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RESONANCE PROBLEM FOR A CLASS OF DUFFING'S EQUATIONS

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Abstract

Consider the Duffing's equation

$$\ddot{x}+g(x)=f(t), \tag{1}$$

where $g\in C(\mathbf{R}, \mathbf{R})$ and $f\in P\equiv\{f\in C(\mathbf{R}, \mathbf{R}); f \text{ is } \omega\text{-periodic for some } \omega>0\}$. The function g is said to be resonant if there exists $f\in P$ such that eq. (1) has no bounded solutions on $[0, \infty)$. Using a generalized version of the Poincaré-Birkhoff fixed point theorem, the authors establish conditions on g which guarantee the following result holds: for any $f\in P$ with period ω, there exists $\bar{k}\geqslant 0$ such that eq. (1) has infinitely many $k\omega$-periodic solutions for every integer $k\geqslant\bar{k}$. In such a case, g is clearly non-resonant.

In the work [1], the author studied the existence of infinitely many harmonic solutions for Duffing's equation

$$\ddot{x}+g(x)=f(t), \tag{1}$$

where f, $g\in C(\mathbf{R}, \mathbf{R})$ and f is ω-periodic. In that study the main tool is a generalized Poincaré-Birkhoff theorem obtained recently (see [2, 3]). It is the aim of the present note to study a resonance problem as well as the existence of infinitely many $k\omega$-periodic solutions for (1) by using the same tool.

Set

$$P=\{f\in C(\mathbf{R}, \mathbf{R}): f \text{ is } \omega\text{-periodic for some } \omega>0\}.$$

Definition 1. *A function $g\in C(\mathbf{R}, \mathbf{R})$ is said to be resonant if there exists $f\in P$ such that Equation* (1) *has no bounded solution on* $[0, +\infty)$.

By this definition any linear function $g(x)=a^2x+b$ $(a>0)$ is resonant, because for $f(t)=\cos at+b$ the equation $\ddot{x}+a^2x+b=\cos at$ has no bounded solution. However, there is a large class of nonlinear functions which are not resonant. For instance, if g satisfies the super-linear condition: $\lim x^{-1}g(x)=+\infty$ as $|x|\to\infty$, then Equation (1) has infinitely many harmonic solutions for any $f\in P$[2]. Thus g is not resonant. In this note we will restrict our attention to a specific class of functions g which satisfies the following conditions:

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(a) Suppose g is locally Lipschitzian, i.e. there are constants k_0, $k_1 \geqslant 0$, such that

$$|g(x)-g(y)| \leqslant k_0|x-y|+k_1, \quad \forall x, y \in \mathbf{R};$$

(b) There are constants A, $M>0$, such that

$$x^{-1}g(x) \geqslant A \quad \text{for} \quad |x| \geqslant M.$$

If $g \in C(\mathbf{R}, \mathbf{R})$ satisfies conditions (a) and (b) we will simply write $g \in Q$.

Remark. Condition (b) is just (H_2) in [1]. But we replace (H_1) in [1] by condition (a). The hypothesis (H_1) requiring that $g \in C^1(\mathbf{R}, \mathbf{R})$ and $|g'(x)| \leqslant K (\forall x \in \mathbf{R})$ is somewhat too strong. It is easy to check that all the results in [1] remain true with (H_1) replaced by (a).

Now, consider the "homogeneous" equation associated with Equation (1):

$$\ddot{u}+g(u)=0. \tag{2}$$

We collect some basic facts concerning Equation (2) in the following Lemma[1].

Lemma 1. *Let $g \in Q$ be given. Consider the equivalent system of Equation* (2)

$$\dot{u}=v, \quad \dot{v}=-g(u), \tag{3}$$

which is an autonomous Hamiltonian system with the Hamiltonian function

$$H(u, v)=\frac{1}{2} v^2+G(u),$$

where $G(u)=\int_0^u g(s)ds$. For $C \in \mathbf{R}$, the set $\Gamma_c(H(u,v)=C)$ is a star-shaped (with respect to the origin) periodic orbit for every $C \geqslant C_0$. The minimal period of this orbit is given by

$$\tau_g(C)=\int_{-h_1(c)}^{h(c)} \frac{\sqrt{2}\,du}{\sqrt{C-G(u)}},$$

where $h(c)$, $h_1(c)>0$ are uniquely determined by $G(h(c))=G(-h_1(c))=C$.

Lemma 2. *Eor each $g \in Q$, $\tau_g(c)$ is bonnded on $[C_0, \infty)$.*

Proof We need only to show the boundedness of $I_1=\int_0^{h(c)} du/\sqrt{C-G(u)}$ and $I_2=\int_{-h_1(c)}^0 du/\sqrt{C-G(u)}$ for large C. Note first that (a) and (b) imply the existence of constants α_1, α_2, β_1, $\beta_2>0$ such that

$$\alpha_1 u^2-\beta_1 \leqslant G(u) \leqslant \alpha_2 u^2+\beta_2, \quad \forall u \in \mathbf{R}. \tag{4}$$

Let $\bar{u} \in (0, h(c))$ satisfy $G(\bar{u})=\frac{C}{2}$ and $G(u) \leqslant G(\bar{u})$ for $u \in [0, \bar{u}]$. Then we have by (4)

$$\bar{u}^2 \leqslant \frac{1}{\alpha_1}(G(\bar{u})+\beta_1)=\frac{C}{2\alpha_1}+\frac{\beta_1}{\alpha_1} \leqslant \left(\frac{1}{2\alpha_1}+\frac{\beta_1}{\alpha_1 C_0}\right)C \equiv \gamma_1^2 C \quad (\gamma_1>0).$$

Therefore, we have

$$\int_0^{\bar{u}} \frac{du}{\sqrt{C-G(u)}} \leqslant \int_0^{\bar{u}} \sqrt{\frac{2}{C}}\, du=\sqrt{\frac{2}{C}}\, \bar{u} \leqslant \sqrt{2}\, \gamma_1. \tag{5}$$

Using (4) again, we find a $\gamma_2>0$ such that

$$\bar{u} \geqslant \gamma_2 \sqrt{C}. \tag{6}$$

Assuming C is so large that $\bar{u} \geqslant M$, we then have

$$\int_{\bar{u}}^{h(c)} \frac{du}{\sqrt{C-G(u)}} = \int_{\bar{u}}^{h(c)} \frac{1}{g(u)} \cdot \frac{g(u)du}{\sqrt{C-G(u)}}$$

$$\leqslant \frac{1}{A\bar{u}} \int_{\bar{u}}^{h(c)} \frac{g(u)du}{\sqrt{C-G(u)}} \quad (\text{by (b)})$$

$$= \frac{1}{A\bar{u}} \cdot \sqrt{2C} \leqslant \frac{\sqrt{2}}{A\gamma_2} \quad (\text{by (6)}). \tag{7}$$

From (5) and (7), we see I_1 is bounded. Since we can pro vethe boundedness of I_2 in the same way, the proof is complete.

Definition 2. *$g \in Q$ is said to be asymptotically resonant iff* $\lim\limits_{C\to+\infty} \tau_g(C)$ exists.

Now we can state our main theorem.

Theorem 1. *Suppose $g \in Q$ is not asymptotically resonant. Then, for any $f \in P$, there exists $\bar{k} > 0$ such that Equation (1) has infinitely many $k\omega$-periodic solutions for each $k \geqslant \bar{k}$ and it has at least one ω-periodic solution.*

As a consequence of Theorem 1, we have:

Corollary. *If $g \in Q$ is resonant, then it is asymptotically resonant.*

An open question is whether we can prove or disprove the converse of the above corollary.

Before proving Theorem 1, let us give the statement of the following generalized Poincaré-Birkhoff theorem.

Theorem A. *Let A be an annular region in $\mathbf{R}^2$ bounded by two disjoint simple closed curves Γ_1 and Γ_2. Let D_i denote the open set bounded by Γ_i, $i=1, 2$. Assume that $0 \in D_1 \subset \bar{D}_1 \subset D_2$. Suppose T: $A \to T(A) \subset \mathbf{R}^2 \setminus \{0\}$ is an area-preserving homeomorphism which has the polar coordinate expression*

$$r^* = f(r, \theta), \quad \theta^* = \theta + g(r, \theta), \tag{8}$$

where (r^*, θ^*) denotes the image of (r, θ) under T, and f and g are continuous and 2π-periodic in θ. Assume

(i) *Γ_1 is star-shaped about the origin;*

(ii) *the "twist" condition:*

$$g > 0 \text{ on } \Gamma_1 \text{ and } g < 0 \text{ on } \Gamma_2 \tag{9}$$

is satisfied;

(iii) *there exists an area-preserving homeomorphism T_1: $\bar{D}_2 \to \mathbf{R}^2$, such that $T_1|A = T$ and $0 \in T_1(D_1)$.*

Then T has at least two fixed points in A.

Remark. For any integer m, the following

$$r^* = f(r, \theta); \quad \theta^* = \theta + g(r, \theta) + 2m\pi \tag{8'}$$

is also a polar coordinate expression for the mapping T. For this expression, the corresponding twist condition should be

$$g > -2m\pi \text{ on } \Gamma_1; \quad g < -2m\pi \text{ on } \Gamma_2. \tag{9'}$$

If we replace (9) in this theorem by

$$\min_{\Gamma_1} g - \max_{\Gamma_2} g > 2\pi, \tag{10}$$

we see that there is an integer m for which condition (9)′ holds. Therefore, Theorem A remains true with (9) replaced by (10). We note also that reversing all the inequalities in (9), (9)′ and (10) is permitted.

Proof of Theorem 1 Consider the equivalent system for Equation (1):

$$\dot{x} = y, \quad \dot{y} = -g(x) + f(t). \tag{11}$$

Let $(\bar{x}(t, x_0, y_0), \bar{y}(t, x_0, y_0))$ be the unique solution of (11) which satisfies $\bar{x}(0) = x_0$, $\bar{y}(0) = y_0$. For each $t \geqslant 0$, define a mapping T_t: $\mathbf{R}^2 \to \mathbf{R}^2$ by

$$T_t(x, y) = (\bar{x}(t, x, y), \bar{y}(t, x, y)).$$

Since (11) is a Hamiltonian system, T_t is an area-preserving homeomorphism. It is also clear that if $k>0$ is an integer, then fixed points of $T_{k\omega}$ correspond to $k\omega$-periodic solutions of (11). Suppose A is an annular region in $\mathbf{R}^2$ such that $0 \notin T_t(A)$ for $t \in [0, k\omega]$. Then there exist two functions $\bar{r}(t, r, \theta)$ and $\bar{\theta}(t, r, \theta)$ which are continuous on $[0, k\omega] \times A$ and satisfy

$$\begin{cases} \bar{x}(t, x, y) = \bar{r}(t, r, \theta) \cos \bar{\theta}(t, r, \theta), \\ \bar{y}(t, x, y) = \bar{r}(t, r, \theta) \sin \bar{\theta}(t, r, \theta), \end{cases} \quad (t \in [0, k\omega], \ (r, \theta) \in A)$$

with the initial condition: $\bar{r}(0, r, \theta) = r$, $\bar{\theta}(0, r, \theta) = \theta$. Moreover, $\bar{r}(t, r, \theta)$ and $\bar{\theta}(t, r, \theta) - \theta$ are 2π-periodic in θ[2]. We will call $(\bar{r}, \bar{\theta})$ the polar coordinate expression for the solution $(\bar{x}, \bar{y})$ of (11). Using these two functions we obtain a polar coordinate expression for $T_{k\omega}$ as follows

$$r^* = \bar{r}(k\omega, r, \theta), \quad \theta^* = \theta + [\bar{\theta}(k\omega, r, \theta) - \theta]. \tag{12}$$

By assumption, g is not asymptotically resonant. Hence

$$a = \varliminf_{C \to \infty} \tau_g(C) < \varlimsup_{C \to \infty} \tau_g(C) = b.$$

Let k be any integer such that

$$k\omega \left(\frac{1}{a} - \frac{1}{b} \right) > 4. \tag{13}$$

Let $\{C_{1j}\}$ and $\{C_{2j}\}$ be two sequences such that $C_{1j}, C_{2j} \geqslant C_0$, $C_{1j} \to +\infty$, $C_{2j} \to +\infty$, and $\tau_g(C_{1j}) \to a$, $\tau_g(C_{2j}) \to b$ as $j \to +\infty$. By Lemma 1, $\Gamma_{C_{1j}}$ and $\Gamma_{C_{2j}}$ are star-shaped periodic orbits of (3). Let A_j be the annular region bounded by $\Gamma_{C_{1j}}$ and $\Gamma_{C_{2j}}$ $(C_{1j} < C_{2j})$. We are going to prove the restriction on A_j of $T_{k\omega}$ satisfies the assumptions of Theorem A, for each sufficiently large j. Hence $T_{k\omega}$ has at least two fixed points in each A_j. It follows that system (11), thus Equation (1), has infinitely many $k\omega$-periodic solutions. The following comparison lemma is crucial for our aim[1].

Lemma 3. *Let $(\bar{u}(t, u_0, v_0), \bar{v}(t, u_0, v_0))$ be the solution of* (3) *which satisfies $\bar{u}(0) = u_0$, $\bar{v}(0) = v_0$. Let $(\bar{\rho}(t, \rho, \varphi), \bar{\varphi}(t, \rho, \varphi))$ be the polar coordinate expression of this solution. Given $t_0 > 0$ and $\varepsilon > 0$, there are $R = R(t_0)$ and $K = K(t_0, \varepsilon)$ such that*

$$|\bar{r}(t, r, \theta)-\bar{\rho}(t, r, \theta)| \leqslant R, \quad \forall t \in [0, t_0], \tag{14}$$

for all $(r, \theta) \in \Gamma_c$, *and* $C \geqslant K$; *and*

$$|\bar{\theta}(t, r, \theta)-\bar{\varphi}(t, r, \theta)| \leqslant \varepsilon, \quad \forall t \in [0, t_0], \tag{15}$$

for all $(r, \theta) \in \Gamma_c$, *and* $C \geqslant K$.

To apply Theorem A to the mapping $T_{k\omega}|A_j$, we notice that condition (i) of Theorem A is satisfied by Lemma 1. It is easy to see from (14) of Lemma 3 that (iii) of Theorem A is also satisfied for large j such that $A_j \subset \mathbf{R}^2 \backslash B_R$, where $B_R=\{(x, y) \in \mathbf{R}^2: x^2+y^2 \leqslant R^2\}$. It remains to prove that condition (ii) is satisfied. By the Remark following Theorem A, we need only to show

$$\min_{\Gamma_{C_{2j}}} \tilde{\theta}(r, \theta)-\max_{\Gamma_{C_{1j}}} \tilde{\theta}(r, \theta)>2\pi, \tag{16}$$

where $\tilde{\theta}(r, \theta)=\bar{\theta}(k\omega, r, \theta)-\theta$ (by (12)).

Now let $\Phi(r, \theta)=\bar{\varphi}(k\omega, r, \theta)-\theta$. We see from (15) of Lemma 3 that (16) will hold for large j as soon as we have the following Lemma.

Lemma 4. *For sufficiently large* j, *we have*

$$\min_{\Gamma_{C_{2j}}} \Phi-\max_{\Gamma_{C_{1j}}} \Phi>3\pi.$$

Proof Since Γ_c is a periodic orbit of (3) with minimal period $\tau_g(C)$ for $C \geqslant C_0$, we have

$$\bar{\varphi}(t+\tau_g(c), r, \theta)=-2\pi+\bar{\varphi}(t, r, \theta), \ (r, \theta) \in \Gamma_c. \tag{17}$$

Let

$$k\omega=m_{1j}\tau_g(C_{1j})+\xi_{1j}=m_{2j}\tau_g(C_{2j})+\xi_{2j}, \tag{18}$$

where m_{ij} are non-negative integers, $0 \leqslant \xi_{ij}<\tau_g(C_{ij})$, $i=1, 2$. By (17) and (18), we have

$$\bar{\varphi}(k\omega, r, \theta)=\bar{\varphi}(m_{1j}\tau_g(C_{1j})+\xi_{1j}, r, \theta)=-2m_{1j}\pi+\bar{\varphi}(\xi_{1j}, r, \theta), \ (r, \theta) \in \Gamma_{C_{1j}}. \tag{19}$$

It is clear that

$$\theta-2\pi<\bar{\varphi}(\xi_{1j}, r, \theta) \leqslant \theta.$$

From (18) and (19) we derive

$$\begin{aligned}
\bar{\varphi}(k\omega, r, \theta) &= -2\pi\left(\frac{k\omega}{\tau_g(C_{1j})}-\frac{\xi_{1j}}{\tau_g(C_{1j})}\right)+\bar{\varphi}(\xi_{1j}, r, \theta) \\
&< -\frac{2\pi k\omega}{\tau_g(C_{1j})}+2\pi+\theta \\
&= -\frac{2\pi k\omega}{a}+2\pi+\theta+\varepsilon_{1j}, \quad (u, \theta) \in \Gamma_{C_{1j}},
\end{aligned} \tag{20}$$

where $\varepsilon_{1j} \to 0$ as $j \to +\infty$. In a similar way, we obtain

$$\bar{\varphi}(k\omega, r, \theta)>-\frac{2\pi k\omega}{b}-2\pi+\theta+\varepsilon_{2j}, \quad (r, \theta) \in \Gamma_{C_{2j}}, \tag{21}$$

where $\varepsilon_{2j} \to 0$ as $j \to \infty$. Thus, for j large enough, we have by (20), (21) and (13)

$$\min_{\Gamma_{C_{2j}}} \Phi-\max_{\Gamma_{C_{1j}}} \Phi>2\pi k\omega\left(\frac{1}{a}-\frac{1}{b}\right)-4\pi+\varepsilon_{2j}-\varepsilon_{1j}>4\pi+\varepsilon_{2j}-\varepsilon_{1j} \geqslant 3\pi.$$

This proves Lemma 4.

Since all the assumptions of Theorem A are fulfilled, we conclude that equation (1) has infinitely many $k\omega$-periodic solutions for any k satisfying (13).

Next, since every solution of the system (11) is continuable on $[0, \infty)$ due to condition (a), and since we have obtained bounded solutions for this system, we can apply a theorem of Massera [4] to obtain at least one ω-periodic solution.

The proof of Theorem 1 is complete.

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