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# ADDENDUM TO "THE STRUCTURES OF GROUPS OF ORDER $2^3p^2$"

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**Abstract**

The present paper is a complement to the paper [1]. It is proved that groups of order $2^37^2$ have 44 types.

In the paper [1] Zhang Yuanda has solved the structures of groups of order $2^3p^2$, where $p$ is a prime different from 3 and 7. We discuss the problem when $p=7$ in the present paper. We shall show the following

**Theorem.** *The groups of order* $2^3\,7^2$ *have* 44 *types.*

Recalling the article [1], we find out that the proof and the result are also valid when the Sylow $p$-subgroup is normal. In other words, we have the following

**Lemma.** *The groups of order* $2^3\,7^2$ *have* 42 *types, when their Sylow* 7-*subgroups are normal.*

Owing to this, we shall only consider groups of order $2^37^2$ without normal Sylow 7-subgroups.

Let $G$ be a group of order $2^3\,7^2$, $P$ the Sylow 7- subgroup of $G$ and $A$ the Sylow 2-subgroup of $G$. Moreover assume that $P$ is not normal in $G$. From the Sylow's theorem we conclude that $|G:N_G(P)|\equiv 1 \pmod 7$. Note that $P$ is not normal, $|G:N_G(P)|=8$, consequently $N_G(P)=P$.

Because of the commutativity of $P$, we have $N_G(P)=P=C_G(P)$. The argument of the Burnside's theorem[2] shows that $G$ has a normal 7-complement, whence $A$ is a normal subgroup.

Since $A$ is normal in $G$, and since the order $|A|$ of $A$ is prime to the order $|P|$ of $P$, the group $G=AP$ may be viewed as a semi-direct product, i. e., $G=P_f\circ A$, where $f$ is the mapping of P into Aut(A) such that each element of $P$ corresponds to the automorphism of $A$ induced by its conjugation. i. e., $f$: $x(\in P)\to(a\to a^x=x^{-1}ax,\ a\in A)\in \mathrm{Aut}(A)$.

Therefore

$$P/\ker f\cong S\leqslant \mathrm{Aut}(A).$$

However, $A$ has the following five possibilities: (i) cyclic group $Z_8$; (ii) abelian

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group $Z_4\times Z_2$; (iii) elementary abelian group $Z_2\times Z_2\times Z_2$; (iv) quaternion group $Q_8$; (v) dihedral group $D_8$. When $A$ lies in the case (i), (ii), (iv) and (v), $|\mathrm{Aut}(A)|$ is 4, 8, 24 and 8 respectively, so $(|\mathrm{Aut}(A)|, |P|)=1$, which means that $P$ acts on $A$ trivially. Then $G=AP$, which contradicts our assumption of non-normality of $P$. Hence $A$ has only one possibility (iii).

Suppose $A=\langle a\rangle\times\langle b\rangle\times\langle c\rangle$, viewed as a vector space over the field $Z_2$. Since we are not discussing the trivial action of $P$ on $A$, kern $f\neq P$. But $|\mathrm{Aut}(A)|=|GL(3, 2)|=7\times 3\times 2^3$, we also have kern $f\neq 1$. Then

$$P/\mathrm{kern} f\cong S\in \mathrm{Syl}_7\mathrm{Aut}(A),\ \mathrm{Aut}(A)=GL(3, 2).$$

Now let $M, N\in GL(3, 2)$, and $|M|=|N|=7$. By the above statement there exists a matrix $T\in GL(3, 2)$ such that $M=TNT^{-1}$. Here the minimal polynomial of $M$ is a factor of the polynomial $X^7+1=(X+1)(X^3+X^2+1)(X^3+X+1)$. However, the polynomials $X^3+X+1$ and $X^3+X^2+1$ are all irreducible over the field $Z_2$, hence the minimal polynomial must be $X^3+X+1$ or $X^3+X^2+1$. So $M$ is similar to the matrix $\begin{bmatrix}0&0&1\\1&0&1\\0&1&0\end{bmatrix}$ or $\begin{bmatrix}0&0&1\\1&0&1\\0&1&0\end{bmatrix}$. Therefore

$$\begin{bmatrix}0&0&1\\1&0&1\\0&1&0\end{bmatrix}=T\begin{bmatrix}0&0&1\\1&0&0\\0&1&1\end{bmatrix}^i T^{-1}$$

for some $T\in GL(3, 2)$ and for some $i$ prime to 7. As $A$ is a vector space over $Z_2$; and since $G=A\langle y\rangle=A\langle y^i\rangle\ (y^{49}=1)$ or $G=A(\langle x\rangle\times\langle y\rangle)=A(\langle x\rangle\times\langle y^i\rangle)$, where $\langle x\rangle=$ kern $f$, we know that in the case of $P$ being non-normal, there exist two types of $G$ in the following:

(i) $G=\langle a, b, c, y\rangle$, $a^2=b^2=c^2=y^{49}=[a, b]=[b, c]=[c, a]=1$.

$$\begin{cases}a^y=c,\\ b^y=ac,\\ c^y=b.\end{cases}$$

(ii) $G=\langle a, b, c, x, y\rangle$, $a^2=b^2=c^2=x^7=y^7=[a, b]=[b, c]=[c, a]=[x, y]=1$

$$\begin{cases}a^x=a,\\ b^x=b,\\ c^x=c,\end{cases}\quad \begin{cases}a^y=c,\\ b^y=ac,\\ c^y=b.\end{cases}$$

Obviously the Sylow 7-subgroups of $G$ in (i) and (ii) are not normal, and therefore by the LEMMA mentioned above we finish the proof of the theorem.

## References

[1] Zhang Yuanda (张远达), The structures of groups of order $2^3p^2$, *Chin. Ann. of Math.*, **4B**: 1 (1983), 77-93.

[2] Zhang Yuanda (张远达) 有限群构造(下) (The Structuyes of finite groups), 科学出版社, 北京 (1982).