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THE WEAK PROJECTION THEORY AND DECOMPOSITIONS OF QUASI-MARTINGALE MEASURES

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Abstract

In this paper it is proved that every bounded, $\mathscr{B}\times\mathscr{F}$-measurable function has a unique predictable projection and that every admissible measure has a unique dual predictable projection. Using this weak projection theory, the author proves a weak version of Doob-Meyer's decomposition theorem for regular quasi- martingale measures.

It is well known that every quasi-martingale X of class [L. D.] has a Doléans measure λ_X and therefore there exists a unique, predictable process $\tilde{X}$ of finite variation such that $\lambda_X=\lambda_{\tilde{X}}$, that is, $M=X-\tilde{X}$ is a martingale (cf. [1, 3]). The process $\tilde{X}$ is called the dual predictable projection of X and $X=M+\tilde{X}$ is called the Doob-Meyer's decomposition of the quasi-marsingale X. When the parameter set $\mathbf{R}_+$ is replaced by a general topological measurable space, it seems to be very difficult to establish a similar theorem since the existence and uniqueness of the dual predictable projection heavily depends on the linear order property of the parameter set $\mathbf{R}_+$. However, we will prove a weak version of the predictable projection theory in this connection and establish a similar decomposition for a wide class of quasi-martingales.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $\mathscr{U}$ be a topological space with its Borel σ-algebra $\mathscr{B}$, $\mathscr{C}$ be a sublattice of $\mathscr{B}$ such that $\mathscr{B}=\sigma(\mathscr{C})$ and $\mathfrak{A}$ be the algebra generated by $\mathscr{C}$. As in [4], let $\{\mathscr{F}_C, C\in\mathscr{C}\}$ be a family of sub-σ-algebras of $\mathscr{F}$ satisfying the following conditions:

(F. 1) $\mathscr{F}_\phi$ contains all P-null sets;

(F. 2) $C_1\subset C_2$, C_1, $C_2\in\mathscr{C}\Rightarrow\mathscr{F}_{C_1}\subset\mathscr{F}_{C_2}$;

(F. 3) $C_n\downarrow C$, $\{C_n\}\subset\mathscr{C}$, $C\in\mathscr{C}\Rightarrow\mathscr{F}_{C_n}\downarrow\mathscr{F}_C$.

Assume that for every $A\in\mathfrak{A}$, there exists a set $t(A)\in\mathscr{C}$ such that

(i) $t(A)\cap A=\varnothing$;

(ii) $C\in\mathscr{C}$, $C\cap A\neq\varnothing\Rightarrow t(A)\subset C$;

(iii) A, $B\in\mathfrak{A}$, $A\subset B\Rightarrow t(B)\subset t(A)$.

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Denote by $\mathscr{R}$ all subsets of the product space $\mathscr{U}\times\Omega$ of the form $A\times\Lambda$ where $A\in\mathfrak{A}$ and $\Lambda\in\mathscr{F}_{t(A)}$. The σ-algebra $\mathscr{P}$ generated by $\mathscr{R}$ is said to be predictable σ-algebra.

For any real valued integrable random set function $X=X(A, w)$ defined on $\alpha\times\Omega$, we define a real valued set function λ_X on $\mathscr{R}$ as follows:

$$\lambda_X(A\times\Lambda)=EI_\Lambda X(A) \text{ for } A\times\Lambda\in\mathscr{R}.$$

If there exists a sequence $\{A_n\times\Lambda_n\}$ of sets in $\mathscr{R}$ such that $A_n\times\Lambda_n\uparrow\mathscr{U}\times\Omega$ and λ_X is finitely additive on the algebra generated by $\mathscr{R}$ and has finite variation on each set $A_n\times\Lambda_n$, then we call X a quasi-martingale measure. If, moreover, the set function λ_X can be extended to a σ-finite signed measure on $\mathscr{P}$, then X is said to be regular and λ_X is called the Doléans measure of X. Clearly, the lodally square integrable martingale μ defined in [4] as well as its square μ^2 are examples of reqular quasi-martingale measures.

We said that a σ-finite signed measure λ on $\mathscr{P}$ is admissible if there exists a σ-finite measure m on $\mathscr{B}$ such that $\lambda\ll m\times P$ on $\mathscr{P}$. This definition is more strict than that defined in [2, 3] in one dimensional case. Denote by $\mathscr{P}_u$ the completion of σ- algebra $\mathscr{P}$ with respect to all admissible measures. A set in $\mathscr{P}_u$ is called a universal null set (u. n. set) if it has measure zero for all admissible measures. To describe the u. n. sets, we have the following lemma:

Lemma. *A set N in $\mathscr{P}_u$ is a u. n. set if and only if for each $u\in\mathscr{U}$, the u-section of N has probability zero, i. e.*

$$P(Nu)=0, \quad \forall u\in\mathscr{U}.$$

Proof The "if" part: Suppose that λ is an arbitrary admissible measure on $\mathscr{P}$ and $\lambda\ll m\times P$ for some measure m on $\mathscr{B}$. Since N is an $m\times P$-null set if and only if almost all its u-sections are P-null sets, it follows that $\lambda(N)=0$.

The "only if" part: Suppose that for some $u\in\mathscr{U}$ we have $P(Nu)>0$. Then, we can find a measure m on $\mathscr{B}$ such that $m(\{u\})>0$. Therefore, we have $(m\times P)(N)>0$, which implies that N is not a u. n. set.

Now we state and prove a weak version of predictable projection theorem:

Theorem 1. *For each bounded, $\mathscr{B}\times\mathscr{F}$-measurable function $h(u, w)$, there exists a unique (up to a u. n. set) $\mathscr{P}$-measurable function $h^{\mathscr{P}}(u,w)$ such that, for every σ-finite measure m on $\mathscr{B}$ and every set S in $\mathscr{P}$, the equality*

$$\int_S h(u, w)d(m\times P)=\int_S h^{\mathscr{P}}(u, w)d(m\times P) \tag{1}$$

holds.

Remark. If we consider the probability measures on $\mathscr{B}\times\mathscr{F}$, the equation (1) means that

$$h^{\mathscr{P}}=E^{m\times P}(h|\mathscr{P})$$

for every probability measure m on $\mathscr{B}$. We will use this notation for general measures. $h^{\mathscr{P}}$ is said to be the predictable projection of h. It is easy to see that the map: $h \to h^{\mathscr{P}}$ has similar properties as those of a conditional expectation.

Proof (Uniqueness) Let $h_1 \geqslant h_2$ be two $\mathscr{B}\times\mathscr{F}$-measurable functions with predictable projections $h_1^{\mathscr{P}}$ and $h_2^{\mathscr{P}}$ respectively. We will prove that the predictable set

$$S=[(u,\ w):h_1^{\mathscr{P}}(u,\ w)<h_2^{\mathscr{P}}(u,\ w)]$$

is a u. n. set. Actually, if for some $u\in\mathscr{U}$, $P(S_u)>0$, then we can find a measure m on $\mathscr{B}$ such that $m(\{u\})>0$ and therefore $(m\times P)(S)>0$. It follows that

$$\int_S h_1 d(m\times P)=\int_S h_1^{\mathscr{P}} d(m\times P)<\int_S h_2^{\mathscr{P}} d(m\times P)=\int_S h_2 d(m\times P),$$

which contradicts the assumption.

(Existence) Firstly we consider the indicator of a product set $A\times\Lambda$, $h(u,\ w)=\mathbf{1}_A(u)\mathbf{1}_\Lambda(w)$, where $A\in\mathfrak{A}$ and $\Lambda\in\mathscr{F}$. For every finite partition of set A,

$$\pi: A=\sum_{i=1}^{n_\pi} A_i^\pi,$$

where $\{A_i^\pi,\ i=1,\ 2,\ \cdots,\ n_\pi\}$ are disjoint sets in $\mathfrak{A}$, define

$$\mathbf{1}_{A\times\Lambda}^\pi(u,\ w)\quad \sum_{i=1}^{n_\pi}\mathbf{1}_{A_i^\pi}(u)E(\mathbf{1}_\Lambda|\mathscr{F}_{t(A_i^\pi)})(w).$$

All the partitions ordered by the "finer" relation constitute a partially ordered set π which is filtering to the right. Every linearly ordered subset in Π is contained in a maximized linearly ordered subset. We choose any one of the maximized linearly ordered subsets, fix it and denote it by Π_0.

For each $u\in\mathscr{U}$, it is easy to prove that

$$\{\mathbf{1}_{A\times\Lambda}^\pi(u,\ \ ),\ \pi\in\Pi_0\}$$

is a martingale bounded in L^1. Accordingly, it will converge almost surely to some limit $\mathbf{1}_{A\times\Lambda}^{\mathfrak{A}}(u,\ w)$ which is defined up to a u. n. set.

For any σ-finite measure m on $\mathscr{B}$ and any set $A_1\times\Lambda_1$ in $\mathscr{R}$, we have

$$\int_{A_1\times\Lambda_1}\mathbf{1}_{A\times\Lambda}^\pi d(m\times P)=\sum_{i=1}^{n_\pi} m(A_1A_i^\pi)E(\mathbf{1}_{\Lambda_1}E(\mathbf{1}_\Lambda|\mathscr{F}_{t(A_i^\pi)})). \tag{2}$$

Since Π_0 is a maximized linearly ordered subset of Π, it follows that there exists some $\pi_0\in\Pi_0$ such that for $\forall\pi\succ\pi_0$, all those sets $A_i^\pi(i=1,\ 2,\ \cdots,\ n_\pi)$ will be eventually either contained in A_1 or disjoint from A_1. If $A_1A_i^\pi=\varnothing$ for some i, then $m(A_1A_i^\pi)=0$; if $A_1\supset A_i^\pi$ for some i, then $t(A_1)\subset t(A_i^\pi)$ and

$$E(\mathbf{1}_{\Lambda_1}E(\mathbf{1}_\Lambda|\mathscr{F}_{t(A_i^\pi)}))=E(E(\mathbf{1}_{\Lambda_1\Lambda}|\mathscr{F}_{t(A_i^\pi)})=E(\mathbf{1}_{\Lambda_1\Lambda})=P(\Lambda_1\Lambda).$$

Hence, by passage to limit in (2), we have

$$\int_{A_1\times\Lambda_1}\mathbf{1}_{A\times\Lambda}^{\mathscr{P}} d(m\times P)=m(A_1A)P(\Lambda_1\Lambda)=\int_{A_1\times\Lambda_1}\mathbf{1}_{A\times\Lambda}d(m\times P).$$

Using a standard reasoning in measure theory, it is easy to see that for each bounded, $\mathscr{B}\times\mathscr{F}$-measurable function $h(u,\ w)$, there exists a $\mathscr{P}$-measurable function $h^{\mathscr{P}}(u,\ w)$

such that (1) holds for every set S in $\mathscr{P}$. So the proof is complete.

Theorem 2. *For every admissible measure λ on $\mathscr{P}$, there exists a unique extension $\tilde{\lambda}$ on $\mathscr{B}\times\mathscr{F}$ such that for every bounded, $\mathscr{B}\times\mathscr{F}$-measurable function $h(u, w)$ and its predicatable projection $h^{\mathscr{P}}(u, w)$, the equality*

$$\int_{\mathscr{U}\times\Omega} h d\tilde{\lambda} = \int_{\mathscr{U}\times\Omega} h^{\mathscr{P}} d\lambda \tag{3}$$

holds.

Remark. This extension $\tilde{\lambda}$ is called the dual predicatable projection of λ.

Proof For $S\in\mathscr{B}\times\mathscr{F}$, define

$$\tilde{\lambda}(S) = \int_{\mathscr{U}\times\Omega} \mathbf{1}_S^{\mathscr{P}} d\lambda,$$

where $\mathbf{1}_S^{\mathscr{P}}$ is the predicatable projection of l_S. Since the map $h \to \int h^{\mathscr{P}} d\lambda$ is a positive linear functional on the Banach space of all bounded, $\mathscr{B}\times\mathscr{F}$-measurable functions whenever λ is a nonnegative admissible measure, it follows that $\tilde{\lambda}(\cdot)$ thus defined is a measure on $\mathscr{B}\times\mathscr{F}$ and for each bounded $\mathscr{B}\times\mathscr{F}$-measurable function h, the equation (3) holds. Since every admissible signed measure is a difference of two nonnegative admissible measures, the theorem follows immediately.

Remark. For any admissible measure λ on $\mathscr{P}$, we can always find an extension by setting

$$\lambda_m(S) = \int_S \frac{d\lambda}{d(m\times P)}\bigg|_{\mathscr{P}} d(m\times P) \tag{4}$$

for $S\in\mathscr{B}\times\mathscr{F}$ provided $\lambda \ll m\times P$ on $\mathscr{P}$ for some σ-finite measure m on $\mathscr{B}$. However, Theorem 2 shows that the extension doesn't depend on the choice of m. Actually, by Theorem 1 we have

$$\begin{aligned}\lambda_m(S) &= \int_{\mathscr{U}\times\Omega} \mathbf{1}_S \cdot \frac{d\lambda}{d(m\times P)}\bigg|_{\mathscr{P}} d(m\times P) = \int_{\mathscr{U}\times\Omega} E^{m\times P}\Big(\mathbf{1}_S \cdot \frac{d\lambda}{d(m\times P)}\bigg|_{\mathscr{P}}\Big|\mathscr{P}\Big) d(m\times P) \\ &= \int_{\mathscr{U}\times\Omega} E^{m\times P}(\mathbf{1}_S|\mathscr{P}) \frac{d\lambda}{d(m\times p)}\bigg|_{\mathscr{P}} d(m\times P) = \int_{\mathscr{U}\times\Omega} l_S^{\mathscr{P}} d\lambda = \tilde{\lambda}(S).\end{aligned} \tag{5}$$

Theorem 3. *For every admissible measure λ on $\mathscr{P}$, there exists a unique (up to equivalence) set function $X(A, w)$ on $\mathfrak{a}\times\Omega$ such that*

i) *for each $w\in\Omega$, $X(\cdot, w)$ can be extended to a σ-finite signed measure on $\mathscr{B}$;*

ii) *for each $A\in\mathfrak{A}$, $X(A, \cdot)$ is $\mathscr{F}_C$-measurable provided $A\subset C\in\mathscr{C}$;*

iii) *the Doléans measure λ_X of X coincides with λ on $\mathscr{P}$.*

Proof Suppose that $\lambda \ll m\times P$ on $\mathscr{P}$ for some σ-finite measure m on $\mathscr{B}$. Combining (4) and (5) we see that

$$\frac{d\tilde{\lambda}}{d(m\times P)}\bigg|_{\mathscr{B}\times\mathscr{F}} = \frac{d\tilde{\lambda}}{d(m\times P)}\bigg|_{\mathscr{P}}.$$

Define

$$X(A, w) = \int_A \frac{d\tilde{\lambda}}{d(m\times P)} dm \text{ for } A\in\mathfrak{A}.$$

Clearly, this set function satisfies (i) and (ii). Moreover, for each $\Lambda\in\mathscr{F}$

$$E\mathbf{1}_A X(A)=\int_{A\times\Lambda}\frac{d\tilde{\lambda}}{d(m\times P)}\,d(m\times P)=\tilde{\lambda}(A\times\Lambda).$$

It follows that (iii) also holds and that the function X actually doesn't depend on the choice of m. In fact, if $\lambda\ll\tilde{m}\times P$ for another σ-finite measure $\tilde{m}$ on $\mathscr{B}$ and if we define

$$\tilde{X}(A,\ w)=\int_A\frac{d\tilde{\lambda}}{d(\tilde{m}\times P)}\,d\tilde{m}\ \text{for}\ A\in\mathfrak{A},$$

then we have

$$E\mathbf{1}_\Lambda\tilde{X}(A)=E\mathbf{1}_\Lambda X(A)=\tilde{\lambda}(A\times\Lambda)$$

for each $\Lambda\in\mathscr{F}$. It follows that for each $A\in\mathfrak{A}$

$$\tilde{X}(A,\ w)=X(A,\ w)\quad \text{a. s.}$$

and thus the proof is completed.

Remark. If the admissible measure λ is the Doléans measure of some regular quasi-martingale measure Y, then we call X the dual predictable projection of Y. According to the above theorem, $\lambda_{Y-X}=0$, i. e. $Y-X$ is a martingale measure in some weak sense. This is a weak version of Doob-Meyer's decomposition theorem. If μ is a locally square integrable martingale measure (cf. [4]), then μ^2 is a regular quasi-martingale measure. If, moreover, the Doléans measure of μ^2 (in [4], it is denoted by $\langle\mu\rangle$) is an admissible measure, then there exists a unique dual predicatable projection of μ^2. We can use this pro jection to define the stochastic integral with respect to μ as many authors have already done. But we prefer to use the Doléans measure directly since the conditions imposed on the existence of Doléans measure are much weaker than those of validity of Doob-Meyer's decomposition theorem on a general topological measurable space.

References

[1] Orey, S., F-processes, Fifth Berkely Symp., **2** (1965), 301—313.

[2] Pellaumail, J., Sur L'intégrale stochastique et la décomposition de Doob-Meyer, Asterisque, **9**, S. M. F. (1973).

[3] Métivier, M., Semimartingales - a course on stochastic processes, Walter de Gruyter, Berlin, New York (1982).

[4] Huang, Z. Y. Stochastic integrals on general topological measurable spaces, Z. Wahr. verw. Gebiete **66** (1984), 25—40.