Chin. Ann. of Math.
6B (4) 1985

ON THE TRANSFORMATIONS OF THE POTENTIALS, INTEGRABLE EVOLUTION EQUATIONS AND BÄCKLUND TRANSFORMATIONS

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Abstract

The generalization of the AKNS method, Calogero method and Konopelchenko method is given in three respects. First, the new fundamental relations associated with the matrix spectral problem and a new explicit expression related to the matrixes B and C which are contained in the transformations of the transition matrixes are obtained. Then the wide classes of the integrable evolution equations are conveniently derived without improperly assuming $B=C$. Finally, an important property of the operator L_A is showed, the conditions connected with the temporal half of the Bäcklund transformations and the new simple expressions of the integrals of motion are deduced.

§ 1. Introduction

The general structure of the nonlinear evolution equations integrable by the arbitrary order linear spectral problem was discussed by Li Yishen[1], Zhu Guocheng[2], Newell[3] and Konopelchenko[4,5]. In a series of papers by Konopelchenko, the generalization of the AKNS[6] method and Calogero[7] method to the matrix polynomial spectral problem of arbitrary order was given, both the integrable evolution equations and their Bäcklund transformations were described.

The purpose of the present work is to extend the theory in Refs. [1-7] in four respects. First, we shall present the new fundamental relations associated with the spectral problem. Using the relations, we obtain conveniently the transformations of the potentials connected with the spectral problem and give a new explicit expression related to the matrixes B and C which are contained in the transformations of the transition matrixes[4,5]. Secondly, by means of the fundamental relations it is easy to find the wide classes of the integrable evolution equations, particular case of which was obtained by not properly assuming $B=C$ in Refs. [4, 5]. Then, we show an important property of the operator L_A. Since only the spatialf half of the Bäcklund

Manuscript received August 4, 1983.

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transformations were obtained in Refs. [4, 5], we give the conditions connected with the temporal half. Using the conditions, we get new simple expressions of the integrals of motion. Finally, by the way, we point out that by virtue of the Li Yishen (one of the authors of the present paper) method[1, 8], the transformations of the potentials can be derived simply without any conditions imposed on the potential matrix at infinity.

For simplicity we consider the following linear matrix spectral problem

$$\frac{\partial \Psi}{\partial x}=(i\lambda A+iP(x,\ t,\ \cdots))\Psi, \tag{1.1}$$

where λ is the spectral parameter, A is the constant diagonal matrix, $A_{ik}=a_i\delta_{ik}$, $a_i\neq a_k$, $i,\ k=1,\ \cdots,\ N$; $P(x,\ t)$ is the potential matrix of order N and $P_{ii}=0$, $i=1,\ \cdots,\ N$. We assume that[4]

$$\lim_{|x|\to\infty} P(x,\ t)=0. \tag{1.2}$$

The method given in this paper can be generalized to other matrix spectral problem.

§ 2. The fundamental Relations and Their Applications

2.1. The fundamental ralations

Let us introduce in the usual way [4] the fundamental matrix solutions F^+ and F^- with asymptotics

$$F^+(x,\ \lambda,\ t)\sim\exp(i\lambda Ax)=E,\ x\to+\infty, \tag{2.1}$$

$$F^-(x,\ \lambda,\ t)\sim\exp(i\lambda Ax)=E,\ x\to-\infty, \tag{2.2}$$

and the transition matrix

$$S(\lambda,\ t):\qquad F^+(x,\ \lambda,\ t)=F^-(x,\ \lambda,\ t)S(\lambda,\ t). \tag{2.3}$$

Let P and P' be two potentials in (1.1) and Ψ and Ψ' two corresponding solutions of (1.1)

$$\frac{\partial \Psi'}{\partial x}=(i\lambda A+iP')\Psi', \tag{2.4}$$

$$\frac{\partial \Psi^{-1}}{\partial x}=-\Psi^{-1}(i\lambda A+iP). \tag{2.5}$$

For an arbitrary matrix function $G(x,\ \lambda,\ t)$ of order N (at least once differentiable), using

$$\int_{-\infty}^{\infty} dx\frac{\partial}{\partial x}[(F^+)^{-1}G(F^+)']=(F^+)^{-1}G(F^+)'\big|_{x=-\infty}^{+\infty}$$

and (2.1)—(2.5), it is not difficult to show that

$$i\int_{-\infty}^{\infty} dx(F^+)^{-1}(GP'-PG-iG_x+\lambda[G,\ A])(F^+)'$$
$$=E^{-1}G(+\infty,\ \lambda,\ t)E-S^{-1}E^{-1}G(-\infty,\ \lambda,\ t)ES'. \tag{2.6}$$

Now, we define matrixes of order $N, \Phi_D, \Phi_F, H_\alpha$ and Δ^{nj} as follows

$$(\Phi_D)_{ik}=\delta_{ik}(\Phi)_{ii}, \quad i, k=1, \cdots, N,$$
$$\Phi_F=\Phi-\Phi_D,$$
$$(H_\alpha)_{ik}=\delta_{\alpha i}\delta_{\alpha k}, \quad i, k, \alpha=1, \cdots, N.$$
$$(\Delta^{nj})_{ik}=\delta_{in}\delta_{kj}, \quad i, k, n, j=1, \cdots, N.$$

Let us now consider only those $G(x, \lambda, t)$ as follows

$$G=B(\lambda, t)=B_D(\lambda, t)=\sum_{\alpha=1}^{N} B_\alpha(\lambda, t)H_\alpha, \tag{2.7}$$

where $B_\alpha(\lambda, t)$ are arbitrary entire functions of λ.

Substituting (2, 7) into (2.6), we get

$$i\int_{-\infty}^{\infty} dx(F^+)^{-1}(BP'-PB)(F^+)'=B-S^{-1}BS'. \tag{2.8}$$

Rewriting equation (2.8) in components and designating

$$\tilde{\Phi}^{(in)}\overset{\text{df}}{=\!=}(F^+)'\Delta^{ni}(F^+)^{-1}, \tag{2.9}$$

we obtain

$$i\int_{-\infty}^{\infty} dx\mathrm{Tr}\Big(\sum_{\alpha=1}^{N}(H_\alpha P'-PH_\alpha)B_\alpha(\lambda, t)\tilde{\Phi}_F^{(in)}\Big)$$
$$=(B-S^{-1}BS')_{in}, \quad i, n=1, \cdots, N, \tag{2.10}$$

where Tr denotes the usual matrix trace.

From (2.1)—(2.5) and (2.9) it is easy to show that[4]

$$\Lambda_A\tilde{\Phi}_F^{(in)}=\lambda\tilde{\Phi}_F^{(in)}+\delta_{in}(P'H_n-H_nP)_A, \quad i, n=1, \cdots, N, \tag{2.11}$$

where $\Lambda_A\Phi_F=(\Lambda\Phi_F)_A$,

$$\Lambda\Phi_F=-i\frac{\partial\Phi_F}{\partial x}-(P'\Phi_F-\Phi_F P)_F+iP'\int_x^{\infty}dx(P'\Phi_F-\Phi_F P)_D$$
$$-i\Big(\int_x^{\infty}dx(P'\Phi_F-\Phi_F P)_D\Big)P.$$

$$(\Phi_{FA})_{ik}\overset{\text{df}}{=\!=}\frac{(\Phi_F)_{ik}}{a_i-a_k}, \quad i\neq k, (\Phi_{FA})_{kk}=0, \quad i, k=1, \cdots, N.$$

Repeatedly using (2.11), we have

$$\lambda^l\tilde{\Phi}_F^{(in)}=\Lambda_A^l\tilde{\Phi}_F^{(in)}+\tilde{\Lambda}_A^l\delta_{in}(H_nP-P'H_n)_A, \tag{2.12}$$

with $\tilde{\Lambda}_A^l\overset{\text{df}}{=\!=}\sum_{j=0}^{l-1}\lambda^j\Lambda_A^{l-1-j}$.

Substituting (2.12) into (2.10) gives

$$i\int_{-\infty}^{\infty} dx\mathrm{Tr}\Big(\sum_{\alpha=1}^{N}(H_\alpha P'-PH_\alpha)(B_\alpha(\Lambda_A, t)\tilde{\Phi}_F^{(in)}+\tilde{B}_\alpha(\Lambda_A, t)\delta_{in}(H_nP-P'H_n)_A)\Big)$$
$$=(B-S^{-1}CS')_{in}, \quad i, n=1, \cdots, N, \tag{2.13}$$

where

$$B_\alpha(\lambda, t)=\sum_{l=0}^{m}\beta_l(t)\lambda^l,$$
$$\tilde{B}_\alpha(\Lambda_A, t)=\sum_{l=0}^{m}\beta_l(t)\tilde{\Lambda}_A^l.$$

Finally, (2.13) can be written as follows

$$i\int_{-\infty}^{\infty} dx \mathrm{Tr}\Big(\widetilde{\Phi}_F^{(in)} \sum_{\alpha=1}^{N} B_\alpha(\Lambda_A^+, t)(H_\alpha P' - PH_\alpha) + \delta_{in}(H_n P - P'H_n)_A \sum_{\alpha=1}^{N} \widetilde{B}_\alpha(\Lambda_A^+, t)(H_\alpha P' - PH_\alpha)\Big) = (B - S^{-1}BS')_{in}, \quad i, n=1, \cdots, N, \tag{2.14}$$

where Λ^+ is the operator adjoint[4] to Λ with respect to the bilinear form

$$\langle \Phi, \Psi \rangle = \int_{-\infty}^{\infty} dx \mathrm{Tr}(\Phi_F(x)\Psi_F(x)),$$

$$\Lambda_A^+ \Phi_F \stackrel{\mathrm{df}}{=\!=} (\Lambda_A)^+ \Phi_F = -\Lambda^+ \Phi_{FA},$$

$$\Lambda^+\Phi_F = i\frac{\partial \Phi_F}{\partial x} - (\Phi_F P' - P\Phi_F)_F + i\Big(\int_{-\infty}^{x} dx(\Phi_F P' - P\Phi_F)_D\Big)P' - iP\int_{-\infty}^{x} dx(\Phi_F P' - P\Phi_F)_D. \tag{2.15}$$

Equations (2.14) are important fundamental relations connected with the spectral problem (1.1), which heve not been obtained in Refs. [4, 5]. These relations are basic for further discussion.

2.2. The transformations of the potentials and the transition matrices

From (2.14) follows that if P and P' satisfy

$$\sum_{\alpha=1}^{N} B_\alpha(\Lambda_A^+, t)(H_\alpha P' - PH_\alpha) = 0, \tag{2.16}$$

then

$$(B - S^{-1}BS')_F = 0. \tag{2.17}$$

Since B is a diagonal matrix, (2.17) means that $C = S^{-1}BS'$ is also a diagonal matrix.

Thus we find out that for any entire function of λ, $B_\alpha(\lambda, t)$, the transformation $P \to P'$ of type (2.16) leads to a simple transformation $S \to S'$:

$$S' = B^{-1}SC, \tag{2.18}$$

where the diagonal matrix C relates to the matrix B.

The transformations of potentials (2.16) and the transformations of transition matrices (2.18) which are connected with (1.1) were given in Ref. [4] in a slightly different way.

2.3. The expression of the matrix C

By virtue of (2.14), (2.16) we get

$$i\int_{-\infty}^{\infty} dx \mathrm{Tr}\Big((H_n P - P'H_n)_A \sum_{\alpha=1}^{N} \widetilde{B}_\alpha(\Lambda_A^+, t)(H_\alpha P' - PH_\alpha)\Big) = (B - S^{-1}BS')_{nn}, \quad n=1, \cdots, N. \tag{2.19}$$

Using (2.17) and (2.19), we have

$$C = B - i\sum_{n=1}^{N} H_n \int_{-\infty}^{\infty} dx \mathrm{Tr}\Big((H_n P - P'H_n)_A \cdot \sum_{\alpha=1}^{N} \widetilde{B}_\alpha(\Lambda_A^+, t)(H_\alpha P' - PH_\alpha)\Big). \tag{2.20}$$

This is the explicit form of C in terms of B. It is easy to see that if B_α are not constant, then $B\neq C$. Thus assuming[4] $B=C$ in (2.16) and (2.18) is not proper. Furthemore the expression (2.20) enables us to obtain the conditions connected with the temporal half of the Bäcklund transformations in 3.3.

§ 3. Results

3.1. The general structure of the integrable evolution equations

Using the fundamental relations (2.14), we can derive conveniently wider classes of integrable evolution equations than those in Ref. [4].

First, take

$$B_\alpha(\lambda, t)=h(\lambda, t)-i\varepsilon\Omega_\alpha(\lambda, t), \tag{3.1}$$

$$P'(x, t)=P(x, t+\varepsilon)=P(x, t)+\varepsilon\frac{\partial P(x, t)}{\partial t}+o(\varepsilon), \tag{3.2}$$

then, from (2.1)—(2.5), it follows easily that

$$S'(\lambda, t)=S(\lambda, t+\varepsilon)=S(\lambda, t)+\varepsilon\frac{\partial S(\lambda, t)}{\partial t}+o(\varepsilon). \tag{3.3}$$

Substituting (3.1)—(3.3) into (2.14) and taking into account only terms of the first order in ε we obtain

$$\begin{aligned}
&i\int_{-\infty}^{\infty} dx\mathrm{Tr}\Big(\Phi_F^{(in)}\Big(h(L_A, t)\frac{\partial P}{\partial t}-i\sum_{\alpha=1}^{N}\Omega_\alpha(L_A, t)[H_\alpha, P]\\
&\quad+\delta_{in}[H_n, P]_A\Big(\tilde{h}(L_A, t)\frac{\partial P}{\partial t}-i\sum_{\alpha=1}^{N}\tilde{\Omega}_\alpha(L_A, t)[H_\alpha, P]\Big)\\
&\quad=\Big(-h(\lambda, t)S^{-1}\partial S/\partial t+iS^{-1}\sum_{\alpha=1}^{N}\Omega_\alpha H_\alpha S-i\sum_{\alpha=1}^{N}\Omega_\alpha H_\alpha\Big)_{in},\quad i, n=1, \cdots, N,
\end{aligned} \tag{3.4}$$

where

$$\Phi^{(in)}=(F^+)\Delta^{ni}(F^+)^{-1},$$

$$L_A\Phi_F=\Lambda_A^+(P'=P)\Phi_F=-L\Phi_{FA},$$

$$L=i\frac{\partial}{\partial x}+[P, \circ]_F+i\Big[P, \int_{-\infty}^{x}[P, \circ]_D dx\Big],$$

$$[\Phi, \Psi]\overset{\mathrm{df}}{=\!=}\Phi\Psi-\Psi\Phi,$$

$h(\lambda, t)$ and $\Omega_\alpha(\lambda, t)$ are arbitrary entire functions of λ.

Secondly, assume now that P depends on other variables besides x and t. Taking

$$B_\alpha(\lambda, t)=f(\lambda, t), \tag{3.5}$$

$$P'(x, y, t)=P(x, y+\varepsilon, t)=P(x, y, t)+\varepsilon\frac{\partial P(x, y, t)}{\partial y}+o(\varepsilon), \tag{3.6}$$

by virtue of (2.1)—(2.5), it is easy to see that

$$S'(\lambda, y, t)=S(\lambda, y+\varepsilon, t)=S(\lambda, y, t)+\varepsilon\frac{\partial s(\lambda, y, t)}{\partial y}+o(\varepsilon). \tag{3.7}$$

Considering (2.14) with (3.5)—(3.7), if we keep terms linear in ε, we have

$$i\int_{-\infty}^{\infty} dx \mathrm{Tr}\left(\Phi_F^{(in)} f(L_A, t)\frac{\partial P}{\partial y} + \delta_{in}[H_n, P]_A \tilde{f}(L_A, t)\frac{\partial P}{\partial y}\right)$$
$$= \left(-f(\lambda, t)S^{-1}\frac{\partial S}{\partial y}\right)_{in}, \quad i, n=1, \cdots, N, \tag{3.8}$$

where $f(\lambda, t)$ is an arbitrary entire function of λ.

Finally, putting

$$B_\alpha(\lambda, t) = g(\lambda, t), \tag{3.9}$$

$$P'(x, \varepsilon, t) = e^{-i\varepsilon Ax}P(x, t)e^{i\varepsilon Ax} = P(x, t) - i\varepsilon x[A, P] + o(\varepsilon), \tag{3.10}$$

by means of (2.1)—(2.5), it is easy to show that

$$F^{+\prime}(x, \lambda, \varepsilon, t) = e^{-i\varepsilon Ax}F^+(x, \lambda+\varepsilon, t),$$
$$F^{-\prime}(x, \lambda, \varepsilon, t) = e^{-i\varepsilon Ax}F^-(x, \lambda+\varepsilon, t),$$

and

$$S'(\lambda, \varepsilon, t) = S(\lambda+\varepsilon, t) = S(\lambda, t) + \varepsilon\frac{\partial S(\lambda, t)}{\partial\lambda} + o(\varepsilon). \tag{3.11}$$

Substituting (3.9)—(3.11) into (2.14) and taking into account only terms of the first order in ε, we get

$$i\int_{-\infty}^{\infty} dx \mathrm{Tr}(\Phi_F^{(in)} g(L_A, t)x[A, P] + \delta_{in}[H_n, P]_A \tilde{g}(L_A, t)x[A, P])$$
$$= \left(g(\lambda, t)S^{-1}\frac{\partial S}{\partial\lambda}\right)_{in}, \quad i, n=1, \cdots, N, \tag{3.12}$$

where $g(\lambda, t)$ is an arbitrary entire function of λ.

By taking a simple linear combination of the equations (3.4), (3.8) and (3.12) there immediately follows that if $P(x, y, t)$ evolves, according to the nonlinear evolution equation,

$$h(L_A, t)\frac{\partial P}{\partial t} + f(L_A, t)\frac{\partial P}{\partial y} + g(L_A, t)x[A, P] = i\sum_{\alpha=1}^{N}\Omega_\alpha(L_A, t)[H_\alpha, P], \tag{3.13}$$

then $S(\lambda, y, t)$ obeys the companion equation

$$h(\lambda, t)\frac{\partial S}{\partial t} + f(\lambda, t)\frac{\partial S}{\partial y} - g(\lambda, t)\frac{\partial S}{\partial\lambda}$$
$$= i\left[\sum_{\alpha=1}^{N}\Omega_\alpha H_\alpha, S\right] - iS\sum_{n=1}^{N}H_n\int_{-\infty}^{\infty} dx \mathrm{Tr}([H_n, P]_A(\tilde{h}(L_A, t)\frac{\partial P}{\partial t}$$
$$+ \tilde{f}(L_A, t)\frac{\partial P}{\partial y} + \tilde{g}(L_A, t)x[A, P] - i\sum_{\alpha=1}^{N}\tilde{\Omega}_\alpha(L_A, t)[H_\alpha, P])\Big). \tag{3.14}$$

We shall show in 3.2 that

$$\sum_{n=1}^{N}H_n\int_{-\infty}^{\infty} dx \mathrm{Tr}([H_n, P]_A\sum_{\alpha=1}^{N}\tilde{\Omega}_\alpha(L_A, t)[H_\alpha, P]) = 0. \tag{3.15}$$

So we obtain

$$h(\lambda,\ t)\frac{\partial S}{\partial t}+f(\lambda,\ t)\frac{\partial S}{\partial y}-g(\lambda,\ t)\frac{\partial S}{\partial \lambda}$$
$$=i\Big[\sum_{\alpha=1}^{N}\Omega_\alpha H_\alpha,\ S\Big]-iS\sum_{n=1}^{N}H_n\int_{-\infty}^{\infty}dx\mathrm{Tr}\Big([H_n,\ P]_A\Big(\tilde{h}(L_A,\ t)\frac{\partial P}{\partial t}$$
$$+\tilde{f}(L_A,\ t)\frac{\partial P}{\partial y}+\tilde{g}(L_A,\ t)x[A,\ P]\Big). \tag{3.16}$$

3.2. The integral relations of the operator L_A

In order to show (3.15), we prove the following integral relations of the operator L_A

$$\int_{-\infty}^{\infty}dx\mathrm{Tr}([H_n,\ P]_A L_A^k[H_j,\ P])=0,\quad n,\ j=1,\ \cdots,\ N,\ k=0,\ 1,\ \cdots. \tag{3.17}$$

Putting $h(\lambda,\ t)=1$, $f(\lambda,\ t)=g(\lambda,\ t)=0$. $\Omega_\alpha(\lambda,\ t)=\delta_{j\alpha}\lambda^{m+1}$ in (3.14), we obtain

$$\frac{\partial S}{\partial t}=i\lambda^{m+1}[H_j,\ S]-S\sum_{n=1}^{N}H_n\int_{-\infty}^{\infty}dx\mathrm{Tr}([H_n,\ P]_A\tilde{L}_A^{m+1}[H_j,\ P]). \tag{3.18}$$

Considering the diagonal part in (3.18), we get

$$\frac{\partial\ln S_D}{\partial t}=-\sum_{n=1}^{N}H_n\int_{-\infty}^{\infty}dx\mathrm{Tr}([H_n,\ P]_A\tilde{L}_A^{m+1}[H_j,\ P]),\quad j=1,\ \cdots,\ N. \tag{3.19}$$

Expanding $\ln S_D(\lambda)$ in a series on λ^{-1} in the usual way[4]

$$\ln S_D(\lambda,\ t)=\sum_{l=0}^{\infty}\lambda^{-l}C^{(l)}(t). \tag{3.20}$$

Substituting (3.20) and

$$\tilde{L}_A^{m+1}[H_j,\ P]=\sum_{k=0}^{m}\lambda^{m-k}L_A^k[H_j,\ P]$$

into (3.19), and then equating the coefficients of λ^{m-k}, $0\leqslant k\leqslant m$, we get

$$\int_{-\infty}^{\infty}dx\mathrm{Tr}([H_nP]_A L_A^k[H_j,\ P])=0,\ n,\ j=1,\ \cdots,\ N,\quad k=0,\ \cdots,\ m. \tag{3.21}$$

Actually, (3.21) is valid for all natural number k since m is arbitrary. Clearly formulae (3.17) have nothing to do with the time evolution; indeed they express the property of spectral problem (1.1).

3.3. The temporal half of the Bäcklund transformations

From (3.13) and (3.16), it follows that the validity of the nonlinear evolution equations for $P(x,\ t)$

$$\frac{\partial P}{\partial t}=i\sum_{\alpha=1}^{N}\Omega_\alpha(L_A,\ t)[H_\alpha,\ P] \tag{3.22}$$

implies the validity of the linear equations for $S(\lambda,\ t)$

$$\frac{\partial S}{\partial t}=i\Big[\sum_{\alpha=1}^{N}\Omega_\alpha(\lambda,\ t)H_\alpha,\ S\Big]. \tag{3.23}$$

Now put $B=B(\lambda)$ and

$$\frac{\partial}{\partial t}\sum_{n=1}^{N}H_n\int_{-\infty}^{\infty}dx\mathrm{Tr}\Big((H_nP-P'H_n)_A\sum_{\alpha=1}^{N}\tilde{B}_\alpha(\Lambda_A^+)(H_\alpha P'-PH_\alpha)\Big)=0. \tag{3.24}$$

Then, according to (2.20), we get

$$\frac{\partial C}{\partial t}=0. \tag{3.25}$$

In this case, if $P'(x, t)$ satisfies (3.22) and P and P' satisfy (2.16), namely

$$\sum_{\alpha=1}^{N} B_\alpha(\Lambda_A^+)(H_\alpha P' - PH_\alpha) = 0, \tag{3.26}$$

from (2.18), (3.23) and (3.25) it follows that $S'(\lambda, t)$ also satisfies (3.23). Ref. [4] pointed out that (3.26) are Bäcklund transformations (BT'S) for equations (3.22). Let us emphasize that (3.26) is only the spatial half of the Bäcklund transformations. It is clear that we must impose condition (3.24) besides (3.26) in order to guarantee that P' also satisfies (3.22). Thus condition (3.24) is equivalent to the temporal half of the Bäcklund trasformations for the equation (3.22).

3.4. The integrals of motion

Substituting the particular solution $P'=0$ of equation (3.22) into (3.24), we obtain

$$\frac{\partial}{\partial t}\sum_{n=1}^{N} H_n \int_{-\infty}^{\infty} dx \mathrm{Tr}((H_n P)_A \sum_{\alpha=1}^{N} \tilde{B}_\alpha(\bar{L}_A) P H_\alpha) = 0, \tag{3.27}$$

where

$$\bar{L}_A = \Lambda_A^+(P'=0),$$

$$\bar{L}_A \Phi_F = -\bar{L}\Phi_{FA},$$

$$\bar{L}\Phi_F = i\frac{\partial \Phi_F}{\partial x} + (P\Phi_F)_F + iP\int_{-\infty}^{x} dx (P\Phi_F)_D. \tag{3.28}$$

Since $B_\alpha(\lambda)$ are arbitrary entire functions of λ, we get the infinite series of the integrals of motion for equations (3.22)

$$\int_{-\infty}^{\infty} dx \mathrm{Tr}((H_n P)_A \sum_{\alpha=1}^{N} \bar{L}_A^l P H_\alpha) = C_n^{(l)}, \quad n=1, \cdots, N,\ l=1, 0, \cdots. \tag{3.29}$$

§ 4. The Li Yishen method

In Refs. [1, 8], Li Yishen presented amethod, by using which the nonlinear evolution equations and the transformations of the potentials associated with the spectral problem can be found simply without any conditions imposed on the potentials at infinity.

Now we use the method to obtain the transformations of the potentials associated with spectral problem (1.1).

If a gauge transformation of the eigenmatrix Ψ

$$\Psi = T\Psi',$$

$$T(x, \lambda, t) = \sum_{k=0}^{m} \lambda^{m-k}(T_{kD} + T_{kFA}), \tag{4.1}$$

transforms the spectral problem (1.1) into

$$\frac{\partial \Psi'}{\partial x} = (i\lambda A + iP'(x, t))\Psi', \tag{4.2}$$

then T satisfies

$$T_x = (i\lambda A + iP)T - T(i\lambda A + iP'). \tag{4.3}$$

Substituting (4.1) into (4.3) and equating the coefficients of λ^{m-k}, we have

$$T_{oF}=0, \tag{4.4}$$

$$\frac{\partial}{\partial x}T_{kD}=i(PT_{kFA}-T_{kFA}P')_D,\quad k=0,\cdots,m, \tag{4.5}$$

$$\frac{\partial}{\partial x}T_{kFA}=i(PT_{kD}-T_{kD}P'+T_{k+1},F)+i(PT_{kFA}-T_{kFA}P')_F,\quad k=0,\cdots,m, \tag{4.6}$$

with

$$T_{m+1,F}=0. \tag{4.7}$$

From (4.5), it follows that

$$T_{kD}=iD^{-1}(PT_{kFA}-T_{kFA}P')+\sum_{\alpha=1}^{N}B_\alpha^{(k)}H_\alpha, \tag{4.8}$$

where D^{-1} is the integral operator, $B_\alpha^{(k)}(t)$ are arbitrary functions independent of x.

By means of (4.6) and (4.8), we obtain

$$T_{k+1,F}=\Lambda_A^+T_{kF}+\sum_{\alpha=1}^{N}B_\alpha^{(k)}(H_\alpha P'-PH_\alpha),\qquad k=0,\cdots,m, \tag{4.9}$$

where

$$\Lambda_A^+\Phi_F=-\Lambda^+\Phi_{FA},$$

$$\Lambda^+\Phi_F=i\frac{\partial\Phi_F}{\partial x}-(\Phi_FP'-P\Phi_F)_F+i(D^{-1}(\Phi_FP'-P\Phi_F)_D)P'-iPD^{-1}(\Phi_FP'-P\Phi_F)_D. \tag{4.10}$$

Repeatedly using (4.9), we easily show that

$$T_{k+1,F}=\sum_{l=0}^{k}\sum_{\alpha=1}^{N}B_\alpha^{(k-l)}(t)\Lambda_A^{+l}(H_\alpha P'-PH_\alpha). \tag{4.11}$$

Putting $B_\alpha(\lambda,t)=\sum_{l=0}^{m}B_\alpha^{(k-l)}(t)\lambda^l$ and using (4.11) and (4.7), we obtain the transformations of the potentials associated with (1.1)

$$\sum_{\alpha=1}^{N}B_\alpha(\Lambda_A^+,t)(H_\alpha P'-PH_\alpha)=0. \tag{4.12}$$

If we assume condition (1.2) and take $D^{-1}=\int_{-\infty}^{x}\cdots dx$, the transformations (4.12) are the same as (2.16).

References

[1] Li Yishen, Proceedings of the 1980 Beijing symposium on differential geometry and differential equations, Vol. 3. 1293—1308, editors S. S. Chern, Wu Wen—tsun, Science press, Beijing, China, 1982.

[2] Zhu Guocheng, Chen Dengyuan, Zeng Yungbo, Zhang Etang and Li Yishen, *Chinese Annals of Mathematics*, **3**:1 (1982), 1—12.

[3] Newell. A. C., *R. Soc. Lond. A.*, 365 (1979), 283—311.

[4] B. G. Konopelchenko. a. J. Phys. A: Math.. Gen **14** (1981), 1237—1259. b. Phys. Lett., **75A** (1980), 447. c. Phys. Lett., **79A** (1980), 39.

[5] Konopelchenko, B. G. and Formusatic, I. B., *J. Phys. A: Math. Gen.*, **15** (1982), 2017—2040.

[6] Ablowitz, M. J., Kaup, D. J., Newell, A. C. and Segnr. H., *Stud. Math.*, 53 (1974), 249—315.

[7] Calogero, F. and Degasperis, A., Nuovo Cimento. 32 B (1976), 201—242. 31 B (1976), 225—249.

[8] Li Yishen., Gauge transformation, Bäcklund transformation and the nonlinear superposition formula, (1982), to submit to *Mathematics advance* (in chinese).