $R_1+R_2+R_3=U$ is quaternary. If we can find a non-defective plane $\pi\subseteq(R_i+R_j)^*$, $\pi\nsubseteq(R_1+R_2+R_3)^*$, for some i, j, $1\leqslant i<j\leqslant 3$, then $(\pi, R_i)=(\pi, R_j)=0$ and $(\pi, R_k)\neq 0$, where k is the remaining index. Hence $(\pi', R'_i)=(\pi', R'_j)=0$ and $(\pi', R'_k)\neq 0$ Py proposition 3.1. Put $R'_i\supset R'_k=L_i$ and $R'_j\cap R'_k=L_j$, which are lines by Proposition 3.2. Then $L_i=L_j$, otherwise $(R'_k, \pi')=(L_i+L_j, \pi')=0$. Hence $R'_i\cap R'_j\cap R'_k$ is a line. Now we attempt to find such a π. Notice $\dim U\cap U^*=2$ or 0.

Suppose $\dim\,U\cap U^*=2$. We can write the ternary subspace $R_1+R_2=R_1\perp Fy$ for some $y\in U\cap U^*$. Since $\dim\,(R_1+R_2)^*\geqslant 3$ and $(R_1+R_2)\cap(R_1+R_2)^*=Fy$, there is a non-defective plane in $(R_1+R_2)^*$. It is easily seen that $(R_1+R_2)^*$ is spanned by non-defective planes. Since $(R_1+R_2)^*\nsupseteq(R_1+R_2+R_3)^*$, we can choose a non-defective plane π in $(R_1+R_2)^*$ which is not in $(R_1+R_2+R_3)^*$. Hence in this case we are done.

Suppose now $\dim\,U\cap U^*=0$, i. e., U is non-defective. Take a non-zero vector U in U^*. Write $R_1=Fx+Fz$. Then $(x, z)\neq 0$ since R_1 is non-defective. Put $R_4=Fx+F(z+u)$ which is a non-defective plane containing Fx. Now both $R_1+R_2+R_4$ and $R_1+R_3+R_4$ are quaternary and defective. Hence by the previous case, $R'_1\cap R'_2\cap R'_4=L'_3$ and $R'_1\cap R'_3\cap R'_4=L'_2$ are lines. Therefore $R'_1\cap R'_2=L'_3=R'_1\cap R'_4=L'_2=R'_1\cap R'_3$, and so $R'_1\cap R'_2\cap R'_3$ is a line.

2) Finally suppose U is ternary. We can choose a non-defective plane R_4 with $Fx\subset R_4\nsubseteq U$. Then $U+R_4=R_1+R_2+R_4=R_1+R_3+R_4$ is quaternary. By step 1), $R'_1\cap R'_2\cap R'_4$ and $R'_1\cap R'_3\cap R'_4$ are lines. Therefore $R'_1\cap R'_2=R'_1\cap R'_4=R'_1\cap R'_3$, and so $R'_1\cap R'_2\cap R'_3$ is a line.

Proposition 3.4. *Suppose $n\geqslant 6$ with $n\neq 8$. Let R_1 and R_2 be two non-defective planes. Then $R_1\cap R_2$ is a singular line if and only if $R'_1\cap R'_2$ is a singular line.*

Proof See 5.12 of [4]

Proposition 3.5. *Suppose $n \geqslant 4$. Let L_1 and L_2 be two distinct lines in V. Then there are non-defective planes R_1 and R_2 in V such that $L_1 \subset R_1$, $L_2 \subset R_2$ and $R_1 \cap R_2 = 0$. Moreover, if $(L_1, L_2) = 0$, we can choose R_1 and R_2 with $(R_1, R_2) = 0$.*

Proof See 7.2 of [4].

We now define a correspondence of the set of lines in V onto itself. Let L be a line in V. Then $L = \bigcap_I R_\mu$, where $\{R_\mu\}_{\mu \in I}$ is the set of all non-defective planes containing L. By Proposition 3.3, $\bigcap_I R'_\mu = L'$ is a line. Let L correspond to L'. Denote $L' = \Lambda L$. Using Propositions 3.5 and 3.2 and the fact that Λ^{-1} is also an automorphism of Δ, we easily see the correspondence is a bijection.

Proposition 3.6. *Suppose $n \geqslant 6$ with $n \neq 8$. Let R_1 and R_2 be non-defective planes in V, and let L, L_1 and L_2 be lines in V with $L_1 \neq L_2$. Then*

(a) $(L_1, L_2) = 0 \Leftrightarrow (L'_1, L'_2) = 0$.

(b) $L = R_1 \cap R_2 \Leftrightarrow L' = R'_1 \cap R'_2$.

(c) $L \subset L_1 + L_2 \Leftrightarrow L' \subset L'_1 + L'_2$.

Proof See 7.4 of [4].

Definition 4. *Let g be a semilinear automorphism of V with field automorphism μ. We say that g preserves Q (or g preserves the quadratic structure) if there is a non-zero α in F such that $Q(gx) = \alpha(Q(x))^\mu$ for all x in V. We say that g preserves orthogonality if $(x, y) = 0$ implies $(gx, gy) = 0$.*

Proposition 3.7. *Let g be a semilinear automorphism of V. Then g preserves Q if and only if g preserves orthogonality and $Q(gx) = 0$ whenever $Q(x) = 0$.*

Proof See 7.5 of [4].

Now consider the may Φ_g: $GL(V) \rightarrow GL(V)$, sending σ to $g\sigma g^{-1}$. This is an automorphism of $GL(V)$.

Proposition 3.8. *Suppose $n \geqslant 4$ and g is a semilinear automorphism of V. If g preserves Q, then Φ_g induces an automorphism of $O(V)$, $O^+(V)$, $O'(V)$, and $\Omega(V)$.*

Proof See 6.3 of [4].

Proposition 3.9 *Suppose $n \geqslant 6$ with $n \neq 8$. Let Λ be an autvmorphism of Δ. If $R' = R$ for all non-defective planes R, then $\Lambda = 1$.*

Proof Proceed as in 4.5 of [9].

Using Proposition 3.6 and the Fundamental Theorem of Projective Geometry, we see that there exists a semilinear automorphism g of V such that $L' = gL$. By Propositions 3.6 and 3.4, g preserves orthogonality and sends singular lines to singular lines, and so g preserves Q by Proposition 3.7. Then Φ_g induces an automorphism of Δ. Consider the automorphism $\Lambda\Phi_g^{-1}$ of Δ. Under this autorphism, $R' = R$ for all non-defective planes R in V. Therefore we obtain the main theorem of this paper.

Theorem 3.10. *Let V be a non-defective n-dimensional quadratic space over a field F of characteristic 2. Suppose $n\geqslant 6$ with $n\neq 8$ and $F\neq \mathbf{F}_2$. Let Λ be any automorphism of $\Omega(V)$ or $O'(V)$. Then Λ has the standard type $\Lambda=\Phi_g$, where g is a semilinear automorphism of V which preserves Q. Hence*

$$\mathrm{Aut}\ \Omega(V)\simeq\mathrm{Aut}\ O'(V)\simeq\Phi\Gamma O(V),$$

where $P\Gamma O(V)$ is the projective group of the semilinear automorphisms of V which preserve Q.

Finally, if Λ is an automorphism of $O(V)$ or $O^+(V)$, then Λ induces an automorphism of $\Omega(V)$, and $\Lambda|_{\Omega(V)}$ induces a bijection of the non-defebtive planes in V as above. We easily see that $\Lambda=1$ if and only if the associated correspondence of the non-defective planes is the identity. Thus, we have

Corollary 3.11. *Let V be a non-defective n-dimensional quadratic space over a field F of characteristic 2. Suppose $n\geqslant 6$ with $n\neq 8$ and $F\neq \mathbf{F}_2$. Then all automorphisms of $O(V)$ and $O^+(V)$ are of standard type Φ_g, and*

$$\mathrm{Aut}\ O(V)\simeq\mathrm{Aut}\ O^+(V)\simeq P\Gamma O(V).$$

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