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COMMUTATORS OF MULTIPLIER OPERATORS

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Abstract

Denote $M^l=\{\omega\in C^\infty(R^K\backslash\{0\}: |\omega^{(\beta)}(\xi)|\leqslant C_\beta|\xi|^{l-|\beta|}\}$, l is an integer. $R^{\{m\}}_{(-\alpha)}$ is the n-fold composition of Taylor series remainder operator, $m=(m_1, \cdots, m_n)\in Z^n$. Z is the set of non-negative integers, $\alpha\in(R^K)^n$.

Denote

$$T_{\sigma(\alpha,\xi)}(a, f)(x)=\int_{(R^K)^{n+1}} e^{ix\xi}\sigma(\alpha, \xi)\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha\, d\xi,$$

where $a=(a_1, \cdots, a_n)$, $a_i, f\in\mathscr{S}(R^K)$, $\hat{a}(\alpha)=\hat{a}_1(\alpha_1)\cdots\hat{a}_n(\alpha_n)$, $[\alpha]=\sum_{i=1}^n\alpha_i$, $d=d\alpha_1\cdots d\alpha_n$.

The main results are as follows:

(i) If $\gamma_1, \gamma_2\in Z^K$ and l is an integer such that $|\gamma_1|+|\gamma_2|+l=|m|=m_1+\cdots+m_n$, $0\leqslant|\gamma_1|\leqslant\min\limits_{1\leqslant i\leqslant n}\{m_i\}$, and $\omega\in M^l$, then we have

$$\|\partial^{\gamma_1}T_{R^{\{m\}}_{(-\alpha)}\omega(\xi)}(a, \partial^{\gamma_2}f)\|_q\leqslant C\|f\|_{p_0}\prod_{i=1}^n\|\nabla^{m_i}a_i\|_{p_i},$$

where $\|\nabla^{m_i}a_i\|_{p_i}=\sum\limits_{|\beta|=m_i}\|a_i^{(\beta)}\|_{p_i}$, $q^{-2}=p_0^{-1}+\sum\limits_{i=1}^n p_i^{-1}\in(0, 1)$, p^{-1}, $p_i^{-1}\in(0, 1)$, $C=C(K, n, m, C_\beta, p_0, p_i, \gamma_1, \gamma_2)$ is a conseant.

(ii) In the same sense of notation as in (i), but now $|m|=1$, we have

$$\|\partial^{\gamma_1}T_{R^{\{m\}}_{(-\alpha)}\omega(\xi)}(a, \partial^{\gamma_2}f)\|_p\leqslant C\|f\|_p\prod_{i=1}^n\nabla^{m_i}a_i\|_\infty p\in(1, \infty),$$

where $C=C(K, n, m, C_\beta, p, \gamma_1, \gamma_2)$.

These results extend the corresponding ones given by coifman-Meyer in [4] and Cohen, J. in [2], and, in a sense, extend those given by Calderón, A. P. in [1].

§ 1. Notation and Terminology

Denote

$$M^l=\{\omega\in C^\infty(R^K\backslash\{0\}):\forall_\beta,\ \beta\in Z^K,\ \exists C_\beta \text{ such that } |\omega^{(\beta)}(\xi)|\leqslant C_\beta|\xi|^{l-|\beta|}\},\quad l\in R^1,$$

$AC^\infty(n, 1)=\{\sigma(\alpha, \xi):\alpha\in(R^K)^n, \xi\in R^K$, σ is defined and has partial derivatives of arbitrary order for a. e. $(\alpha, \xi)\}$.

For $g\in AC^\infty(0, 1)$, define

$$R^{m_i}_{-\alpha_i}g(\xi)=g(\xi-\alpha_i)-\sum_{|\beta|<m_i}\frac{g^{(\beta)}(\xi)}{\beta!}(-\alpha_i)^\beta,\ m_i\in Z,\ \xi,\ \alpha_i\in R^K,$$

$$P_{m_i}(g, x, y)=g(x)-\sum_{|\beta|<m_i}\frac{g^{(\beta)}(y)}{\beta!}(x-y)^\beta,\quad m_i\in Z,\ x,\ y\in R^K,$$

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$$R^{(m)}_{(-\alpha)}g(\xi)=R^{m_1}_{-\alpha_1}\cdot\cdots\cdot R^{m_n}_{-\alpha_n}g(\xi),$$

where $m=(m_1, \cdots, m_n)\in Z^n, \alpha=(\alpha_1, \cdots, \alpha_n)\in(R^K)^n$, and $\beta=(\beta_1, \cdots, \beta_n)\in Z^K$, $\beta!=\beta_1!\cdots\beta_n!$, $|\beta|=|\beta_1|+\cdots+|\beta_n|$, $g^{(\beta)}=\partial^\beta g$.

For $\sigma\in AC^\infty(n, 1)$, define

$$T_{\sigma(\alpha,\xi)}(a, f)(x)=\int_{(R^K)^{n+1}}e^{ix\xi}\sigma(\alpha, \xi)\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha\, d\xi,$$

where $a=(a_1, \cdots, a_n)$, $a_i, f\in\mathscr{S}(R^K)$, $\hat{a}(\alpha)=\hat{a}_1(\alpha_1)\cdots\hat{a}_n(\alpha_n)$, $[\alpha]=\alpha_1+\cdots+\alpha_n$, $d\alpha=d\alpha_1\cdots d\alpha_n$.

If $m_1=\cdots=m_n=1$, it is easy to see that

$$T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a, f)(x)=C[a_n, \cdots, [a_1, \omega(D)]\cdots]f(x),$$

which is the nth commutator of $\omega(D)$, where $a_i(f)(x)=(a_if)(x)$. Therefore we can extend the notation of commutator and call $T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a, \cdot)$ a commutator of order $|m|$ ([2, 3]).

We will suppose $a_i, f\in\mathscr{S}(R^K)$ below aod discuss two kinds of indexes:

(i) $p_0, p_i, q\in(1, \infty)$, $q^{-1}=p_0^{-1}+\sum_{i=1}^n p_i^{-1}$;

(ii) $p_0, q\in(1, \infty)$, $\forall i$, $p_i=\infty$, $p^{-1}=p_0^{-1}+\sum_{i=1}^n p_i^{-1}$.

Let us introduce

$$\|\nabla^{m_i}a_i\|_{p_i}=\sum_{|\beta|=m_i}\|a_i^{(\beta)}\|_{p_i},\quad m_i\in Z,\ \beta\in Z^K,$$

and

$$\|\nabla^m a\|_p=\prod_{i=1}^n\|\nabla^{m_i}a_i\|_{p_i},$$

where $p=(p_1, \cdots, p_n)$, $m=(m_1, \cdots, m_n)$ and $a=(a_1, \cdots, a_n)$.

§ 2. The First Kind of Indexes

Theorem 1. *If $\omega\in M^{|m|}$, then for the first kind of indexes we have*

$$\|T_{R^{(m)}_{(-\alpha)}\omega(\xi)}, (a, f)\|_q\leqslant C\|f\|_{p_0}\|\nabla^m a\|_p,$$

where $C=C(K, n, m, C_\beta, p_0, p_i)$ is a constant.

The theorem can be proved by using the same method as in [8], Theorem 2, but instead of applying Coifman-Meyer's theorem ([3], Thereom 1) we now apply the following theorem, which can be obtained similarly.

Theorem A. *Let $\sigma\in C^\infty((R^K)^n\backslash\{0\})$ and for $\forall\beta=(\beta_1, \cdots, \beta_n)\in(Z^K)^n$, $\forall\xi=(\xi_1, \cdots, \xi_n)\in(R^K)^n\exists$ constant C_β such that $|\sigma^{(\beta)}(\xi)|\leqslant C_\beta|\xi|^{-|\beta|}$. Then with*

$$T(f_1, \cdots, f_n)(x)=\int_{(R^K)^n}e^{ix[\xi]}\sigma(\xi)\hat{f}_1(\xi_1)\cdots\hat{f}_n(\xi_n)d\xi,$$

we have

$$\|T(f_1, \cdots, f_n)\|_q\leqslant C\|f_1\|_{p_1}\cdots\|f_n\|_{p_n},$$

where $q, p_i\in(1, \infty)$, $q^{-1}=\sum_{i=1}^n p_i^{-1}$, $C=C(K, n, C_\beta, p_i)$ is a constant.

Theorem 1 extends [2], Theorem I, which is for $K=1$, $\omega(\xi)=|\xi|^{|m|}\operatorname{sgn}\xi$. A further extension of them is as follows.

Theorem 2. *If* $\gamma_1, \gamma_2\in Z^K$ and *l is an integer such that* $|\gamma_1|+|\gamma_2|+l=|m|$, and $\omega\in M^l$, *then for the first kind of indexes we have*

$$\|\partial^{\gamma_1}T_{R^{(m)}_{(-a)}\omega(\xi)}(a, \partial^{\gamma_2}f)\|_q\leqslant C\|f\|_{p_0}\|\nabla^m a\|_p,$$

where $C=C(K, n, m, C_\beta, p_0, p_i, \gamma_i)$ *is a constant.*

Proof Denote

$$\overline{T}_{\sigma(\alpha,\xi)}(a, f)(x)=\int_{(R^K)^{n+1}} e^{ix(\xi+[\alpha])}\hat{a}(\alpha)\hat{f}(\xi)d\alpha\, d\xi,$$

$\overline{M}(m)=\{\alpha, \xi)\in AC^\infty(n, 1)$: For the first kind of indexes p_0, p_i, q, $\|\overline{T}_{\sigma(\alpha,\xi)}(a,f)\|_q \leqslant C\|f\|_{p_0}\|\nabla^m a\|_p$, $C=C(K, n, m, p_0, p_i, \sigma)$ is a constant$\}$.

We should prove

$$(\xi+[\alpha])^{\gamma_1}\xi^{\gamma_2}R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])\in\overline{M}(m). \tag{2.1}$$

We use induction for (γ_1, γ_2) in the following manner.

First we prove (2.1) for $\gamma_1=\gamma_2=0$. Then we reduce every other couple (γ_1, γ_2) to the cases $(0, \gamma_2')$, $(\gamma_1', 0)$, (γ_1', γ_2'), where $0\leqslant|\gamma_1'|<|\gamma_1|$, $0\leqslant|\gamma_2'|<|\gamma_2|$, for which we suppose (2.1) has been proved.

Theorem 1 shows that (2, 1) is correct for $\gamma_1=0$, $\gamma_2=0$. For other (γ_1, γ_2) there are three cases:

(i) $\gamma_1=0$, $\gamma_2\neq 0$.

If $|m|=0$, there exists

$$T_{R^{(m)}_{(-a)}\omega(\xi)}(a, \partial^{\gamma_2}f)(x)=\int_{(R^K)^{n+1}} e^{ix(\xi+[\alpha])}\xi^{\gamma_2}R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])\hat{a}(\alpha)\hat{f}(\xi)d\alpha\, d\xi$$

$$=\int_{(R^K)^{n+1}} e^{ix(\xi+[\alpha])}\xi^{\gamma_2}\omega(\xi)\hat{a}(\alpha)\hat{f}(\xi)d\alpha\, d\xi.$$

Because $\xi^{\gamma_2}\omega(\xi)\in M^0$, we obtain (2.1) by applying Mihlin multiplier theorem and Hölder inequality.

If $|m|>0$, denote $J=\{i: m_i=0\}$, $J'=\{1, \cdots, n\}\backslash J$. It follows that

$$T_{R^{(m)}_{(-a)}\omega(\xi)}(a, \partial^{\gamma_2}f)(x)=Ca_J(x)T_{R^{(m_{J'})}_{(-a_{J'})}\omega(\xi)}(a_{J'}, \partial^{\gamma_2}f)(x),$$

where $a_J(x)=\prod_{i\in J}a_i(x)$, $m_{J'}=(m_{j_1}, \cdots, m_{j_r})$, $a_{J'}=(a_{j_1}, \cdots, a_{j_r})$, $J'=(j_1, \cdots, j_r)$.

Therefore, without loss of generality we can suppose $\forall i$, $m_i\geqslant 1$.

As in [8], Lemma 1, we can verify the following equality

$$R^{(m)}_{(-\alpha)}(\xi_j\omega(\xi+[\alpha]))=\xi_j R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])-\sum_{i=1}^n \alpha_{i,j}R^{(m^i)}_{(-\alpha)}\omega(\xi+[\alpha]), \tag{2.2}$$

where $m^i=(m_1, \cdots, m_i-1, \cdots, m_n)$. Therefore

$$\xi^{\gamma_2'}R^{(m)}_{(-\alpha)}(\xi_j\omega(\xi+[\alpha]))=\xi^{\gamma_2}R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])-\sum_{i=1}^n\alpha_{i,j}\xi^{\gamma_2'}R^{(m^i)}_{(-\alpha)}\omega(\xi+[\alpha]),$$

where $\xi^{\gamma_2}=\xi_j\xi^{\gamma_2'}$, $|\gamma_2'|=|\gamma_2|-1$, $|\gamma_2'|+1=|m^i|$.

From the induction hypothesis, $\forall i$

$$\alpha_{i,j}\xi^{\gamma_2}R^{(m^i)}_{(-\alpha)}\omega(\xi+[\alpha]\in\overline{M}(m).$$

So, because $R^{(m)}_{(-\alpha)}$ is a linear operator, what we need to prove is

$$\forall i,\ \alpha_{i,j}\xi^{\gamma_2}R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])\in\overline{M}(m).$$

And then it follows that

$$\xi^{\gamma_2}R^{(m)}_{(-\alpha)}(\xi_j\omega(\xi+[\alpha]))=\xi^{\gamma_2}R^{(m)}_{(-\alpha)}(((\xi+[\alpha])_j\omega(\xi+[\alpha]))$$
$$-\sum_{i=1}^{n}\xi^{\gamma_2}R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])\in\overline{M}(m).$$

For the assertion we have

$$\alpha_{i,j}\xi^{\gamma_2}R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])=\alpha_{i,j}\xi^{\gamma_2}R^{(m^i)}_{(-\alpha)}\omega(\xi+[\alpha])-\alpha_{i,j}\xi^{\gamma_2}\frac{1}{(m_i-1)!}$$
$$\cdot\sum_{|\beta|=m_i-1}\frac{(-\alpha_i)^\beta}{\beta!}R^{m_1}_{-\alpha_1}\cdot\cdots\cdot R^{m_{i-1}}_{-\alpha_{i-1}}\cdot R^{m_{i+1}}_{-\alpha_{i+1}}\cdot\cdots\cdot R^{m_n}_{-\alpha_n}\omega(\xi+[\alpha]),\tag{2.3}$$

and then apply the induction hypothesis to each term on the right hand of (2.3).

(ii) $\gamma_1\neq 0,\ \gamma_2=0$.

The hypothesis on γ_1 implies $\forall i,\ m_i\geq 1$. Therefore

$$(\xi+[\alpha])^{\gamma_1}R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])=(\xi+[\alpha])^{\gamma_1}R^{(m)}_{(-\alpha)}(((\xi+[\alpha])_j\omega(\xi+[\alpha]))$$
$$+\sum_{i=1}^{n}\alpha_{i,j}(\xi+[\alpha])^{\gamma_1}R^{(m^i)}_{(-\alpha)}\omega(\xi+[\alpha]),\tag{2.4}$$

and then the induction hypothesis can be applied to each term on the right hand of (2.4).

(iii) $\gamma_1\neq 0,\ \gamma_2\neq 0$.

In this case by repeatedly applying formulas (2.4), (2.2) and (2.3) we reduce the assertion to the induction hypothesis and to the cases mentioned above.

A corollary to the theorem and [7], Theroem 1 is as follows.

Corollary 1. *Let* $\Omega\in C^\infty(R^K\backslash\{0\})$, *homogeneous of degree* 0, $1\in Z$, $\gamma_i\in Z^K$, $i=1,2$, $0\leq|\gamma_1|\leq\min_{1\leq i\leq n}\{m_i\}$, *and* $\beta\in Z^K$, $|\beta|\leq 1\Rightarrow\int_{S^{K-1}}\Omega(y)y^\beta d\sigma(y)=0$. *Then for* $|m|\leq l+n$ *and*

$$T^{(m)}_{l,\Omega}(f,a)(x)=\text{P. V.}\int\prod_{i=1}^{n}P_{m_i}(a_i,x,y)\cdot\frac{\Omega(x-y)}{|x-y|^{K+l}}f(y)dy,$$

we have

$$\|\partial^{\gamma_1}T^{(m)}_{l,\Omega}(a,\partial^{\gamma_2}f)\|_q\leq C\|f\|_{p_0}\|\nabla^m a\|_p,$$

where $\{p_0, p_i, q\}$ *satisfies index condition* (i), $C=C(K,n,m,l,\Omega,p_0,p_i)$ *is a constant.*

The corresponding conclusions for the cases $K=n=1$, $\gamma_1=0$ or $\gamma_2=0$ were obtained by Calderón, A. P. in [1]. Though the conditions upon Ω were weaker ones in the paper, the complex method used there can not be easily extended to the case $K>1$, $n>1$.

§ 3. The Second Kind of Indexes

At first we prove the following proposition.

Proposition 1. *Suppose* $m\in Z^n$, $\omega\in M^{|m|}$, $\varphi\in C_0^\infty(R^K)$ *svch that* supp $\varphi\subset\{y: 1\leqslant|y|\leqslant 2\}$ *and* $\xi\neq 0\Rightarrow\sum_{j=-\infty}^{\infty}\varphi(2^{-j}\xi)=1$. *Denote* $\omega_j(\xi)=\varphi(2^{-j}\xi)\omega(\xi)$, $K_j=\omega_j^v$, $g_j(x,y)=\left(\prod_{i=1}^{n}P_{m_i}(a_i, x, y)\right)K_j(x-y)$ *and* $G_N(x, y)=\sum_{-N}^{N}g_j(x, y)$. *Then* G_N *satisfies the following condition*

$$\int_{x\in B'(y_0, 2t)}|G_N(x, y)-G_N(x, y_0)|dx\leqslant C, \quad y\in B(y_0, t),$$

where the constant C *is indepfndent of* N *and* t, $B(y_0, t)$ *is the ball of center* y_0 *and radius* t, $B'(y_0, 2t)=R^K\backslash B(y_0, 2t)$.

Proof Because $\omega_j\in M^{|m|}$ with the uniform constants C_β, we have, for $k_0=\left[\frac{K}{2}\right]+1$,

$$2^{-Kj}\int_{R^K}\sum_{|\beta|\leqslant k_0}|2^{j|\beta|}\partial^{\beta+\alpha}\omega_j|^2d\xi\leqslant C2^{2j(|m|-|\alpha|)}. \tag{3.1}$$

From the Parseval equation it follows that

$$\int_{R^K}(1+2^{2j}|x|^2)^{k_0}|x^\alpha K_j(x)|^2dx\leqslant C2^{Kj}2^{2j(|m|-|\alpha|)}. \tag{3.2}$$

Applying the Höler inequality, since $2k_0>K$, we obtain

$$\int_{R^K}|x^\alpha K_j(x)|dx\leqslant\left(\int_{R^K}(1+2^{2j}|x|^2)^{k_0}|x^\alpha K_j(x)|^2dx\right)^{1/2}\cdot\left(\int_{R^K}\frac{1}{(1+2^{2j}|x|^2)^{k_0}}dx\right)^{1/2}$$
$$\leqslant C2^{j(|m|-|\alpha|)}. \tag{3.3}$$

Similarly, for j: $2^jt>1$,

$$\int_{|x|>t}|x^\alpha K_j(x)|dx\leqslant C\left(\int_{|x|>t}\frac{2^{jK}}{(1+2^{2j}|x|^2)^{k_0}}dx\right)^{1/2}\leqslant C(2^jt)^{(K-2k_0)/2}2^{j(|m|-|\alpha|)}. \tag{3.4}$$

Since $|x|^{|\alpha|}\leqslant C\sum_{|\beta|=|\alpha|}|x^\beta|$, there exists

$$\int_{R^K}|x|^{|\alpha|}|K_j(x)|dx\leqslant C2^{j(|m|-|\alpha|)}, \tag{3.5}$$

and for j: $2^jt>1$ we deduce

$$\int_{|x|>t}|x|^{|\alpha|}|K_j(x)|dx\leqslant C(2^jt)^{(K-2k_0)/2}2^{j(|m|-|\alpha|)}. \tag{3.6}$$

To prove the assertion we estimate the integral

$$\int_{x\in B'(y_0, 2t)}|g_j(x, y)-g_j(x, y_0)|dx,$$

and this reduces to the estimates of the following two integrals

$$I_1=\int_{x\in B'(y_0,2t)}\left|\prod_{i=1}^{l-1}P_{m_i}(a_i,x,y)P_{m_l}(a_l,x,y)-P_{m_l}(a_l,x,y_0))\right.$$
$$\left.\cdot\Big(\prod_{i=l+1}^{n}P_{m_i}(a_i,x,y_0)\Big)K_j(x-y_0)\right|dx \tag{3.7}$$

and

$$I_2=\int_{x\in B'(y_0,2t)}\left|\prod_{i=1}^{n}P_{m_i}(a_i,x,y)(K_j(x-y)-K_j(x-y_0))\right|dx. \tag{3.8}$$

To see I_1, applying the equation

$$\nabla_y P_{m_i}(a_i,x,y)=\frac{-1}{(m_i-1)!}\Big(\sum_{j=1}^{n}(x_j-y_j)D_j\Big)^{m_i-1}\nabla a_i(y) \tag{3.9}$$

and the estimate

$$\frac{|P_{m_i}(a_i,x,y)|}{|x-y|^{m_i}}\leqslant C\|^{m_i}a_i\|_\infty, \tag{3.10}$$

we conclude

$$I_1\leqslant C\int_{x\in B'(y_0,2t)}|x-y|^{|m|-1}|y-y_0||K_j(x-y)|dx$$
$$\leqslant Ct\int_{R^K}|x-y|^{|m|-1}|K_j(x-y)|dx\leqslant C2^jt, \tag{3.11}$$

where the last inequality is obtained in terms of (3.5) for $|\alpha|=|m|-1$.

For I_2, firstly we have

$$I_2\leqslant C\int_{R^K}|x|^{|m|}|K_j(x)-K_j(x-\bar{y})|dx \tag{3.12}$$

where $\bar{y}=y-y_0$, $|\bar{y}|\leqslant t$.

Write

$$K_j(x)-K_j(x-\bar{y})=C\int_{R^K}(1-e^{-i\bar{y}\xi})\omega_j(\xi)e^{ix\xi}d\xi.$$

When $2^jt\leqslant1$, since

$$|\partial_\xi^\gamma((1-e^{-i\bar{y}\xi})\omega_j(\xi))|\leqslant Ct2^{j(|m|-|\gamma|+1)},$$

we derive

$$2^{-Kj}\int_{R^K}\sum_{|\beta|\leqslant k_0}|2^{j\beta}\partial^{\beta+\alpha}((1-e^{-i\bar{y}\xi})\omega_j(\xi)|^2d\xi\leqslant Ct2^j2^{j(|m|-|\alpha|)}, \tag{3.13}$$

and

$$\int_{R^K}(1+2^{2j}|x|^2)^{k_0}|x^\alpha(K_j(x)-K_j(x-\bar{y}))|^2dx\leqslant Ct2^j2^{j(|m|-|\alpha|)}2^{Kj}.$$

Then, with a method similar to that used in getting (3.5), we obtain

$$\int_{R^K}|x|^{|m|}|K_j(x)-K_j(x-\bar{y})|dx\leqslant Ct2^j,$$

where the constant C is independent of j, and $2^jt\leqslant1$.

When $2^jt>1$, because

$$\int_{|x-y|>t}\left|\prod_{i=1}^{n}P_{m_i}(a_i,x,y)K_j(x-y)\right|dx\leqslant C\int_{|x-y|>t}|x-y|^{|m|}|K_j(x-y)|dx$$
$$\leqslant C(2^jt)^{(K-2k)/2},$$

it leads to

$$\int_{x\in B'(y_0,2t)} |g_j(x, y)-g_j(x, y_0)|\,dx \leqslant C(2^j t)^{(K-2k)/2}. \tag{3.15}$$

Then (3.11), (3.14) and (3.15) give

$$\int_{x\in B'(y_0,2t)} |G_N(x, y)-G_N(x, y_0)|\,dx \leqslant C\sum_{-\infty}^{\infty} \min\{2^j t, (2^j t)^{(K-2k)/2}\}$$
$$\leqslant C\Big(\sum_{-\infty}^{l_0} 2^j t + \sum_{l_0+1}^{\infty} (2^j t)^{(K-2k)/2}\Big) \leqslant C,$$

where $2^{l_0}t\approx 1$. The proof is thus finished.

The following theorem is obtained.

Theorem 3. *If $\gamma_1, \gamma_2 \in Z^K$, l is an integer, $|\gamma_1|+|\gamma_2|+l=|m|=1$, $0\leqslant|\gamma_1|\leqslant \min\limits_{1\leqslant i\leqslant n}\{m_i\}$ and $\omega\in M^l$, then for p_0: $1<p_0<\infty$ there exists*

$$\|\partial^{\gamma_1} T_{R\{^{m}_{\alpha}\},\omega(\xi)}(a, \partial^{\gamma_2} f)\|_{p_0} \leqslant C\|f\|_{p_0}\|\nabla^m a\|_\infty.$$

where $C=C(K, n, m, C_\beta, p_0, \gamma_i)$ is a constant.

For the proof, the following theorems are needed.

Theorem B. ([6], Theorem 8)

If a linear oprator T satisfies the following conditions:

(i) $f\in C_0^\infty(R^K)$, $x\overline{\in}\operatorname{supp} f \Rightarrow T(f)(x)=\int_{R^K} K(x, y)f(y)dy$;

(ii) $f\in L^2(R^K)\Rightarrow \|T(f)\|_2\leqslant C\|f\|_2$, *$C$ is a constant;*

(iii) *the kernel K satisfies*

$$\int_{x\in B'(y_0,2t)} |K(x, y)-K(x, y_0)|\,dx \leqslant C, \quad y\in B(y_0, t),$$

where C is a constant independent of t; then for T there are the inequalities of strong type (p, p), $1<p<\infty$.

Theorem C. ([4], Theorem 35)

If $\omega\in M^0$, $|\beta|=1$, $m\in Z^n$ and $|m|=1$, then for the operator $\overline{T}_{\xi^\beta R\{^m_\alpha\},\omega(\xi+[\alpha])}(a, \cdot)$ there exists the inequality of strong *type* $(2, 2)$.

Remark. The original form of the theorem is for $K=n=1$. In this more general situation it holds too.

Proof of Theorem 3 Choose $\varphi_1, \cdots, \varphi_K \in C^\infty(R^K\backslash\{0\})$ such that $\forall j$, φ_j is homogeneous of degoee 0, $1=\varphi_1+\cdots+\varphi_K$ on $R^K\backslash\{0\}$, and $\varphi_j(\xi)\neq 0 \Rightarrow |\xi_j|\geqslant \frac{1}{2}\sup(|\xi_1|, \cdots, |\xi_K|)$. We write $\omega(\xi)=\omega_1(\xi)+\cdots+\omega_K(\xi)$ with $\omega_j(\xi)=\omega(\xi)\varphi_j(\xi) =\xi_j\overline{\omega}^j(\xi)$. Now suppose $\omega\in M^1$, and then it follows that $\overline{\omega}^j\in M^0$. We will prove that the conclusion holds in the case of $|l|=|m|=1$, $\gamma_1=\gamma_2=0$. Then by using the same method as in the proof of Theorem 2, we derive the desired results in the extended cases $|\gamma_1|+|\gamma_2|+l=|m|=1$.

From the decomposition of ω mentioned above, we can restrict ourself to the case of $\omega=\xi_j\overline{\omega}$, where $\overline{\omega}\in M^0$. Let $\omega^N=\sum\limits_{j=-N}^{N}\omega_j$, where ω_j are obtained from ω as in

Proposition 1.

Denote $T(f)(x)=T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a, f)(x)=\overline{T}_{R^{(m)}_{(-\alpha)}\omega(\xi+[\alpha])}(a, f)(x)$ and $T_N(f)(x)=T_{R^{(m)}_{(-\alpha)}\omega^N(\xi)}(a, f)(x)$. From the equality

$$T_N(f)=\overline{\overline{T}}_{\xi_{j_0}R^{(m)}_{(-\alpha)}\overline{\omega}^N(\xi+[\alpha])}(a, f)-\overline{T}_{a_{i_0}j_0\overline{\omega}^N(\xi+[\alpha])}(a, f), \tag{3.16}$$

where i_0 satisfies $m_{i_0}=1$, by applying Theorem C to the first term and Mihlin multiplier theorem to the second term on the right hand of (3.16), we obtain

$$\|T_N(f)\|_2\leqslant C\|f\|_2, \tag{3.17}$$

where the constant C is independent of N.

Resorting to [8], Theorem 1, we have

$$T_N(f)(x)=\int_{R^K} G_N(x, y)f(y)dy, \tag{3.18}$$

where G_N is introduced from ω as in Proposition 1. Then the Proposition can be applied to T_N. From Fatou's Lemma we conclude the inequality

$$\|T(f)\|_p\leqslant C\|f\|_p, \quad 1<p<\infty.$$

By virtue of linearity of $T(a, f)$ in a_i, the constant $C=C_0\|\nabla^m a\|_\infty$ where $C_0=C_0(K, n, m, C_B, p)$ is another constant.

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