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ON PRIMALITY OF THE COMBINATION OF EXPONENTIAL FUNCTIONS

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Abstract

The author discusses in this paper the transcendental unsolvability of the functional equation $F(z)=f\circ g(z)$ with f being meromorphic and g entire, for the function of the form

$$F(z)=\sum_{j=1}^{m} Q_j(z)e^{P_j(z)},$$

where Q_j's are rational, P_j's are polynomials. The main results are:

a) $F(z)$ is pseudo-prime, i. e. $F=f\circ g$ has no transcendental solutions f and g;

b) If $0\leqslant n_1<n_2<\cdots<n_m$ with $n_j=\deg P_j$, then $F(z)$ is prime (i. e. $F=f\circ g$ implies that either f or g is linear), unless there exists a nonlinear polynomial $g(z)$ such that $P_j=p_j(g)$, $Q_j=q_j(g)$ with p_j's being polynomials and q_j's rational.

These results generalize some theorems due to Prokopovich and other authors.

§ 1. Introduction

A meromorphic function $F(z)$ is said to be composite if $F(z)$ can be factorized as

$$F(z)=(f\circ g)(z)(\equiv f(g(z))), \tag{1}$$

where f is meromorphic and g is entire (g may be meromorphic when f is rational), none of which is (fractional) linear.

$F(z)$ is said to be prime (pseudo-prime) if every factorization of the form (1) implies that either f is fractional linear or g is linear (either f is rational or g is a polynomial). Also, a transcendental meromorphic function $F(z)$ is said to be left (right) prime if every factorization of the form (1) implies that f is fractional linear whenever g is transcendental (g is linear whenever f is transcendental). Further, two (or more) functions F and G are said to have a common right factor α, if there exist functions f and g such that

$$F=f\circ\alpha \text{ and } G=g\circ\alpha,$$

where α is a non-linear entire function.

When f in (1) is rational and g is restricted to entire functions, or when $F=f\circ g$ is entire, and f and g are restricted to entire functions, it is called to be a factorization

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in entire sense.

The factorization theory of meromorphic functions has been developed in various aspects. It is an interesting problem to determine the primality or pseudo-primality of entire and meromorphic functions. Many results in this topic are known. In an earlier paper[8], Rosenbloom pointed out that the function $F(z)=z+e^z$ is prime. Baker and Gross[1] proved that the function

$$F(z)=e^z+P(z)$$

is prime, where $P(z)$ is a non-constant polynomial. Gol'dberg and Prokopovich[3] generalized these results. They considered the function

$$F(z)=P(z)+Q(z)e^{R(z)},$$

where P, Q, R are polynomials, P, $R\not\equiv$const., $Q\not\equiv 0$. Later on, Prokopovich[7] dealt with functions of the form

$$F(z)=\sum_{j=1}^{m} Q_j(z)e^{P_j(z)},\quad F(z)\neq C+Q_2(z)e^{P_2(z)}, \tag{2}$$

where $Q_j(z)$ and $P_j(z)$ are polynomials, and C is a constant. He proved

Theorem A. *Let P_j and Q_j be polynomials, and* deg $P_j=n_j$ *with* $0\leqslant n_1<n_2<\cdots<n_m$, *where $Q_j\not\equiv 0$ for $j=1,\cdots,m$. Let $F(z)$ be of the form* (2). *Then F is composite if and only if all P_j and Q_j have a common right factor.*

Recent results of N. Steinmetz[9] make the discussion of the pseudo-primality of (2) much easier. We shall in this note discuss the primality of functions of the form (2) with all Q_j being rational functions and without the restriction that $F(z)\neq C+Q_2(z)e^{P_2(z)}$. Here are the main results.

Theorem 1. *Let $P_1(z),\cdots,P_m(z)$ be polynomials, and $Q_1(z),\cdots,Q_m(z)$ be rational functions. Then the function*

$$F(z)=\sum_{j=1}^{m} Q_j(z)e^{P_j(z)} \tag{3}$$

is pseudo-prime.

In the following theorems, the factorization is restricted in entire sense. That is, if $F=f\circ g$ and f is rational, then only entire factor g is considered.

Theorem 2. *Suppose that the hypotheses of Theorem* 1 *are satisfied, and that in addition* $n_m=$deg $P_m>$deg $P_j=n_j$ *for* $j=1,\cdots,m-1$ $(m\geqslant 2)$. *Then the function* (3) *with $F(z)\neq C+Q_2(z)e^{P_2(z)}$ is left prime.*

Remark 1. The example

$$G(z)=1+2e^z+e^{2z}=(1+e^z)^2$$

shows that the hypothesis $n_m>n_j\,(j<m)$ cannot be omitted.

Remark 2. The demand $F(z)\neq C+Q_2e^{P_2(z)}$ in Theorem 2 cannot be canceled either, as is shown in the example below

$$H(z)=C+e^{2z},$$

since we have $H=f\circ g$ with $f(\zeta)=C+\zeta^2$ and $g(z)=e^z$.

Theorem 3. *Suppose that in addition to the hypotheses of Theorem 2, the function $F(z)$ of the form (3) satisfies $0\leqslant n_1<n_2<\cdots<n_m$. Then $F(z)$ is prime, unless all $P_1, \cdots, P_m$ and $Q_1, \cdots, Q_m$ have a common right factor.*

Corollary. *Let f be meromorphic and g be entire such that (i) f and g are nonlinear, (ii) $F=f\circ g$ is transcendental and of finite order, and (iii) F has only finitely many poles. Then F has infinitely many fix-points.*

This is a slightly refined form of a result in [5, Theorem 1].

Theorem 4. *Let*

$$h(z)=C+Q(z)e^{P(z)}, \tag{4}$$

where $Q(z)\not\equiv 0$ is rational, and $P(z)$ is a non-constant polynomial. Then

(i) *$h(z)$ is right prime if and only if P and Q have no common right factor;*

(ii) *$h(z)$ is left prime, unless $Q(z)$ is of the form*

$$Q(z)=\beta(z)^n,$$

where $\beta(z)$ is rational, and n is an integer with $n\geqslant 2$.

Remark 3. From this theorem, we conclude that if the function (4) has a factorization $h=f\circ g$, then either

$$f(\zeta)=C+\zeta^n \text{ and } g(z)=\beta(z)e^{\frac{1}{n}P(z)} \text{ with } Q(z)=\beta(z)^n$$

or

$$f(\zeta)=C+q(\zeta)e^{p(\zeta)}$$

and $g(z)$ is a polynomial of degree $\geqslant 2$ such that

$$Q(z)=q(g(z)) \text{ and } P(z)=p(g(z)).$$

Throughout this note the standard notation of Nevanlinna theory will be employed without explanation.

§ 2. Preliminary Lemmas

Lemma 1[9]. *Let $h(z)$ be a transcendental meromorphic function satisfying the linear differential equation*

$$w^{(n)}+a_{n-1}(z)w^{(n-1)}+\cdots+a_0(z)w=a(z),$$

where $a(z), a_0(z), \cdots, a_{n-1}(z)$ are rational. Then $h(z)$ is pseudo-prime.

Lemma 2. *Let $P_1(z), \cdots, P_m(z)$ be polynomials ($m\geqslant 1$) such that P_j-P_l is not a constant for $j\neq l$, $1<j$, $l<m$ (when $m=1$, no restriction arises for polynomial $P_1(z)$). Let $Q_1(z), \cdots, Q_m(z)$ be rational. Then the function*

$$F(z)=\sum_{j=1}^{m} Q_j(z)e^{P_j(z)}$$

is identically zero only if $Q_j(z)\equiv 0$ for $j=1, \cdots, m$.

This is an immediate consequence of a result in [2, Theorem 1].

Lemma 3[6]. *Let $g(z)$ be a transcendental meromorphic function such that*

$$N(r, g)=o(T(r, g)) \text{ as } r\to\infty.$$

Let $R(z)$ be a rational function of degree k and $\alpha(z)\neq C$ be a meromorphic function satisfying the condition $T(r, \alpha)=o(T(r, g))$ as $r\to\infty$. Then

$$\varliminf_{\substack{r\to\infty \\ r\notin E}} \frac{N\left(r, \frac{1}{R(g)-\alpha}\right)}{T(r, g)}\geqslant k-1+\delta(0, g), \tag{5}$$

where mes $E<\infty$.

Lemma 4 (See [4, Theorem 5.1]). *Let $F(z)$ be a transcendental meromorphic function of finite order which has only a finite number of poles and zeros. Then if F has a factorization $F=f\circ g$, where f is rational, then f must be of the form*

$$f(\zeta)=a(\zeta-b)^n,$$

where n is an integer with $|n|\geqslant 2$, and a, b are constants.

§ 3. Proofs of Theorems

Proof of Theorem 1 Without loss of generality, we may assume that P_j-P_l is not a constant for $j\neq l$, $1\leqslant j, l\leqslant m$, and that $Q_j\not\equiv 0$ for $j=1, \cdots, m$.

We have

$$F^{(k)}(z)=\sum_{j=1}^{m} Q_{k,j}(z)e^{P_j(z)},\ 0\leqslant k\leqslant m-1 \tag{6}$$

and

$$F^{(m)}(z)=\sum_{j=1}^{m} Q_{m,j}e^{P_j(z)}, \tag{7}$$

where

$$F^{(0)}=F,\ Q_{0,j}=Q_j$$

and

$$Q_{k,j}=Q_{k-1,j}P_j'+Q_{k-1,j}' \text{ for } 1\leqslant j,\ k\leqslant m.$$

By Lemma 2, the set of functions $Q_1e^{P_1}, \cdots, Q_me^{P_m}$ is linear independent over complex numbers. Therefore, the Wronskian of $Q_1e^{P_1}, \cdots, Q_me^{P_m}$

$$\det(Q_{k,j})\exp\left(\sum_{j=1}^{m} P_j\right)$$

is not identically zero, and $\det(Q_{k,j})$ is not either. Solving the system of linear equations (6) for functions $e^{P_1}, \cdots, e^{P_m}$, we see that each of these functions can be expressed by the linear combinations of $F^{(j)}(z)$, $j=0, \cdots, m-1$, with rational functions as coefficients. Substituting all these expressions of $e^{P_1}, \cdots, e^{P_m}$ into (7), we conclude that $F(z)$ satisfies a linear differential equation

$$w^{(m)}+a_{m-1}(z)w^{(m-1)}+\cdots+a_0(z)w=a(z),$$

where $a(z)$, $a_0(z)$, $\cdots$, $a_{n-1}(z)$ are rational functions. By Lemma 1, $F(z)$ is therefore pseudo-prime as is claimed in the theorem.

Proof of Theorem 2 By Theorem 1, $F(z)$ is pseudo-prime. Suppose that $F=R\circ g$, when $R(\zeta)$ is a rational function of degree k, and g is entire. Apparently, g is of order n_m, since F is. Let

$$\alpha(z)=\sum_{j=1}^{m-1}Q_j(z)e^{P_j(z)}.$$

Clearly, $T(r,\alpha)=o(T(r,g))$ as $r\to\infty$. Hence, by Lemma 3, there exists a sequence $r_n\to\infty(n\to\infty)$ such that

$$N\left(r_n,\frac{1}{Q_me^{P_m}}\right)\geqslant(k-1+o(1))T(r_n,g)\quad(n\to\infty).$$

But, this can hold only if $k=1$, namely, R is fractional linear, so that $F(z)$ is left prime. The theorem follows.

Proof of Theorem 3 By Theorem 2, it suffices to consider the case when $F=f\circ g$ with g being a polynomial of degree$\geqslant 2$. In this case, we want to show that g is a common right factor of all P_j and Q_j.

Indeed, since F has only finitely many poles, so does f. Hence, we can write

$$f\circ g=\frac{f_1\circ g}{R\circ g},$$

where f_1 is entire and R a polynomial. Let

$$F_1(z)=F(z)R(g(z)).$$

Then $F_1(z)$ is entire and

$$F_1(z)=\sum_{j=1}^{m}\tilde{Q}_j(z)e^{P_j(z)},$$

where

$$\tilde{Q}_j=(R\circ g)Q_j,\ j=1,\cdots,\mathrm{m}. \tag{8}$$

It is clear that $\tilde{Q}_1,\cdots,\tilde{Q}_m$ have to be polynomials. Now that

$$F_1=(f\cdot R)\circ g,$$

by Theorem A, there exist polynomials $\tilde{q}_j$ and p_j such that

$$\tilde{Q}_j=\tilde{q}_j\circ g \text{ and } P_j=p_j\circ g,\ j=1,\cdots,m. \tag{9}$$

From (8) and (9) we obtain

$$Q_j=q_j\circ g \text{ and } P_j=p_j\circ g,$$

where

$$q_j=\tilde{q}_j/R,\ j=1,\cdots,m.$$

And this is what we needed.

Proof of Corollary. If F had only finitely many fix-points, then by the assumption, we would have

$$F(z)-z=Q(z)e^{P(z)},$$

where $Q(z)$ is rational and $P(z)$ is a polynomial. This means that the function

$$F(z)=z+Q(z)e^{P(z)}$$

would be composite, which violates the conclusion of Theorem 3.

Proof of Theorem 4 By Theorem 1, $h(z)$ is pseudo-prime. Suppose that $h=f\circ g$. We deal with two cases.

Case 1 g is a polynomial of degree $\geqslant 2$. Assume first that $C\neq 0$. Put

$$\tilde{h}(z)=g(z)h(z)=Cg(z)+\tilde{Q}(z)e^{P(z)},$$

where

$$\tilde{Q}(z)=g(z)Q(z).$$

From the proof of Theorem 3, we see that $g(z)$ is a common right factor of $\tilde{Q}$ and P. Hence $\tilde{Q}=\tilde{q}\circ g$, where $\tilde{q}$ is rational. We thus obtain

$$Q(z)=\tilde{Q}(z)/g(z)=(q\circ g)(z),$$

where $g(\zeta)=\tilde{q}(\zeta)/\zeta$ is rational. Therefore, $g(z)$ is a common right factor of P and Q.

In the case when $C=0$, by setting

$$h_1(z)=1+h(z)$$

and to $h_1(z)$ applying the result just proved, we can easily reach the same conclusion.

Case 2 f is rational of degree $\geqslant 2$. Assume first that $C=0$. By Lemma 4, f is of the form

$$f(\zeta)=a(\zeta-b)^n,$$

where n is an integer with $|n|\geqslant 2$, and we may take $a=1$. Then we have

$$Q(z)e^{P(z)}=[g(z)-b]^n.$$

Hence

$$Q(z)=[(g(z)-b)e^{-\frac{1}{n}P(z)}]^n.$$

Let

$$\beta(z)=[g(z)-b]e^{-\frac{1}{n}P(z)}.$$

Then $\beta(z)$ must be rational and

$$Q(z)=\beta(z)^n. \tag{10}$$

For the case when $C\neq 0$, we set

$$h_1(z)=Q(z)e^{P(z)}.$$

Then $h_1=f_1\circ g$, where $f_1=f-C$ is a rational function of degree $\geqslant 2$. So we also deduce (10). The theorem is complete.

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