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ON THE METRICS OF THE RIEMANNIAN MANIFOLDS WHICH ADMIT ISOMETRIC IMBEDDING INTO SPACE OF ANY CONSTANT CURVATURE*

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Abstract

In this paper the term "Riemannian manifold" means that the fundamental quadratic differential form may be indefinite.

Theorem. If for any constant K the Riemannian manifold M^n ($n\geqslant 4$) admits isometric imbedding into certain spaces $S^{n+p}(K)$ of constant curvature K, and if the metric of M^n is analytic, then the metric is expressible either in the form

$$ds^2=\left\{c+\sum c_m x^m+\frac{1}{2}az+f(y)\right\}^{-2}\sum e_i(dx^i)^2,$$

or

$$ds^2=\{c+\sum c_m x^m+\varphi(z)\}^{-2}\sum e_i(dx^i)^2 \quad (e_i=\pm 1),$$

where $y=\sum a_m x^m$, $z\equiv\sum e_m(x^m)^2$, c, a, a_m, c_m: consts., and $f(y)$ and $\varphi(z)$ are analytic functions of the arguments y and z respectively. The converse is true for any constant K, when $n\geqslant 5$ and

$$(\sum e_m a_m^2)f(y)+\frac{1}{2}ay^2+(\sum e_m c_m a_m)y\neq\text{const}.$$

or

$$\{\varphi(z)+c\}/\sqrt{z}\neq\text{const}.$$

1. It is well known that any Riemannian manifold M^n of n dimensions does not admit in general isometric imbedding into an $(n+1)$-dimensional space S^{n+1} of constant curvature. Although M^n admits isometric imbedding into a space S^{n+1} of constant curvature K_0, yet it does not admit in general isometric imbedding into a space S^{n+1} of constant curvature $K_1(\neq K_0)$. We denote, for simplicity, by $S^{n+1}(K_0)$ an $(n+1)$-dimensional space S^{n+1} of constant curvature K_0. In a recent paper[1] we have proved for any Riemannian manifold with indefinite fundamental quadratic form that if M^n ($n\geqslant 4$) admits isometrie imbedding both into S^{n+1} (K_0) and S^{n+1} (K_1), M^n admits in general isometric imbedding into space S^{n+1} of any constant curvature. We have also shown that the space S^n (K) of constant curvature K

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admits isometric imbedding into space $S^{n+1}(K_1)$ of any constant curvature K_1. Consequently, if M^n admits isometric imbedding into any space $S^{n+1}(K)$ of constant curvature K, it admits also isometric imbedding into space $S^{n+p}(K)$ $(p>0)$, of any constant curvature K. This class of Riemannian manifolds M^n is a generalization of the class of Riemannian manifolds of constant curvature. In this paper, we determine completely the Riemannian metrics of this class of manifolds M^n.

The Gauss equations of a hypersurface M^n of the space $S^{n+1}(K_0)$ are

$$R_{ijkl}=e(b_{ik}b_{jl}-b_{il}b_{jk})+K_0(g_{ik}g_{jl}-g_{il}g_{jk}), \tag{1}$$

where $e=\pm 1$, g_{ij} is the Riemannian metric of M^n and R_{ijkl} the Riemannian curvature. The metric of M^n is $ds^2=eg_{ij}dx^i dx^j$, where $e=1$ when the fundamental quadratic form $g_{ij}dx^i dx^j$ is positive definite. In this paper we assume that $g_{ij}dx^i dx^j$ may be positive definite or not.

We put

$$T_{ijkl}(K_0)\equiv R_{ijkl}-K_0(g_{ik}g_{jl}-g_{il}g_{jk}), \tag{2}$$

$$T_{jk}(K_0)\equiv g^{il}T_{ijkl}(K_0),\ T(K_0)\equiv g^{jk}T_{jk}=R+n(n-1)K_0, \tag{3}$$

$$d_{jk}\equiv\frac{1}{n-2}R_{jk}-\frac{R}{2(n-1)(n-2)}g_{jk}, \tag{4}$$

$$\Delta\equiv g^{ij}d_{ij}, \tag{5}$$

$$D_{ij}\equiv g^{pk}d_{pi}d_{kj},\ D\equiv g^{ij}D_{ij}. \tag{6}$$

About this class of manifolds M^n which we shall consider throughout this paper we have already established the following theorem[1]:

Theorem A. *If a Riemannian manifold M^n $(n\geqslant 4)$ admits isometric imbedding both into $S^{n+1}(K_0)$ and $S^{n+1}(K_1)$, M^n is conformally flat. Conversely, if any conformally flat M^n $(n\geqslant 4)$ admits isometric imbedding into $S^{n+1}(K_0)$, and if for another constant K, the rank r of the matrix $(T_{ijkl}(K))$ is $\geqslant 4$, $T(K)\neq 0$, M^n admits also isometric imbedding into an $S^{n+1}(K)$ and this class of M^n is characterized by the following conditions:*

$$R_{ijkl}=g_{il}d_{jk}+g_{jk}d_{il}-g_{ik}d_{jl}-g_{jl}d_{ik}, \tag{7}$$

$$(n-2)(\rho-\Delta)R_{jlim}=(n-2)^2(d_{ji}d_{lm}-d_{jm}d_{li})+(\rho-\Delta)^2(g_{ji}g_{lm}-g_{jm}g_{li}). \tag{8}$$

2. We establish the following theorems, which are useful in the determination of the metrics of the Riemannian manifolds M^n which admit isometric imbedding into space of any constant curvature.

Theorem 1. *If M^n $(n\geqslant 4)$ is a Riemannian manifold whose Riemannian curvature R_{ijkl} satisfies the conditions (7) and (8), then the metric of M^n is expressible in the form*

$$ds^2=e^{2\sigma}\sum_{i=1}^{n}e_i(dx^i)^2 \quad (e_i=\pm 1), \tag{9}$$

and there exist $n+1$ functions $\lambda, \lambda_1, \cdots, \lambda_n$ such that the function σ is a solution of the *following system of partial differential equations*

$$\frac{\partial^2\sigma}{\partial x^i\partial x^j}-\frac{\partial\sigma}{\partial x^i}\frac{\partial\sigma}{\partial x^j}=-\lambda_i\lambda_j+e_i\delta_{ij}\lambda \quad (i,\ j=1,\ \cdots,\ n). \tag{10}$$

Conversely, if σ is any solution of (10) *in which $\lambda_1, \cdots, \lambda_n, \lambda$ are $n+1$ arbitrary functions, then the Riemannian manifold M^n with the corresponding metric* (9) *satisfies the conditions* (7) *and* (8).

From Theorem 1, it follows that the metric of any Riemannian manifold M^n which admits isometric imbedding into space of any constant curvature is expressible in the form of (9) with σ in the sense of the theorem.

We proceed to prove this theorem as follows.

Contracting (7) by g^{il} we obtain (4). Contracting (8) by g^{jm} we have

$$(n-2)(\rho-\Delta)R_{il}=(n-2)^2(D_{il}-\Delta d_{il})-(n-1)(\rho-\Delta)^2g_{il}. \tag{11}$$

Contracting this equation again by g^{il} we have

$$(n-2)(\rho-\Delta)R=(n-2)^2(D-\Delta^2)-(\rho-\Delta)^2n(n-1). \tag{12}$$

When (4) is contracted by g^{jk}, we have

$$R=2(n-1)\Delta. \tag{13}$$

When R is eliminated from (12) and (11), we get a quadratic equation in ρ

$$\rho^2-\frac{4}{n}\Delta\rho+\frac{1}{n-1}\left[\Delta^2-\frac{(n-2)^2}{n}D\right]=0, \tag{14}$$

that is, the function ρ in (8) must be a root of this equation.

From the condition (7) it follows that M^n ($n\geqslant 3$) is conformally flat and its metric is reducible to (9). Conversely, if the metric of M^n is reducible to (9), the condition (7) is satisfied. In this case

$$g_{ij}=e_ie^{2\sigma}\delta_{ij},\ g^{ij}=e_ie^{-2\sigma}\delta_{ij}, \tag{15}$$

where δ_{ij} denote the Kronecker deltas.

The components of the Riemannian curvature tensor R_{pqrs} of M^n are obtained from (15) by a direct computation as follows:

$$R_{hijk}=0 \quad (h,\ i,\ j,\ k\neq), \tag{16}$$

$$R_{hiik}=e_ie^{2\sigma}(\sigma_{hk}-\sigma_h\sigma_k), \quad (h,\ i,\ k\neq), \tag{17}$$

$$R_{hiih}=e^{2\sigma}\left\{e_i(\sigma_{hh}-\sigma_h^2)+e_h(\sigma_{ii}-\sigma_i^2)+e_ie_h\sum_{m=1}^{n}e_m\sigma_m^2\right\} \quad (i\neq h), \tag{18}$$

where

$$\sigma_h\equiv\frac{\partial\sigma}{\partial x^h},\ \sigma_{hk}\equiv\frac{\partial^2\sigma}{\partial x^h\partial x^k}.$$

From (15)—(18) we have

$$R_{ij}=(n-2)(\sigma_{ij}-\sigma_i\sigma_j) \quad (i\neq j), \tag{19}$$

$$R_{ii}=(n-2)(\sigma_{ii}-\sigma_i^2)+e_i\sum e_m\sigma_{mm}+e_i(n-2)\sum e_m\sigma_m^2, \tag{20}$$

$$R=(n-1)e^{-2\sigma}[2\sum e_m\sigma_{mm}+(n-2)\sum e_m\sigma_m^2]. \tag{21}$$

From (4) and (19) we have

$$d_{jk}=\sigma_{jk}-\sigma_j\sigma_k \quad (j\neq k). \tag{22}$$

From (4), (20) and (21) we have

$$d_{jj}=\sigma_{jj}-\sigma_j^2+\frac{1}{2}e_j\sum e_m\sigma_m^2. \tag{23}$$

From (13) and (21) we have

$$\Delta=e^{-2\sigma}\Big[\sum e_m\sigma_{mm}+\frac{1}{2}(n-2)\sum e_m\sigma_m^2\Big]. \tag{24}$$

From (8), (16) and (15) we have

$$d_{ji}d_{lm}-d_{jm}d_{li}=0 \quad (j,\ l,\ i,\ m\neq). \tag{25}$$

From (8), (15) and (17) we have

$$(n-2)(\rho-\Delta)e_ie^{2\sigma}(\sigma_{jm}-\sigma_j\sigma_m)=(n-2)^2(d_{ji}d_{im}-d_{jm}d_{ii}) \quad (i,\ j,\ m\neq). \tag{26}$$

From (8) and (18) we have

$$\begin{aligned}&(n-2)(\rho-\Delta)e^{2\sigma}\{e_i(\sigma_{hh}-\sigma_h^2)+e_h(\sigma_{ii}-\sigma_i^2)+e_ie_h\sum e_m\sigma_m^2\}\\&=(n-2)^2(d_{hi}^2-d_{hh}d_{ii})-e_ie_he^{4\sigma}(\rho-\Delta)^2 \quad (i\neq h).\end{aligned} \tag{27}$$

Hence the conditions (7) and (8) are equivalent to (9) and (25)—(27).

From (25) we find that d_{ji}/d_{jm} is independent of j and the n functions $\lambda_1, \cdots, \lambda_n$ can be chosen such that

$$d_{jk}=-\lambda_j\lambda_k \quad (j\neq k). \tag{28}$$

From (28), (22) and (26) become

$$\sigma_{jk}-\sigma_j\sigma_k=-\lambda_j\lambda_k \quad (j\neq k), \tag{29}$$

and

$$-(n-2)(\rho-\Delta)(e_ie^{2\sigma}\lambda_j\lambda_m=(n-2)^2(\lambda_i^2+d_{ii})\lambda_j\lambda_m, \tag{30}$$

respectively.

From (4) and (13) we have

$$R_{ij}=(n-2)d_{ij}+\Delta g_{ij}. \tag{31}$$

Then (11) reduces to

$$(n-2)^2D_{ij}-(n-2)^2\rho d_{ij}-(\rho-\Delta)[(n-1)\rho-\Delta]g_{ij}=0.$$

Hence

$$D_{ij}=\rho d_{ij}=-\rho\lambda_i\lambda_j \quad (i\neq j). \tag{32}$$

We have by definition for $j\neq k$,

$$D_{jk}=g^{lp}d_{lj}d_{pk}=g^{ll}d_{lj}d_{lk}=g^{jj}d_{jj}d_{jk}+g^{kk}d_{jk}d_{kk}+\sum_{l\neq j,k}g^{ll}d_{lj}d_{lk}.$$

Substituting (15), (23) and (28) in the above equation we have

$$\begin{aligned}e^{2\sigma}D_{jk}=&(\sum e_m\sigma_m^2-\sum e_m\lambda_m^2+e_j\lambda_j^2+e_j\sigma_{jj}-e_j\sigma_j^2+e_k\lambda_k^2\\&+e_k\sigma_{kk}-e_k\sigma_k^2)(-\lambda_j\lambda_k), \quad (j\neq k).\end{aligned} \tag{33}$$

Comparing (32) with (33) we obtain

$$\rho e^{2\sigma}=\sum e_m\sigma_m^2-\sum e_m\lambda_m^2+e_j\lambda_j^2+e_j\sigma_{jj}-e_j\sigma_j^2+e_k\lambda_k^2+e_k\sigma_{kk}-e_k\sigma_k^2.$$

Since the left-hand member is independent of j and k, it follows consequently that

$$e_k\lambda_k^2+e_k\sigma_{kk}-e_k\sigma_k^2=e_l\lambda_l^2+e_l\sigma_{ll}-e_l\sigma_l^2, \tag{34}$$

and the equation becomes

$$\rho e^{2\sigma}=\sum e_m\sigma_m^2-\sum e_m\lambda_m^2+\frac{2}{n}\left(\sum e_m\lambda_m^2+\sum e_m\sigma_{mm}-\sum e_m\sigma_m^2\right),$$

or

$$\rho e^{2\sigma}=\frac{1}{n}\left[(n-2)\sum e_m\sigma_m^2+2\sum e_m\sigma_{mm}-(n-2)\sum e_m\lambda_m^2\right]. \tag{35}$$

We have by definition

$$\begin{aligned}D_{jj}&=g^{ll}d_{lj}d_{lj}=g^{jj}(d_{jj})^2+\sum_{l\neq j}g^{ll}d_{lj}d_{lj}\\&=e^{-2\sigma}e_j\left(\sigma_{jj}-\sigma_j^2+\frac{1}{2}e_j\sum e_m\sigma_m^2\right)^2+\sum_{l\neq j}e^{-2\sigma}e_l\lambda_l^2\lambda_j^2,\end{aligned}$$

or

$$\begin{aligned}e^{2\sigma}D_{jj}&=\left(\sum e_m\lambda_m^2\right)\lambda_j^2-e_j\lambda_j^4+e_j\left(\sigma_{jj}-\sigma_j^2+\frac{1}{2}e_j\sum e_m\sigma_m^2\right)^2\\&=\left(\sum e_m\lambda_m^2\right)\lambda_j^2-e_j\lambda_j^4+e_j\left(e_j\sigma_{jj}-e_j\sigma_j^2+\frac{1}{2}\sum e_m\sigma_m^2\right)^2.\end{aligned}$$

By means of (34) we have

$$e_j\sigma_{jj}-e_j\sigma_j^2=\frac{1}{n}\left(\sum e_m\lambda_m^2+\sum e_m\sigma_{mm}-\sum e_m\sigma_m^2\right)-e_j\lambda_j^2.$$

Hence

$$\begin{aligned}e^{2\sigma}D_{jj}&=\left(\sum e_m\lambda_m^2\right)\lambda_j^2-e_j\lambda_j^4+e_j\left[\frac{1}{n}\left(\sum e_m\lambda_m^2+\sum e_m\sigma_{mm}+\frac{n-2}{2}\sum e_m\sigma_m^2\right)-e_j\lambda_j^2\right]^2\\&=\frac{1}{n}\left[(n-2)\sum e_m\lambda_m^2-2\sum e_m\sigma_{mm}-(n-2)\sum e_m\sigma_m^2\right]\lambda_j^2\\&\quad+e_j\frac{1}{n^2}\left(\sum e_m\lambda_m^2+\sum e_m\sigma_{mm}+\frac{n-2}{2}\sum e_m\sigma_m^2\right)^2.\end{aligned}$$

From the equation (6) for the definition of D, we obtain

$$\begin{aligned}e^{4\sigma}D&=\frac{1}{n}\left[(n-2)\sum e_m\lambda_m^2-2\sum e_m\sigma_{mm}-(n-2)\sum e_m\sigma_m^2\right]\cdot\left(\sum e_m\lambda_m^2\right)\\&\quad+\frac{1}{n}\left(\sum e_m\lambda_m^2+\sum e_m\sigma_{mm}+\frac{n-2}{2}\sum e_m\sigma_m^2\right)^2\\&=\frac{n-1}{n}\left(\sum e_m\lambda_m^2\right)^2+\frac{1}{n}\left(\sum e_m\sigma_{mm}+\frac{n-2}{2}\sum e_m\sigma_m^2\right)^2.\end{aligned}$$

If we put

$$A\equiv\sum_{m=1}^n e_m\lambda_m^2,\quad B\equiv\sum_{m=1}^n e_m\sigma_m^2,\quad C\equiv\sum_{m=1}^n e_m\sigma_{mm}, \tag{36}$$

we can write the above equation and (24), (35) respectively as follows:

$$De^{4\sigma}=\frac{n-1}{n}A^2+\frac{1}{n}\left(C+\frac{n-2}{n}B\right)^2, \tag{37}$$

$$\Delta e^{2\sigma}=C+\frac{n-2}{2}B, \tag{38}$$

$$\rho e^{2\sigma}=\frac{1}{n}\left[(n-2)B+2C-(n-2)A\right]. \tag{39}$$

We observe that (14) is an algebraic consequence of (37)—(39).

From (34) we have

$$e_i\lambda_i^2+e_i\sigma_{ii}-e_i\sigma_i^2=\frac{1}{n}(A+C-B). \tag{40}$$

In consequence of (23), (38)—(40) the equation (30) is satisfied identically. Thus we have proved that the equations (25) and (26) are an algebraic consequence of the equations (29) and (40). Moreover, we can show that (27) is also an algebraic consequence of (29) and (40).

From (38) and (39) we have

$$(\rho-\Delta)e^{2\sigma}=-\frac{n-2}{n}\left(A+C+\frac{n-2}{2}B\right). \tag{41}$$

From (23), (40) and (39) we have

$$e_i d_{ii}=e_i\sigma_{ii}-e_i\sigma_i^2+\frac{1}{2}B=\frac{1}{n}(A+C-B)-e_i\lambda_i^2+\frac{1}{2}B$$
$$=-\frac{1}{n-2}(\rho-\Delta)e^{2\sigma}-e_i\lambda_i^2. \tag{42}$$

By a direct calculation it is easy to show that both the left and right-hand members of (27) are equal to

$$-2e^{4\sigma}(\rho-\Delta)^2-(n-2)(\rho-\Delta)e^{2\sigma}(e_i\lambda_i^2+e_h\lambda_h^2),$$

in consequence of (28), (40) and (42).

If we put

$$\lambda\equiv\frac{1}{n}(A+C-B), \tag{43}$$

it is clearly that the system of equations (29) and (40) are equivalent to (10).

We have shown that the conditions (7) and (8) are equivalent to the system of equations (9), (25)—(27) which are equivalent to the system of equations (9), (29) and (40). Hence the conditions (7) and (8) are equivalent to (9) and the system of partial differential equations (10). Thus the first part of Theorem 1 is proved.

Conversely, if σ is any solution of the system of partial differential equation (10) corresponding to the $n+1$ arbitrary functions $\lambda, \lambda_1, \cdots, \lambda_n$, by means of the functions σ and $\lambda_1, \cdots, \lambda_n$, we define A, B, C by (36). From (10) we have $\lambda=\frac{1}{n}(A+C-B)$. Define Δ and ρ by (38) and (39). Define d_{jk} $(j\neq k)$ and d_{jj} by (28) and (23). From (9) we have $g_{ij}=e^{2\sigma}e_i\delta_{ij}$. From the preceding considerations we have shown that Δ, ρ, d_{ij} and the Riemannian curvature R_{ijkl} formed with respect to g_{ij} satisfy (7) and (8) identically.

Thus Theorem 1 is proved.

If we effect the transformation $e^{-\sigma}=\mu$, (9) and (10) become

$$ds^2=\frac{1}{\mu^2}\sum_{i=1}^{n}e_i(dx^i)^2, \quad (e_i=\pm1), \tag{9'}$$

and

$$\frac{\partial^2\mu}{\partial x^i\partial x^j}=\mu(\lambda_i\lambda_j-e_i\delta_{ij}\lambda) \quad (i, j=1, \cdots, n). \tag{10'}$$

3. In order to determine the solutions of the system of partial differential

equations (10′) or (10), we put

$$y \equiv \sum_{m=1}^{n} a_m x^m \quad (a_m = \text{consts.},\ m=1, \cdots, n),$$

$$z \equiv \sum_{m=1}^{n} e_m (x^m)^2, \tag{44}$$

and prove the following

Theorem 2. *Any analytic solution μ of the system of partial differential equations (10′) is reducible either to the form*

$$\mu = c + \sum c_m x^m + az + f(y), \tag{45}$$

or to the form

$$\mu = c + \sum c_m x^m + \varphi(z), \tag{46}$$

where $c, c_1, \cdots, c_n$ and a are arbitrary constants, f and φ are arbitrary analytic functions of the arguments y and z respectively.

It is easy to show that both (45) and (46) are solutions of (10′). Since we have for (45)

$$\mu_i = \frac{\partial \mu}{\partial x^i} = c_i + 2ae_i x^i + f'(y) a_i,$$

$$\mu_{ij} = \frac{\partial^2 \mu}{\partial x^i \partial x^j} = f''(y) a_i a_j \quad (i \neq j),$$

$$\mu_{ii} = \frac{\partial^2 \mu}{\partial x^{i2}} = 2ae_i + f''(y) a_i^2,$$

we see that μ satisfies (10′) for $\lambda_i = \sqrt{\frac{f''(y)}{\mu}}\, a_i$ and $\lambda = -\frac{2a}{\mu}$; Since we have for (46)

$$\begin{aligned} &\mu_i = c_i + \varphi'(z) 2e_i x^i,\ \mu_{ij} = \varphi''(z) 4e_i e_j x^i x^j \\ &\mu_{ii} = \varphi''(z) 4(x^i)^2 + 2\varphi'(z) e_i, \end{aligned} \quad (i \neq j),$$

we see that μ satisfies (10′) for $\lambda_i = \sqrt{\frac{\varphi''(z)}{\mu}}\, 2e_i x^i$ and $\lambda = -\frac{2}{\mu}\varphi''(z)$.

Now we prove that any analytic solution of (10′) is expressible either to (45) or to (46).

We prove first a series of lemmas.

Lemma 1. *A necessary and sufficient condition that $A_{ij} = a_i a_j$ is that*

$$A_{ij} A_{lk} - A_{lj} A_{ik} = 0 \quad (i, j, l, k = 1, \cdots, n;\ i, j, l, k \neq). \tag{47}$$

Since from (47) it follows that A_{ij}/A_{ik} is independent of i and A_{ij} is symmetric with respect to the subscripts, we can choose n functions $a_1, \cdots, a_n$ such that $A_{ij} = a_i a_j$ $(i, j = 1, \cdots, n)$.

Lemma 2. *If $A_{ij} = a_i a_j$, $B_{ij} = b_i b_j$ and $A_{ij} + B_{ij} = c_i c_j$ $(i, j = 1, \cdots, n;\ j \neq j)$, we have $b_i = \alpha a_i$.*

Since

$$(a_i a_j + b_i b_j)(a_l a_k + b_l b_k) - (a_l a_j + b_l b_j)(a_i a_k + b_i b_k) = 0 \quad (i, j, l, k \neq),$$

we have

$$(a_k b_j - a_j b_k)(a_i b_l - a_l b_i) = 0.$$

Lemma 3. *If $P_{ij}^k (i, j=1, \cdots, n; k=1, 2, \cdots; i \neq j)$ are homogeneous polynomials in the x's of the k-th order, $P_{ij}^k = P_{ji}^k$, and if $\sum_{k=r}^{\infty} P_{ij}^k$ is a convergent power series in the x's and*

$$\sum_{k=r}^{\infty} P_{ij}^k(x) = g_i(x) g_j(x), \quad P_{ij}^{r+1}(x) = k_i(x) k_j(x),$$

we have

$$P_{ij}^r = f_i(x) f_j(x), \quad P_{ij}^r + P_{ij}^{r+1} = h_i(x) h_j(x)$$

and

$$k_i(x) = \alpha f_i(x).$$

Proof. From $\sum_{k=r}^{\infty} P_{ij}^k = g_i(x) g_j(x)$ and Lemma 1 we have

$$\Big(\sum_{t=r}^{\infty} P_{ij}^t\Big)\Big(\sum_{t=r}^{\infty} P_{lk}^t\Big) - \Big(\sum_{t=r}^{\infty} P_{lj}^t\Big)\Big(\sum_{t=r}^{\infty} P_{ik}^t\Big) = 0 \quad (i, j, l, k \neq)$$

for any values of the x's, and consequently

$$P_{ij}^r P_{lk}^r - P_{lj}^r P_{ik}^r = 0,$$

$$P_{ij}^r P_{lk}^{r+1} + P_{lk}^r P_{ij}^{r+1} - P_{lj}^r P_{ik}^{r+1} - P_{lk}^r P_{jr}^{r+1} = 0.$$

By the assumption we have

$$P_{ij}^{r+1} P_{lk}^{r+1} - P_{lj}^{r+1} P_{ik}^{r+1} = 0.$$

Hence

$$(P_{ij}^r + P_{ij}^{r+1})(P_{lk}^r + P_{lk}^{r+1}) - (P_{lj}^r + P_{lj}^{r+1})(P_{ik}^r + P_{ik}^{r+1}) = 0.$$

By Lemma 1 there exist functions $h_i(x)$ and $h_j(x)$ such that

$$P_{ij}^r + P_{ij}^{r+1} = h_i(x) h_j(x).$$

Since $P_{ij}^{r+1} = k_i(x) k_j(x)$ and $P_{ij}^r P_{lk}^r - P_{lj}^r P_{ik}^r = 0$, we have $P_{ij}^r = f_i(x) f_j(x)$. By Lemma 2 there is a function α such that $k_i(x) = \alpha f_i(x)$.

Lemma 4. *If μ is any solution of (10'), then $\omega = \mu + c + \sum c_m x^m + a \sum e_m (x^m)^2$ for any arbitrary constants $c, c_1, \cdots, c_n$ and a are also solutions of (10').*

Proof Putting

$$\nu \equiv c + \sum c_m x^m + a \sum e_m (x^m)^2,$$

we have

$$\nu_i = 2a e_i x^i, \ \nu_{ii} = 2a e_i, \ \nu_{ij} = 0, \quad (i \neq j),$$

where

$$\nu_i \equiv \frac{\partial \nu}{\partial x^i}, \quad \nu_{ij} \equiv \frac{\partial^2 \nu}{\partial x^i \partial x^j}, \text{ etc.}$$

The notations such as $\mu_{ij} \equiv \dfrac{\partial^2 \mu}{\partial x^i \partial x^j}$ will be used frequently throughout this paper, if no ambiguity would arise.

From (10') $\mu_{ij} = \lambda_i \lambda_j \mu$ $(i \neq j)$. Hence $\omega_{ij} = \mu_{ij} + \nu_{ij} = \lambda_i \lambda_j \mu$. If we put

$$\lambda_i' = \sqrt{\frac{\mu}{\omega}}\, \lambda_i,$$

we have

$$\omega_{ij} = \lambda_i' \lambda_j' \omega.$$

Also from (10')

$$\omega_{ii}=\mu_{ii}+\nu_{ii}=(\lambda_i^2-e_i\lambda)\mu+2ae_i=\lambda_i^2\mu-e_i(\lambda\mu-2a)=(\lambda_i'^2-e_i\lambda')\omega,$$

where $\lambda'=\frac{1}{\omega}(\lambda\mu-2a)$.

Lemma 5. *In order that a power series of the form*

$$\mu=c+\sum_m c_m x^m+\frac{1}{2!}\sum_{i,j}a_{ij}x^i x^j+\frac{1}{3!}\sum_{i,j,k}a_{ijk}x^i x^j x^k+\cdots$$

be a solution of (10′), *the constants* $c_1,\cdots,c_n$ *and* c *may be chosen arbitrary, but the terms of the second order* $\sum a_{ij}x^i x^j$ *must have the form*

$$\sum a_{ij}x^i x^j=a\sum e_m(x^m)^2+(\sum a_m x^m)^2. \tag{48}$$

Proof Let the coefficients of the power series a_{ij}, a_{ijk}, $\cdots$, are symmetric with respect to their subscripts. We have

$$\mu_i=c_i+\sum_j a_{ij}x^j+\frac{1}{2!}\sum_{j,k}a_{ijk}x^j x^k+\cdots,$$

$$\mu_{ij}=a_{ij}+\sum_k a_{ijk}x^k+\cdots,\quad (i\neq j),$$

$$\mu_{ii}=a_{ii}+\sum_p a_{iip}x^p+\cdots.$$

If μ is a solution of (10′), we have $\mu_{ij}=\lambda_i\lambda_j\mu$ for $i\neq j$. By Lemma 1 it is necessary and sufficient that

$$\mu_{ij}\mu_{lk}-\mu_{lj}\mu_{ik}=0\quad (i,\ j,\ l,\ k\neq).$$

Hence

$$a_{ij}a_{lk}-a_{lj}a_{ik}=0.$$

By Lemma 1 there are n constants $a_1,\cdots,a_n$ such that

$$a_{ij}=a_i a_j\quad (i\neq j).$$

From $\mu_{ij}=\lambda_i\lambda_j\mu$, it is seen that $\lambda_i\sqrt{\mu}$ must take the form

$$\lambda_i\sqrt{\mu}=(a_i^2+\sum_k p_{ik}x^k+\cdots)^{\frac{1}{2}}.$$

On the other hand, from (10′)

$$\mu_{ii}=a_{ii}+\sum_p a_{iip}x^p+\cdots=\lambda_i^2\mu-e_i\lambda\mu=a_i^2+\sum_k p_{ik}x^k+\cdots-e_i\lambda\mu,$$

that is

$$(a_{ii}-a_i^2)+\sum_p(a_{iip}-p_{ip})x^p+\cdots=-e_i\lambda\mu,$$

consequently $e_i(a_{ii}-a_i^2)=a$ and

$$\begin{aligned}\mu&=c+\sum c_m x^m+\frac{1}{2}\sum a_{ij}x^i x^j+\cdots\\&=c+\sum c_m x^m+\frac{1}{2}\sum a_{mm}(x^m)^2+\frac{1}{2}\sum_{i\neq j}a_i a_j x^i x^j+\cdots\\&=c+\sum c_m x^m+\frac{1}{2}\sum a_{mm}(x^m)^2+\frac{1}{2}[(\sum a_m x^m)-\sum a_m^2(x^m)^2]+\cdots\\&=c+\sum c_m x^m+\frac{1}{2}\sum(a_{mm}-a_m^2)(x^m)^2+\frac{1}{2}(\sum a_m x^m)^2+\cdots\\&=c+\sum c_m x^m+\frac{1}{2}ae\sum_m(x^m)^2+\frac{1}{2}(\sum a_m x^m)^2+\cdots.\end{aligned}$$

Lemma 6. *If $\mu(x^1, \cdots, x^n)\equiv\mu(x)$ is a solution of (10′), then $\mu(x^1+c^1, \cdots, x^n+c^n)\equiv\mu(x+c)$ for arbitrary constants $c^1, \cdots, c^n$ is also a solution of (10′).*

Lemma 6 follows from the fact that the metric (9′) is invariant under the transformations

$$x^i=x'^i-c^i \quad (i=1, \cdots, n),$$

for arbitrary values of the c's.

We proceed now to prove Theorem 2.

Let the power series

$$\mu(x)=c+\sum c_m x^m+\frac{1}{2!}a\sum e_m(x^m)^2+\frac{1}{2!}(\sum a_m x^m)^2$$
$$+\frac{1}{3!}\sum a_{ijk}x^i x^j x^k+\frac{1}{4!}\sum a_{ijkl}x^i x^j x^k x^l+\cdots \tag{49}$$

be a solution of (10′). Let the coefficients a_{ijk}, a_{ijkl}, $\cdots$, which are to be determined, be symmetric with respect to their subscripts. In what follows we use the ordinary summation convention, if no ambiguity would arise.

From (49) we have

$$\mu(x+c)=c'+c'_m x^m+\frac{1}{2!}a'_{ij}x^i x^j+\frac{1}{3!}a'_{ijk}x^i x^j x^k+\frac{1}{4!}a'_{ijkl}x^i x^j x^k x^l+\cdots, \tag{50}$$

where

$$a'_{ij}=\frac{1}{2}ae_i\delta_{ij}+a_i a_j+a_{ijp}c^p+\frac{1}{2}a_{ijpq}c^p c^q+\cdots, \tag{51}$$

$$a'_{ijp}=a_{ijp}+a_{ijpq}c^q+\frac{1}{2}a_{ijpqr}c^q c^r+\cdots, \tag{52}$$

$$a'_{ijpq}=a_{ijpq}+a_{ijpqr}c^r+\cdots. \tag{53}$$

From (50) we have

$$\mu_{ij}(x+c)=a'_{ij}+a'_{ijp}x^p+\frac{1}{2!}a'_{ijpq}x^p x^q+\cdots. \tag{54}$$

By Lemma 1 and $\mu_{ij}=\lambda_i\lambda_j\mu(i\neq j)$, we have

$$\mu_{ij}\mu_{lk}-\mu_{lj}\mu_{ik}=0 \quad (i, j, l, k\neq). \tag{55}$$

By Lemma 6, (55) is satisfied for any set of constants (c). Hence

$$a'_{ij}a'_{lk}-a'_{lj}a'_{ik}=0, \tag{56}$$

and then $a'_{ij}=a'_i a'_j$. It is possible to choose (c) such that $a'_{ij}=0$ $(i, j=1, \cdots, n;\ i\neq j)$. In fact, then n constants $c_1, \cdots, c_n$ are subjected to only $n-1$ conditions and therefore at least one of the c's is arbitrary. In this case (51) becomes

$$-a_{ij}=a_{ijp}c^p+\cdots, \quad (i\neq j).$$

By the analogeous equations (56) for a_{ij}, we have

$$(a_{ijp}a_{lkp}-a_{ljp}a_{ikp})(c^p)^2=0.$$

It follows from Lemma 1 that

$$a_{ijp}c^p=b_i b_j.$$

If $a_i a_j\neq 0$, appling Lemma 3 to (51) we have

$$b_i b_j = \alpha(c) a_i a_j.$$

Hence

$$a_{ijp} c^p = (\sum r_p c^p) a_i a_j,$$

or

$$a_{ijp} = a_i a_j r_p.$$

Since a_{ijp} is symmetric with respect to i, j and p, we have

$$r_p = \alpha a_p$$

for $i, j, p \neq$. Therefore

$$a_{ijp} = \alpha a_i a_j a_p \quad (i, j, \mathrm{p} = 1, \cdots, n;\ i \neq j), \tag{57}$$

and a_{iii} $(i = 1, \cdots, n)$ is to be determined.

If $a_i a_j = 0$ $(i, j = 1, \cdots, n;\ i \neq j)$, but there is at least a set of indices i, j, p such that $a_{ijp} \neq 0$, we can prove by the analogeous process that $a_{ijp} = \alpha b_i b_j b_p$, that is, the relations (57) hold for any case.

If (57) is substituted in (51) we obtain

$$a'_{ij} = (1 + \alpha \sum a_m a^m) a_i a_j + \frac{1}{2} a_{ijpq} c^p c^q + \frac{1}{3!} a_{ijpqr} c^p c^q c^r + \cdots \quad (i \neq j). \tag{58}$$

If we choose (c) such that $1 + \alpha \sum a_m c^m = 0$, (58) becomes

$$a'_{ij} = \frac{1}{2} a_{ijpq} c^p c^q + \cdots.$$

Since $n-1$ of the c's are arbitrary, we have from (56)

$$a_{ijpq} c^p c^q = \rho(c) b_i b_j = (\sum \beta_{pq} c^p c^q) b_i b_j.$$

Substitute these relations in (51) we have

$$a'_{ij} = a_i a_j + \alpha (\sum a_m c^m) a_i a_j + \frac{1}{2} (\sum \beta_{pq} c^p c^q) b_i b_j + \cdots.$$

From (56) and by making use of Lemma 3 or by direct computation we see that $b_i b_j$ must be proportional to $a_i a_j$ or $a_{ijpq} = \gamma_{pq} a_i a_j$.

When $i, j, p, q \neq$, we have

$$a_{ijpq} = \gamma_{pq} a_i a_j = \gamma_{ij} a_p a_q,$$

that is, $\gamma_{pq} = \rho a_p a_q$.

From $a_{ijpp} = \gamma_{pp} a_i a_j = \gamma_{jp} a_i a_p = \rho a_i a_j a_p^2$, we have $\gamma_{pp} = \rho a_p^2$. Hence $a_{ijpq} = \rho a_i a_j a_p a_q$ $(i \neq j)$, and a_{iiii} is to be determined.

In like manner we obtain

$$a_{ijpqr} = \gamma a_i a_j a_p a_q a_r, \cdots, \quad (i \neq j).$$

Hence if there are a_i and a_j $(i \neq j)$ such that $a_i a_j \neq 0$, the solution μ in power series (49) of (10′) must take the form

$$\begin{aligned} \mu = c + \sum c_m x^m + \frac{a}{2} \sum e_m (x^m)^2 + \frac{\alpha}{2!} (\sum a_m x^m)^2 \\ + \frac{\beta}{3!} \sum a_{mmm} (x^m)^3 + \frac{\beta}{3!} [(\sum a_m x^m)^3 - \sum a_m^3 (x^m)^3] \\ + \frac{\gamma}{4!} \sum a_{mmmm} (x^m)^4 + \frac{\gamma}{4!} [(\sum a_m x^m)^4 - \sum a_m^4 (x^m)^4] + \cdots. \end{aligned}$$

It follows that

$$\mu_{ij}=\alpha a_i a_j+\beta(\sum a_m x^m)a_i a_j+\frac{\gamma}{2}(\sum a_m x^m)^2 a_i a_j+\cdots.$$

From (10′) $\mu_{ij}=\lambda_i\lambda_j\mu$ $(i\neq j)$. We have

$$\lambda_i\sqrt{\mu}=\left[\alpha+\beta\sum a_m x^m+\frac{\gamma}{2}(\sum a_m x^m)^2+\cdots\right]^{\frac{1}{2}}a_i. \tag{59}$$

Also from (10′)

$$\mu_{ii}=ae_i+\alpha a_i^2+\beta a_{iii}x^i+\beta(\sum a_m x^m)a_i^2-\beta a_i^3 x^i+\cdots=\lambda_i^2\mu-e_i\lambda\mu.$$

Hence

$$\mu_{ii}-\lambda_i^2\mu=ae_i+\beta(a_{iii}-a_i^3)x^i+\cdots=-e_i\lambda\mu,$$

that is, $e_i\beta(a_{iii}-a_i^3)x^i$ is independent of i and consequently $a_{iii}=a_i^3$.

Similarly we have $a_{iiii}=a_i^4,\cdots$.

Hence we have:

If among the a's in the term $(\sum a_m x^m)^2$ of the second order of μ there exist at least two of the a's, say a_i and a_j, which are different from zero, that is, $a_i a_j\neq 0$ $(i\neq j)$, or otherwise, if $a_{ij}=0(i, j=1, \cdots, n;\ i\neq j)$ but there is at least one of the a_{ijk} $(i\neq j)$ which is not zero, then any solution μ in power series of (10′) is expressible as follows:

$$\mu=c+\sum c_m x^m+\frac{1}{2}a\sum e_m(x^m)^2+\frac{1}{2}\alpha(\sum a_m x^m)^2$$
$$+\frac{1}{3!}\beta(\sum a_m x^m)^3+\frac{1}{4!}\gamma(\sum a_m x^m)^4+\cdots, \tag{60}$$

where α, β, γ, $\cdots$ are arbitrary constants, but in the later case we have $\alpha=0$.

From (54) we have

$$\mu_{ij}=a_{ij}+a_{ijp}x^p+\frac{1}{2!}a_{ijpq}x^p x^q+\frac{1}{3!}a_{ijpqr}x^p x^q x^r+\cdots. \tag{61}$$

We have shown above that if $a_{ij}=0$ $(i, j=1, \cdots, n;\ i\neq j)$ but there is $a_{ijp}(i\neq j)$ which is not zero, any solution μ in power series of (10′) is expressible in the form (60) with $\alpha=0$. If $a_{ij}=0$, $a_{ijp}=0$ $(i\neq j,\ i, j, p=1, \cdots, n)$, it is evidently that the function μ in (60) with $\alpha=\beta=0$ is also a solution of (10′). However, in the later case we show that there exist other solutions of (10′) which can not be reduced to the form (60).

In fact, when $a_{ij}=0$, $a_{ijp}=0$ $(i\neq j)$, we have from (61) and (55)

$$\sum a_{ijkl}x^k x^l=a_i a_j\sum a_{kl}x^k x^l \quad (a_i a_j\neq 0)$$

for any values of the x's. If there is $a_{ijkl}\neq 0$ $(i, j, k, l\neq)$, we can show by an analogeous process that $a_{ijkl}=\rho a_i a_j a_k a_l$ and a_{iiii} is to be determined. But by (10′) $\mu_{ii}=\lambda_i^2\mu-e_i\lambda\mu$, and it follows that $a_{iiii}=\rho a_i^4$. Hence in this case any analytic solution μ which is expressible in a power series (49) is reducible to the form (60).

If all of the a_{ijkl} but a_{ijij} are zero, we have

$$\frac{1}{2}\sum a_{iijkl}x^k x^l = a_{iijj}x^i x^j.$$

From (56) we have

$$a_{iijj}a_{llkk} - a_{iijl}a_{ilkk} = 0, \quad (i, j, l, k \neq).$$

There are $a_{kk}(k=1, \cdots, n)$ such that

$$a_{iijj} = a_{ii}a_{jj}, \quad (i \neq j).$$

From (49) we have

$$\begin{aligned}\mu &= c + \sum c_m x^m + \frac{1}{2} a \sum e_m (x^m)^2 + \frac{1}{2} a_k^2 (x^k)^2 \\ &\quad + \frac{1}{6} \sum a_{mmm}(x^m)^3 + \frac{1}{12} \sum a_{pqpq}(x^p)^2 (x^q)^2 + \cdots \\ &= c + \sum c_m x^m + \frac{1}{2} a \sum e_m (x^m)^2 + \frac{1}{2} a_k^2 (x^k)^2 \\ &\quad + \frac{1}{6} \sum a_{mmm}(x^m)^3 + \frac{1}{12} \sum a_{mmmm}(x^m)^4 \\ &\quad + \frac{1}{12}\{[\sum a_{mm}(x^m)^2]^2 - \sum a_{mm}^2 (x^m)^4\} + \cdots,\end{aligned}$$

$$\begin{aligned}\mu_i &= c_i + a e_i x^i + a_k^2 \delta_{ik} x^k + \frac{1}{2} a_{iii}(x^i)^2 \\ &\quad + \frac{1}{3}[a_{iiii} - a_{ii}^2](x^i)^3 + \frac{1}{3}[\sum a_{mm}(x^m)^2] a_{ii} x^i + \cdots,\end{aligned}$$

$$\mu_{ij} = \frac{2}{3} a_{ii} a_{jj} x^i x^j + \cdots \quad (i \neq j).$$

Hence

$$\lambda_i \sqrt{\mu} = \sqrt{\frac{2}{3}}\, a_{ii} x^i + \cdots,$$

$$\begin{aligned}\mu_{ii} &= a e_i + a_k^2 \delta_{ik} + a_{iii} x^i + (a_{iiii} - a_{ii}^2)(x^i)^2 \\ &\quad + \frac{1}{3}[\sum a_{mm}(x^m)^2] a_{ii} + \frac{2}{3} a_{ii}^2 (x^i)^2 + \cdots = \lambda_i^2 \mu - l_i \lambda \mu.\end{aligned}$$

We have

$$e_i(a e_i + a_k^2 \delta_{ik}) = b, \; e_i a_{iii} = 0,$$
$$e_i a_{ii} = d, \; e_i(a_{iiii} - a_{ii}^2) = 0,$$

that is, $b = a$, $a_k = 0$, $a_{ii} = e_i d$, $a_{iiii} = d^2$. Hence

$$\mu = c + \sum c_m x^m + \frac{1}{2} a \sum e_m (x^m)^2 + \frac{1}{12} d^2 [\sum e_m (x^m)^2]^2 + \cdots.$$

If $d \neq 0$, we can prove in like manner that the terms of the fifth order of μ are zero, and the terms of the sixth order take the form $b^2[\sum e_m(x^m)^2]^3$, $\cdots$. Since in this case

$$\mu_{ij} = \frac{2}{3} d^2 e_i e_j x^i x^j + \cdots \quad (i \neq j),$$

by Lemma 3 the second term in the right-hand members must be proportional to $e_i e_j x^i x^j$, and the proportional factor is a constant multiple of $\sum e_m (x^m)^2$.

In fact, by Lemma 3 the term of the third order in the right-hand member of μ_{ij} must have the form $(\sum_p a_p x^p)e_ie_jx^ix^j$, that is, all of the a_{ijpqr} but a_{ijijp} are zero. But if $p\neq i$, among a_{ijijp}'s only a_{ipipj} may be not zero, and it is impossible, unless $a_{ijpqr}=0$ $(i\neq j)$. Similarly we can prove that $a_{iijj}=0$. Hence $a_{ijpqr}=0$ $(i\neq j)$. But in this case we can prove by (10′), $a_{iiii}=0$ $(i=1, \cdots, n)$ and hence $a_{ijpqr}=0$. The term of the third order in the right-hand is zero. The term of the fourth order must take the form $(\sum a_{pq}x^px^q)e_ie_jx^ix^j$, that is, all of the a_{ijpqrs} but a_{ijijpq} are zero. But, if $p\neq q$, among the a_{ijijpq}'s only the a's of the form a_{pqpqij} may be not zero, that is, $p=i$, $q=j$ or $p=q$. In other words, all of the a_{ijpqrs} but a_{ijijpp} are zero. Since $a_{iijjpp}=e_ie_ja_{pp}$ and the left-hand member is symmetric with respect to i, j, p, we have $a_{pp}=ke_p$ or $a_{iijjpp}=ke_ie_je_p$ and the term considered takes the form

$$k[\sum l_m(x^m)^2]e_ie_jx^ix^j,$$

that is, the terms of the sixth order of μ take the form

$$b^2\sum e_ie_je_p(x^i)^2(x^j)^2(x^p)^2=b^2[\sum e_m(x^m)^2]^3.$$

Continuing this process and by Lemmr 4, we see that in addition to (60) there exist other analytic solutions μ in power series of (10′) which are reducible to the form

$$\mu=c+\sum c_mx^m+\frac{1}{2!}a\sum e_m(x^m)^2+\frac{1}{4!}b[\sum e_m(x^m)^2]^2+\frac{1}{6!}k[\sum e_m(x^m)^2]^3+\cdots. \quad (62)$$

In summary, we have shown that any analytic solution μ of (10′) is expressible in power series either to (60) or to (62). In other words, any analytic solution μ of (10′) is expressible either to (45) or to (46).

Thus Theorem 2 is proved.

By certain transformations

$$x'^i=x^i+c^i \quad (i=1, \cdots, n),$$

we can show that both the types of solutions (45) and (46) are in fact contained in a more general type as follows:

$$\mu=c+\sum c_mx^m+\frac{1}{2}a\sum e_m(x^m)^2+f(y), \quad (63)$$

where

$$y\equiv b+\sum a_mx^m+a\sum e_m(x^m)^2, \quad (64)$$

a, b, c, $a_1, \cdots, a_n$, $c_1, \cdots, c_n$, are arbitrary constants and $f(y)$ is any analytic function of y.

Since when $a=0$ (63) is equivalent to (45), when $a\neq 0$ we can reduce $a_m=0$ $(m=1, \cdots, n)$ by a transformation $x'^i=x^i+c^i(i=2, \cdots, n)$ for certain c's. It follows that the analytic metric of any Riemannian manifold which admits isometric imbedding into space of any constant curvature is expressible in the form (9′), where μ is a function of the form (63).

We now consider the converse problem, that is, to find out what conditions should be subjected to the solutions of (10′) or to the function $f(y)$ of (63) so that the corresponding metric (9′) defines a Riemannian manifold M^n which admits isometric imbedding into space of any constant curvature.

When M^n is a hypersurface of the space $S^{n+1}(K_0)$, the Gauss equation is given by the equation (1)

$$R_{ijkl}=e(b_{ik}b_{jl}-b_{il}b_{jk})+K_0(g_{ik}g_{jl}-g_{il}g_{jk}), \tag{1}$$

and b_{jl} is expressed by the equation (22) in the paper [1], namely

$$b_{jl}=\frac{P_{jlhk}}{\sqrt{e(2P_{hphk}T^p_k(K_0)-P_{hkhk}T(K_0))}}. \tag{65}$$

If M^n admits isometric imbedding into two spaces $S^{n+1}(K)$ and $S^{n+1}(K_0)$, we have by the equation (55) of the paper [1]

$$P_{jlhk}=\frac{n-3}{n-2}\left\{(n-2)\lambda_{jl}+\left[(n-2)K_0+\frac{2(n-1)}{n}\Delta-\rho\right]g_{jl}\right\}$$
$$\cdot\left\{(n-2)\lambda_{hk}+\frac{n-1}{n-3}\left[(n-2)K_0+\frac{2(n-3)}{n}\Delta+\rho\right]g_{hk}\right\}, \tag{66}$$

where

$$\lambda_{ij}\equiv d_{ij}-\frac{\Delta}{n}g_{ij}. \tag{67}$$

From the equation (56) of the paper [1] we have

$$2P_{hphk}T^p_k(K_0)-P_{hkhk}T(K_0)$$
$$=-\frac{(n-3)^2}{n-2}\{(n-2)K_0+2(\Delta-\rho)\}\cdot\left\{(n-2)\lambda_{hk}\right.$$
$$\left.+\frac{n-1}{n-3}\left[(n-2)K_0+\frac{2(n-3)}{n}\Delta+\rho\right]g_{hk}\right\}^2. \tag{68}$$

If (66)—(68) are substituted in (65) we have in $S^{n+1}(K_0)$

$$b_{jl}=\frac{d_{jl}+\left(K_0+\frac{\Delta-\rho}{n-2}\right)g_{jl}}{\sqrt{-e\left(K_0+\frac{2(\Delta-\rho)}{n-2}\right)}}, \tag{69}$$

where

$$e=-\operatorname{sgn}\left(K_0+\frac{2(\Delta-\rho)}{n-2}\right). \tag{70}$$

Theorem 3. *If μ is any solution of the system of partial differential equations (10′) corresponding to the functions $\lambda, \lambda_1, \cdots, \lambda_n (n\geqslant 5)$, and if*

$$\sum e_m\mu_m^2+2\lambda\mu^2+K_0\neq 0, \tag{71}$$

the Riemannian manifold M^n with the metric (9′) admits isometric imbedding into a space $S^{n+1}(K_0)$.

For M^4, if in addition to the condition (71) it also satisfies the following condition

$$\sum e_m\mu_m^2+2\lambda\mu^2-\mu^2\sum e_m\lambda_m^2+K_0\neq 0, \tag{72}$$

M^n admits isometric imbedding into $S^5(K_0)$.

Proof From the equations (10) and (36) we have

$$c=\sum e_m\sigma_{mm}=B-A+n\lambda. \tag{73}$$

From (38) and (39) we have

$$\Delta e^{2\sigma}=-A+\frac{n}{2}B+n\lambda,\ \rho e^{2\sigma}=-A+B+2\lambda. \tag{74}$$

From (42) we have

$$d_{ii}=\frac{1}{2}e_i(B+2\lambda)-\lambda_i^2. \tag{75}$$

From (28), (74), (75) and (69) we have

$$b_{jl}=\frac{-\lambda_j\lambda_l+Qe^{2\sigma}e_j\delta_{jl}}{\sqrt{-eQ}}, \tag{76}$$

where

$$Q\equiv K_0+(B+2\lambda)e^{-2\sigma}. \tag{77}$$

Since $\mu=e^{-\sigma}$, we have

$$B=\sum e_m\sigma_m^2=\frac{1}{\mu^2}\sum e_m\mu_m^2,\ Q=\sum e_m\mu_m^2+2\lambda\mu^2+K_0.$$

From the condition (71), that is, $Q\neq 0$, we can choose $e=\pm 1$ such that $-eQ\geqslant 0$. Hence in this case b_{jl} is real and finite. The existence of the set of functions b_{jl} means that if in addition to the Gauss equation (1) we have for M^n also the Codazzi equations, then M^n admits isometric imbedding into $S^{n+1}(K_0)$. Moreover, it is well known that if the rank of $(T_{ijkl}(K_0))$ is $\geqslant 4$, the Codazzi equations are an algebraic consequence of the Gauss equation (1). But it is also known that the rank of $(T_{ijkl}(K_0))$ is equal to the rank of (b_{ij}). Hence if the rank of (b_{ij}) is $\geqslant 4$, M^n admits isometric imbedding into $S^{n+1}(K_0)$ and the theorem is proved.

We now proceed to estimate the rank of (b_{ij}). From (76) we have

$$b_{ij}=\frac{1}{|Q|^{\frac{n}{2}}}\begin{pmatrix}-\lambda_1^2+e_1e^{2\sigma}Q & -\lambda_1\lambda_2 & \cdots & -\lambda_1\lambda_n\\ -\lambda_2\lambda_1 & -\lambda_2^2+e_2e^{2\sigma}Q & \cdots & -\lambda_2\lambda_n\\ \cdots & \cdots & \cdots & \cdots\\ -\lambda_n\lambda_1 & -\lambda_n\lambda_2 & \cdots & -\lambda_n^2+e_ne^{2\sigma}Q\end{pmatrix}. \tag{78}$$

We have in (b_{ij}) the minor determinant of the fourth order of the following form

$$\Delta(i,j,k,l)=\begin{vmatrix}-\lambda_i^2+e_ie^{2\sigma}Q & -\lambda_i\lambda_j & -\lambda_i\lambda_k & -\lambda_i\lambda_l\\ -\lambda_j\lambda_i & -\lambda_j^2+e_je^{2\sigma}Q & -\lambda_j\lambda_k & -\lambda_j\lambda_l\\ -\lambda_k\lambda_i & -\lambda_k\lambda_j & -\lambda_k^2+e_ke^{2\sigma}Q & -\lambda_k\lambda_l\\ -\lambda_l\lambda_i & -\lambda_l\lambda_j & -\lambda_l\lambda_k & -\lambda_l^2+e_le^{2\sigma}Q\end{vmatrix}. \tag{79}$$

It is easy to show that

$$\Delta(i,j,k,e)=e_ie_je_ke_l(e^{2\sigma}Q)^3(e^{2\sigma}Q-e_i\lambda_j^2-e_j\lambda_j^2-e_k\lambda_k^2-e_l\lambda_l^2)$$
$$(i,j,k,l=1,\cdots,n;\ i,j,k,l\neq). \tag{80}$$

When $n=4$,

$$\det(b_{ij})=\Delta(1,2,3,4)=e_1e_2e_3e_4(e^{2\sigma}Q)^3\left(e^{2\sigma}Q-\sum_{i=1}^{4}e_i\lambda_i^2\right).$$

If the conditions (71) and (72) are satisfied, $\Delta(1,2,3,4)\neq 0$, the rank of $\det(b_{ij})=4$, and the theorem is proved for this case.

When $n\geqslant 5$, we have in (b_{ij}) the minor determinant $\Delta(i, j, k, l, h)$ of the fifth orber which takes the analogeous form as $\Delta(i, j, k, l)$. We have

$$\Delta(i, j, k, l, h)=e_ie_je_ke_le_h(e^{2\sigma}Q)^4(e^{2\sigma}Q-e_i\lambda_i^2-e_j\lambda_j^2-e_k\lambda_k^2-e_l\lambda_l^2-e_h\lambda_h^2). \quad (81)$$

If the rank of (b_{ij}) is <4, all of the $\Delta(i, j, k, l)=0$, $\Delta(i, j, k, l, h)=0$, that is

$$e^{2\sigma}Q-e_i\lambda_i^2-e_j\lambda_j^2-e_k\lambda_k^2-e_l\lambda_l^2=0,$$
$$e^{2\sigma}Q-e_i\lambda_i^2-e_j\lambda_j^2-e_k\lambda_k^2-e_l\lambda_l^2-e_h\lambda_h^2=0$$

for $i, j, k, l, h=1, \cdots, n$; $i, j, k, l, h\neq$. Hence $\lambda_i=0$ $(i=1, \cdots, n)$, $Q=0$. It is a contradiction. Therefore the rank of (b_{ij}) is $\geqslant 4$. Thus the theorem is proved.

It should be observed that the condition $T(K_0)\neq 0$ stated in Theorem A is immaterial for the present case, since when $Q\neq 0$ the set of functions b_{jl} exist so that the condition $T(K_0)\neq 0$ is unnecessary (Cf. Lemma 3, [1]).

Corollary 1. *If μ is any solution of the system of partial differential equations* (10′) *with the functions $\lambda, \lambda_1, \cdots, \lambda_n$ $(n\geqslant 5)$*, and if

$$\sum e_m\mu_m^2+2\lambda\mu^2\neq \text{const.}, \quad (82)$$

the Riemannian manifolds M^n with the metric (9′) *admit isometric imbedding into a space $S^{n+1}(K_0)$ of any constant curvature K_0.*

For M^4, if in addition to (82) *it satisfies also*

$$\sum e_m\mu_m^2+2\lambda\mu^2-\mu^2\sum e_m\lambda_m^2\neq \text{const.}, \quad (83)$$

M^4 admits isometric imbedding into S^5 (K_0) of any constant K_0.

Corollary 2. *If μ is any solution of the system of partial differential equations* (10′), *any Riemannian manifold M^n $(n\geqslant 4)$ with the metric* (9′) *admits isometric imbedding into a space $S^{n+1}(K)$ of constant curvature K, where the values of K may be chosen in infinitly many ways.*

5. From Theorems 1—3 we can determine completely the metrics of the Riemannian manifolds M^n $(n\geqslant 4)$ which admit isometric imbedding into a space S^{n+1} (K) of any constant curvature K.

Theorem 4. *If the Riemannian manifold M^n $(n\geqslant 4)$ admits isometric imbedding into spaces $S^{n+1}(K)$ of any constant curvature K, and if the metric of M^n is analytic, then the metric is expressible either in the form*

$$ds^2=\left[c+\sum c_mx^m+\frac{1}{2}a\sum e_m(x^m)^2+f(y)\right]^{-2}\sum e_i(dx^i)^2, \quad (84)$$

or in the form

$$ds^2=[c+\sum c_mx^m+\varphi(z)]^{-2}\sum e_i(dx^i)^2, \quad (85)$$

where

$$y\equiv\sum a_mx^m,\ z\equiv\sum e_m(x^m)^2, \quad (86)$$

$c, a, c_1, \cdots, c_n$ are constants, $f(y)$ and $\varphi(z)$ are analytic functions of the arguments y and z respectively.

Conversely, when $n\geqslant 5$, (i) for any constants $c, a, c_1, \cdots, c_n$ and any analytic function $f(y)$ but

$$(\sum e_m a_m^2) f(y) + \frac{1}{2} ay^2 + (\sum e_m c_m a_m) y \neq \text{const}.$$

the metric (84) *defines a Riemannian manifold* M^n *whih admits isometric imbedding into spaces* $S^{n+1}(K)$ *of any constant curvature* K; (*ii*) *for any constants* $c, c_1, \cdots, c_n$ *and any analytic function* $\varphi(z)$ *but*

$$[\varphi(z)+c]/\sqrt{z} \neq \text{const}.$$

the metric (85) *defines a Riemannian manifold* M^n *which admits isometric imbedding into spaces* S^{n+1} (K) *of any constant curvature* K.

Proof The first part of the theorem follows from Theorems 1—3 directly.

In order to prove the converse part of the theorem, we estimate the values of Q for both the cases (84) and (85).

For (84), we have

$$\mu = c + \sum c_m x^m + \frac{1}{2} a \sum e_m (x^m)^2 + f(y),$$

$$\mu_i = c_i + a e_i x^i + f'(y) a_i, \ \mu_{ij} = f''(y) a_i a_j \quad (i \neq j),$$

$$\lambda_i \sqrt{\mu} = \sqrt{f''} a_i, \ \mu_{ii} = a e_i + f''(y) a_i^2, \ \lambda \mu = -a.$$

Hence

$$Q = (\sum e_m a_m^2) [f'(y)]^2 + 2(ay + \sum e_m c_m a_m) f'(y) - 2af(y) + (\sum e_m c_m^2 - 2ac + K_0).$$

A necessary and sufficient condition that $Q \neq 0$ (for any K_0) is

$$(\sum e_m a_m^2) [f'(y)]^2 + 2(ay + \sum e_m c_m a_m) f'(y) - 2af(y) \neq \text{const}.$$

or

$$[(\sum e_m a_m^2) f'(y) + ay + \sum e_m c_m a_m] f'' \neq 0.$$

When $f''=0$, M^n is of constant curvature. Hence the required condition reduces to

$$(\sum e_m a_m^2) f'(y) + ay + \sum e_m c_m a_m \neq 0$$

or

$$(\sum e_m a_m^2) f(y) + \frac{1}{2} ay^2 + (\sum e_m c_m a_m) y \neq \text{const}.$$

For (85), we have $\quad \mu = c + \sum c_m x^m + \varphi(z),$

$$\mu_i = c_i + \varphi'(z) 2 e_i x^i, \ \mu_{ij} = 4 e_i e_j x^i x^j \varphi''(z) \quad (i \neq j),$$

$$\lambda_i \sqrt{\mu} = 2\sqrt{\varphi''(z)} e_i x^i, \ \mu_{ii} = 2 e_i \varphi'(z) + 4\varphi''(z)(x^i)^2, \ \lambda\mu = -2\varphi'(z).$$

Hence $\quad Q = 4z[\varphi'(z)]^2 - 4[c+\varphi(z)]\varphi'(z) + \sum e_m c_m^2 + K_0.$

A necessary and sufficient condition that $Q \neq 0$ (for any K_0) is

$$[2z\varphi'(z) - c - \varphi(z)]\varphi''(z) \neq 0.$$

When $\varphi''(z)=0$, M^n is of constant curvature. Hence the required condition reduces to $2z\varphi'(z) - c - \varphi(z) \neq 0$ or $\dfrac{\varphi(z)+c}{\sqrt{z}} \neq \text{const}.$

Thus Theorem 4 is proved.

References

[1] Bai Zheng-guo, Local isometric imbedding of Riemannian manifolds M^n into a space of constant curvature S^{n+1}, *Chin. Ann. of Math.*, 3:4(1982), 471—482.