Chin. Ann. of Math.
6B (4) 1985

INTEGRAL FORMULAS FOR SUBMANIFOLDS IN EUCLIDEAN SPACE AND THEIR APPLICATIONS TO UNIQUENESS THEOREM

LI AH MIN (李安民)*

Abstract

In this paper, the author derives some integral formulas for a pair of submanifolds in Euclidean Space E^{n+p}, and applies these formulas to generalize the Christoffel theorem and the Hilbert Liebmann-Hsiung theorem.

§ 1. A Generalization of the Christoffel Theorem

Let Σ, Σ' be two n-dimensional compact submanifolds (without boundary) in E^{n+p}, $f: \Sigma \to \Sigma'$ be a diffeomorphism such that Σ and Σ' have parallel tangent spaces at $x \in \Sigma$ and $x' = f(x)$, i. e. they have the same Gauss image. Choose a local frame field x, $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$ over Σ such that $e_1, \cdots, e_n$ are tangent to Σ. Then it is also a local frame field over Σ' and $e_1, \cdots, e_n$ are tangent to Σ'.

Throughout this paper we shall agree on the indices of the following ranges:

$$1 \leqslant i, j, \cdots, \leqslant n, \quad n+1 \leqslant \alpha, \beta, \cdots, \leqslant n+p, \quad 1 \leqslant A, B, \cdots, \leqslant n+p.$$

Let $\widetilde{w}_A$ be the field of dual frames, the structure equations of E^{n+p} are given by

$$\begin{gathered} dx = \widetilde{w}_A e_A, \quad de_A = \widetilde{w}_{AB} e_B, \\ d\widetilde{w}_A = \widetilde{w}_B \wedge \widetilde{w}_{BA}, \quad \widetilde{w}_{AB} = -\widetilde{w}_{BA}, \\ d\widetilde{w}_{AB} = \widetilde{w}_{AC} \wedge \widetilde{w}_{CB}. \end{gathered} \tag{1.1}$$

Restricting them to Σ and Σ', we have

$$\begin{gathered} dx = w_i e_i, \quad dx' = w'_i e'_i, \\ w_\alpha = 0, \quad w'_\alpha = 0, \\ w_{i\alpha} = h_{i\alpha j} w_j, \quad w'_{i\alpha} = h'_{i\alpha j} w'_j, \\ dw_{ij} = w_{jk} \wedge w_{kj} + \Omega_{ij}, \quad dw'_{ij} = w'_{jk} \wedge w'_{kj} + \Omega'_{ij}, \\ \Omega_{ij} = w_{j\alpha} \wedge w_{\alpha j}, \quad \Omega'_{ij} = w'_{j\alpha} \wedge w'_{\alpha j}. \end{gathered} \tag{1.2}$$

We introduce the differential forms

$$D_{rs} = \delta^{i_1 \cdots i_n}_{1 \cdots n} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{e-1} i_e} \wedge \underbrace{w_{i_{e+1}} \wedge \cdots \wedge w_{i_{e+r}}}_{r} \wedge \underbrace{w'_{i_{e+r+1}} \wedge \cdots \wedge w'_{i_n}}_{s}, \tag{1.3}$$

Manuscript received September 3, 1983. Revised November, 14, 1983.
* Department of Mathematies, Sichan University, Chengdu, china.

$$C^{\alpha}_{r-1,s}=\delta^{i_1\cdots i_n}_{1\cdots n}\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{e-1}i_e}\wedge w_{i_{e+1},\alpha}\wedge\underbrace{w_{i_{e+2}}\wedge\cdots\wedge w_{i_{e+r}}}_{r-1}\wedge\underbrace{w'_{i_{e+r+1}}\wedge\cdots\wedge w'_{i_n}}_{s},$$

$$C_{r-1,s}=C^{\alpha}_{r-1,s}e_{\alpha},$$

where $e=$ even, $r+s=n-e$.

Consider the deformation of Σ

$$\Sigma_t\colon x_t=(1+t)x,\quad t\in(-\varepsilon,\ \varepsilon),\ x\in\Sigma.$$

Since

$$dx_t=(1+t)dx=(1+t)w_ie_i,$$

we see that $e_1, e_2, \cdots, e_n$ are tangent to Σ_t and

$$w_i(t)=(1+t)w_i. \tag{1.4}$$

Let

$$D_{rs}(t)=\delta^{i_1\cdots i_n}_{1\cdots n}\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{e-1}i_e}\wedge w_{i_{e+1}}(t)\wedge\cdots\wedge w_{i_{e+r}}(t)\wedge w'_{i_{e+r+1}}\wedge\cdots\wedge w'_{i_n}.$$

It is a differential form containing the parameter t on Σ. From (1.4) we have

$$D_{rs}(t)=(1+t)^rD_{rs},$$

$$\left.\frac{\partial D_{rs}(t)}{\partial t}\right|_{t=0}=rD_{rs}. \tag{1.5}$$

Now we compute $\left.\frac{\partial D_{rs}(t)}{\partial t}\right|_{t=0}$ in another way. We consider Σ and Σ' to be two imbeddings

$$\Sigma\colon x\colon M\to E^{n+p},$$

$$\Sigma'\colon x'\colon M\to E^{n+p},$$

where M is a compact manifold and $x'=f\circ x$. Then x_t can be considered to be an immersion

$$x_t\colon M\times(-\varepsilon,\varepsilon)\to E^{n+p}.$$

Pulling $\widetilde{w}_A$ back to $M\times(-\varepsilon,\ \varepsilon)$, we have (see [5, 6])

$$\widetilde{w}_A=dx_t\cdot e_A=d_mx_t\cdot e_A+\frac{\partial x_t}{\partial t}\cdot e_Adt=w_A+x\cdot e_Adt.$$

Noting $w_{\alpha}=0$, we have

$$\begin{aligned}\widetilde{w}_i(t)&=w_i(t)+x_idt,\\ \widetilde{w}_{\alpha}(t)&=x_{\alpha}dt,\\ \widetilde{w}_{AB}(t)&=w_{AB},\end{aligned} \tag{1.6}$$

where w_i, w_{AB} do not contain dt, and

$$x_A=x\cdot e_A,$$

Let

$$\widetilde{D}_{rs}(t)=\delta^{i_1\cdots i_n}_{1\cdots n}\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{e-1}i_e}\wedge\widetilde{w}_{i_{e+1}}(t)\wedge\cdots\wedge\widetilde{w}_{i_{e+r}}(t)\wedge w'_{i_{e+r+1}}\wedge\cdots\wedge w'_{i_n}. \tag{1.7}$$

It can be written as

$$\widetilde{D}_{rs}(t)=D_{rs}(t)+dt\wedge\phi_{rs}(t), \tag{1.8}$$

where

$$\phi_{rs}(t)=\gamma\delta^{i_1\cdots i_n}_{1\cdots n}\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{e-1}i_e}x_{i_{e1}}\wedge w_{i_{e+2}}(t)\wedge\cdots\wedge w_{i_{e+r}}(t)\wedge w'_{i_{e+r+1}}\wedge\cdots\wedge w'_{i_n}.$$

We choose $e_1, e_2, \cdots, e_n$ such that $w_{ij}|_x=0$ for a fixed point $x\in\Sigma$. We write the operator d on $M\times(-\varepsilon,\ \varepsilon)$ as $d=d_m+dt\wedge\frac{\partial}{\partial t}$ (see [5]) Taking the exterior derivative of (1.7), we get

$$d\widetilde{D}_{rs}(t) = -\gamma\delta_{1\cdots n}^{i_1\cdots i_n}\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{e-1}i_e}\wedge\widetilde{w}_\alpha(t)\wedge\widetilde{w}_{i_{e+1},\alpha}(t)$$
$$\wedge\widetilde{w}_{i_{e+2}}(t)\wedge\cdots\wedge\widetilde{w}_{i_{e+r}}(t)\wedge w'_{i_{e+r+1}}\wedge\cdots\wedge w'_{i_n}.$$

On the other hand, we have

$$d\widetilde{D}_{rs}(t) = d_M D_{rs}(t) + dt\wedge\frac{\partial D_{rs}(t)}{\partial t} - dt\wedge d_M\phi_{rs}(t).$$

Equating the terms involving dt and setting $t=0$, we get

$$\left.\frac{\partial D_{rs}(t)}{\partial t}\right|_{t=0} = d_M\phi_{rs} - r x_\alpha C^\alpha_{r-1,s}. \tag{1.9}$$

Comparing (1.9) with (1.5) and integrating over M, we get the following integral formulas

$$\int D_{rs} + x_\alpha C^\alpha_{r-1,s} = 0. \tag{1.10}$$

Similarly, we have

$$\int D_{rs} + x'_\alpha C^\alpha_{r\ s-1} = 0. \tag{1.11}$$

From (1.10), (1.11) it follows that

$$\int x_\alpha C^\alpha_{i-1,s} - x'_\alpha C^\alpha_{r,s-1} = 0. \tag{1.12}$$

The formulas (1.10), (1.11), (1.12) are generalizations of the formulas (17) in [1]. In particular, we have

$$\int D_{r0} + x_\alpha C^\alpha_{r-1,0} = 0. \tag{1.13}$$

Let

$$I_e = \frac{(-1)^{e/2}(n-e)!}{2^{e/2}n!}\delta_{j_1\cdots j_e}^{i_1\cdots i_e}R_{i_1i_2j_1j_2}\cdots R_{i_{e-1}i_ej_{e-1}j_e},$$

$$\underset{e+1}{H} = \frac{(-1)^{e/2}(n-e-1)!}{2^{e/2}n!}\delta_{j_1\cdots j_{e+1}}^{i_1\cdots i_{e+1}}R_{i_1i_2j_1j_2}\cdots R_{i_{e-1}i_ej_{e-1}j_e}h_{i_{e+1},\alpha j_{e+1}}\,e_\alpha,$$

where R_{ijkl} are the Riemannian curvature tensors of Σ. The formula (1.13) can be written as

$$\int (I_e + x\cdot\underset{e+1}{H})\omega_1\wedge\cdots\wedge\omega_n = 0, \tag{1.14}$$

this is the integral formula (19) in [2].

Suppose that there exists a unit normal vector field e_{n+p} over Σ (it is also a unit normal vector field over Σ') such that the second fundamental forms of both Σ and Σ' at e_{n+p} are positive definite symmetric matrices. Let

$$h_{ij} := h_{i,n+p,j},\quad h'_{ij} := h'_{i,n+p,j},$$
$$K = \det(h_{ij}),\quad K' = \det(h'_{ij}),$$
$$H = \frac{1}{n}h_{ii\alpha}e_\alpha,\quad H' = \frac{1}{n}h'_{ii\alpha}e_\alpha,$$
$$(\lambda_{ij}) = (h_{ij})^{-1},\quad (\lambda'_{ij}) = (h'_{ij})^{-1},$$
$$P_{rs} = \frac{(n-r-s)!}{n!}\delta_{j_1\cdots j_{r+s}}^{i_1\cdots i_{r+s}}\lambda_{i_1j_1}\cdots\lambda_{i_rj_r}\lambda'_{i_{r+1}j_{r+1}}\cdots\lambda'_{i_{r+s}j_{r+s}}.$$

Theorem 1. *Let Σ, Σ' be two closed n-dimensional submanifolds in E^{n+p}, f: $\Sigma \to \Sigma'$ be a diffeomorphism such that Σ and Σ' have parallel tangent spaces at $x \in \Sigma$ and $x' = f(x)$. suppose that there exists a unit normal vector field e_{n+p} over Σ and Σ' such that the second fundamental forms of both Σ and Σ' at e_{n+p} are positive. If $K = K'$ and $H = H'$, then f is a translation.*

Proof From (1.10) it follows that

$$\int D_{1,n-1} + x_\alpha C^\alpha_{0,n-1} = 0,$$

$$\int D_{n,0} + x_\alpha C^\alpha_{n-1,0} = 0.$$

From these we obtain

$$\int (D_{1,n-1} - D_{n,0}) + \int x_\alpha (C^\alpha_{0,n-1} - C^\alpha_{n-1,0}) = 0.$$

According to our choice of the frame fields over Σ and Σ', we have

$$w_{i\alpha} = w'_{i\alpha}.$$

Since $w_i = \lambda_{ij} w_{j,n+p}$, $w'_i = \lambda'_{ij} w'_{j,n+p} = \lambda'_{ij} w_{j,n+p}$, we have

$$D_{1,n-1} = \delta^{i_1 \cdots i_n}_{1 \cdots n} w_{i_1} \wedge w'_{i_2} \wedge \cdots \wedge w^{i'}_n = \delta^{i_1 \cdots i_n}_{1 \cdots n} \lambda_{i_1 j_1} \lambda'_{i_2 j_2} \cdots \lambda'_{i_n j_n} w_{j_1 n+p} \wedge \cdots \wedge w_{j_n, n+p}$$

$$= \frac{1}{n!} \delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} \lambda_{i_1 j_1} \lambda'_{i_2 j_2} \cdots \lambda'_{i_n j_n} w_{1,n+p} \wedge \cdots \wedge w_{n,n+p} = P_{1,n-1} dV,$$

where $dV = w_{1,n+p} \wedge \cdots \wedge w_{n,n+p}$.

Similarly, we have

$$D_{n,0} = P_{n,0} dV,$$

$$C_{0,n-1} = \delta^{i_1 \cdots i_n}_{1 \cdots n} w_{i_1 \alpha} \wedge w'_{i_2} \wedge \cdots \wedge w'_{i_n} c_\alpha = \delta^{i_1 \cdots i_n}_{1 \cdots n} w'_{i_1 \alpha} \wedge w'_{i_2} \wedge \cdots \wedge w'_{i_n} e_\alpha$$

$$= H' w'_1 \wedge \cdots \wedge w'_n = \frac{H'}{K'} dV,$$

$$C_{n-1,0} = \frac{H}{K} dV.$$

Hence we get

$$\int (P_{1,n-1} - P_{n0}) dV + \int x \cdot \left(\frac{H'}{K'} - \frac{H}{K} \right) dV = 0,$$

where $dV = w_{1,n+p} \wedge \cdots \wedge w_{n,n+p}$.

From the hypotheses $K = K'$, $H = H'$, we get

$$\int (P_{1,n-1} - P_{n0}) dV = 0.$$

Since $P_{0n} = \frac{1}{K'} = \frac{1}{K} = P_{n0}$, by Gårding's inequality we get

$$P_{1,n-1} \geqslant P_{n0}.$$

The equality sign holds only if $\lambda_{ij} = \rho \lambda'_{ij}$. Since $P_{0n} = P_{n0}$, we have $\rho = 1$. Hence Σ and Σ' differ from each other by a translation.

Theorem 2. *Let Σ, Σ' be two submanifolds descrrbed in Theorem 1, and* $\dim(\Sigma) = \dim(\Sigma') = 2$. *If $\frac{H}{K} = \frac{H'}{K'}$, then f is a translation.*

Proof From (1.10) it follows that

$$\int D_{11}+x\cdot C_{01}=0,$$

$$\int D_{20}+x\cdot C_{10}=0,$$

$$\int D_{11}+x'\cdot C_{10}=0,$$

$$\int D_{02}+x'\cdot C_{01}=0.$$

From these we get

$$\int (2D_{11}-D_{02}-D_{20})=\int (x-x')\cdot(C_{10}-C_{01}).$$

It can be written as

$$\int (2p_{11}-p_{02}-p_{20})dV=\int (x-x')\cdot\left(\frac{H}{K}-\frac{H'}{K'}\right)dV.$$

From the hypothesis $\frac{H}{K}=\frac{H'}{K'}$, we have

$$\int (2P_{11}-P_{02}-P_{20})dV=0.$$

On the other hand we have

$$\begin{aligned}2P_{11}-P_{02}-P_{20}&=(\lambda_{11}\lambda'_{22}+\lambda_{22}\lambda'_{11}-\lambda_{12}\lambda'_{21}-\lambda_{21}\lambda'_{12})\\&\quad-(\lambda'_{11}\lambda'_{22}-\lambda'_{12}\lambda'_{21})-(\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21})\\&=(\lambda_{11}-\lambda'_{11})(\lambda'_{22}-\lambda_{22})+(\lambda_{12}-\lambda'_{12})^2\\&=-\frac{1}{2}[(\lambda_{11}+\lambda_{22})-(\lambda'_{11}+\lambda'_{22})]^2+\frac{1}{2}(\lambda_{11}-\lambda'_{11})^2\\&\quad+\frac{1}{2}(\lambda_{22}-\lambda'_{22})^2+(\lambda_{12}-\lambda'_{12})^2.\end{aligned}$$

Since $\frac{H}{K}=\frac{H'}{K'}$, we have $(\lambda_{11}+\lambda_{22})-(\lambda'_{11}+\lambda'_{22})=0$. Hence

$$2P_{11}-P_{02}-P_{20}\geqslant 0.$$

The equality sign holds only if $\lambda_{ij}=\lambda'_{ij}$.

§ 2. Some Generalizations of the Hilbert–Liebmann–Hsiung theorem

Definition. *Let e_{n+p} be a unit normal vector field over Σ. Σ is called convex with respect to e_{n+p} if, for each $x\in\Sigma$, Σ is contained in one of the closed half spaces*

$$H_x^+=\{y\in E^{n+p}:\ (y-x)\cdot e_{n+p}(x)\geqslant 0\}$$

and

$$H_x^-=\{x\in E^{n+p}: (y-x)\cdot e_{n+p}(x)\leqslant 0\}.$$

If Σ is convex with respect to e_{n+p} and

$$\Sigma\cap\{y\in E^{n+p}: (y-x)\cdot e_{n+p}(x)=0\}=\{x\},$$

then Σ is said to be strictly convex with respect to e_{n+p}. This deffinition is a generalization of the convexity of hypersurface. It is easy to prove that if Σ is strictlyconvex with respect to e_{n+p}, then $(h_{i,n+p,j})$ is definitee and when we choose the origin $0\in\Sigma$, $x\cdot e_{n+p}(x)$ are of the same sign over Σ.

In the following we assume

A. There exist P unit normal vector fields $e_{n+1}, e_{n+2}, \cdots, e_{n+p}$ over Σ such that Σ is umbilical with respect to e_τ $(\tau=n+1, \cdots, n+p-1)$, and Σ is strictly convex with respect to e_{n+p}.

Denote the principal curvature at e_τ by $C_\tau(x)$, the principal curvatures at e_{n+p} by $\lambda_1, \lambda_2, \cdots, \lambda_n$. We put

$$S_r=\frac{1}{\binom{n}{r}}\Sigma\lambda_1\lambda_2\cdots\lambda_r. \tag{2.1}$$

Theorem 3. *If Σ satisfies assnmption A and $I_e=const.$ for a fixed even e, $2<e<n$, then Σ is a sphere.*

Proof From the integaal formulas (1.14) it follows that

$$\int dm+\int x\cdot \underset{1}{H}dm=0,$$

$$\int I_e dm+\int x\cdot \underset{e+1}{H}\,dm=0 \quad (dm=w_1\wedge\cdots\wedge w_n).$$

Since $I_e=$const., we get

$$\int (I_e H_1-\underset{e+1}{H})\cdot x\,dm=0. \tag{2.2}$$

Let

$$C^2(x)=\sum_{\tau=n+1}^{n+p-1}c_\tau^2(x),$$

$$h_{ij}:=h_{i,n+p,j},$$

$$\underset{e+1}{h}=\frac{(n-e-1)!}{n!}\delta^{i_1\cdots i_{e+1}}_{j_1\cdots j_{e+1}}(c^2\delta_{i_1j_1}\delta_{i_2j_2}+h_{i_1j_1}h_{i_2j_2})\cdots(c^2\delta_{i_{e-1}j_{e-1}}\delta_{i_ej_e}+h_{i_{e-1}j_{e-1}}h_{i_ej_e})h_{i_{e+1}j_{e+1}}.$$

Then

$$\underset{1}{H}=(C_{n+1}, C_{n+1}\cdots, C_{n+p-1}, S_1),$$

$$\underset{e+1}{H}=(C_{n+1}I_e, \cdots, C_{n+p-1}I_e, h_{e+1}).$$

From (2.2) we get

$$\int (I_e S_1-\underset{e+1}{h})x_{n+p}dm=0, \tag{2.3}$$

where

$$x_{n+p}=x\cdot e_{n+p}(x).$$

Since

$$\begin{aligned}I_e&=\frac{(n-e)!}{n!}\delta^{i_1\cdots i_e}_{j_1\cdots j_e}(c^2\delta_{i_1j_1}\delta_{i_2j_2}+h_{i_2j_2})\cdots(c^2\delta_{i_{e-1}j_{e-1}}\delta_{i_ej_e}+h_{i_{e-1}j_{e+1}}h_{i_ej_e})\\&=\sum_{r=0}^{e/2}\binom{e/2}{r}S_{2r}C^{e-2r},\end{aligned} \tag{2.4}$$

$$\underset{e+1}{h}=\sum_{r+0}^{e/2}\binom{\frac{e}{2}}{r}S_{2r+1}C^{e-2r},$$

we get

$$\sum_{r=0}^{e/2}\binom{e/2}{r}\int(S_{2r}S_1-S_{2r+1})x_{n+p}C^{e-2r}\,dm=0. \tag{2.6}$$

Since (h_{ij}) is positive definite, we have

$$S_{2r}S_1-S_{2r+1}\geqslant 0 \text{ for } 0\leqslant r\leqslant\frac{e}{2}.$$

The equality sign holds only if $\lambda_1=\lambda_2=\cdots=\lambda_n$. Hence Σ is also umbilical with respect to e_{n+p}. Therefore Σ is a sphere (see[3, 4]).

Theorem 4. *If Σ satisfies assumption A, and there exist two evens e, τ, $2\leqslant\tau<e\leqslant n$, such that $\frac{I_e}{I_\tau}=$const.$=a$, then Σ is a spere.*

Proof From (1.14) it follows that

$$\int x\cdot(\underset{e+1}{H}-a\underset{\tau+1}{H})\,dm=-\int(I_e-aI_\tau)dm=0$$

or

$$\int(\underset{e+1}{h}-a\underset{\tau+1}{h})x_{n+p}\,dm=0. \tag{2.7}$$

The following inequality is valid

$$\frac{I_\tau}{\underset{\tau+1}{h}}\leqslant\frac{I_{\tau+2}}{\underset{\tau+3}{h}}.$$

In fact, from (2.5), (2.6) we have

$$I_{\tau+2}\underset{\tau+1}{h}=\sum_{i-0}^{\tau/2}\sum_{j=0}^{\tau/2}\binom{\frac{\tau}{2}+1}{i}\binom{\frac{\tau}{2}}{j}S_{2i}S_{2j+1}C^{2(\tau+1-i-j)}+\sum_{j=0}^{\tau/2}\binom{\tau/2}{j}S_{2j+1}S_{\tau+2}C^{\tau-2j},$$

$$I_\tau\underset{\tau+3}{h}=\sum_{i=0}^{\tau/2}\sum_{j=0}^{\tau/2}\binom{\frac{\tau}{2}+1}{i}\binom{\frac{\tau}{2}}{j}S_{2i+1}S_{2j}C^{2(\tau+1-i-j)}+\sum_{j=0}^{\tau/2}\binom{\frac{\tau}{2}}{j}S_{2j}S_{\tau+3}C^{\tau-2j}.$$

Hence

$$I_{\tau+2}\underset{\tau+1}{h}-I_\tau\underset{\tau+3}{h}=\frac{1}{2}\sum_{i+0}^{\tau/2}\sum_{j=0}^{\tau/2}\left[\binom{\frac{\tau}{2}+1}{i}\binom{\frac{\tau}{2}}{j}-\binom{\frac{\tau}{2}+1}{j}\binom{\frac{\tau}{2}}{i}\right](S_{2i}S_{2j+1}-S_{2j}S_{2i+1})$$

$$\cdot C^{2(\tau+1-i-j)}+\sum_{j=0}^{\tau/2}\binom{\tau/2}{j}(S_{2j+1}S_{\tau+2}-S_{2j}S_{\tau+3})C^{\tau-2j}.$$

Since

$$\binom{\frac{\tau}{2}+1}{i}\binom{\frac{\tau}{2}}{j}-\binom{\frac{\tau}{2}+1}{j}\binom{\frac{\tau}{2}}{i}=\frac{\left(\frac{\tau}{2}\right)!^2\left(\frac{\tau}{2}+1\right)(i-j)}{i!j!\left(\frac{\tau}{2}+1-i\right)!\left(\frac{\tau}{2}+1-j\right)!}$$

and

$$\frac{S_{2i}}{S_{2i+1}} \leqslant \frac{S_{2j}}{S_{2j+1}} \text{ for } i<j,$$

it is easy to obtain

$$I_{\tau+2} \underset{\tau+1}{h} - I_\tau \underset{\tau+3}{h} \geqslant 0.$$

Thus

$$\frac{I_\tau}{\underset{\tau+1}{h}} \leqslant \frac{I_{\tau+2}}{\underset{\tau+3}{h}} \leqslant \cdots \leqslant \frac{I_e}{\underset{e+1}{h}}.$$

Since

$$\underset{e+1}{h} - a \underset{\tau+1}{h} = \frac{1}{I_\tau}\left(I_\tau \underset{e+1}{h} - I_e \underset{\tau+1}{h}\right) \leqslant 0,$$

from (2.7) we get $\underset{e+1}{h} - a \underset{\tau+1}{h} = 0$, which is possible only when $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Hence Σ is a sphere.

In the following we further assume

B. $C_{n+1}^2(x) + \cdots + C_{n+p-1}^2(x) = C^2(x) = \text{const.}$

Under hypothesis B we can derive the following integral formulas by induction

$$\int\left(1+\sum_{\tau=n+1}^{n+p-1} C_\tau x_\tau\right) S_e\, dm + \int x_{n+p} S_{e+1}\, dm = 0. \tag{2.8}$$

Using (2.8) we can prove the following

Theorem 5. *If Σ satisfies A, B and $S_r=$const., then Σ is a sphere.*

Theorem 6. *If Σ satisfies A, B and $S_e/S_\tau=$*const. *for two evens τ and e, then Σ is a sphere ($\tau \neq e$).*

When Σ is a strictly convex hypersurface in S^{n+1}, Σ satisfies A and B automatically. Hence Σ is a sphere if $S_r=$const. for a fixed even e $\left(\text{or } \frac{S_e}{S_\tau}=\text{const. for two evens } e \text{ and } \tau,\ 2\leqslant\tau<e\leqslant n\right)$.

References

[1] Chern, S. S. Integral formulas for hypersurfaces in euclidean space and their applications to uniqueness theorems, *Jour. Math. and Mech.*, **8** (1959), 947—965.

[2] Chen Weihuan, Some integral formulas on submanifolds in Euclidean space, Proc. of Beijing Symp. on Diff. Geom. and Diff. Equa., Vol. 3 (1980), 1127—1140.

[3] Shing Tungyau, Submanifolds with constant mean curvature. *Amer. Jour. of Math.*, 96. 1, 346—366.

[4] Chen Bangyen & Yano, K., Integral formulas for submanifolds and their applications, *Jour. ofPiff. Geom.*, **5** (1971), 467—477.

[5] Chern, S. S., Minimal submanifilds in a Riemannian manifold, Mimeographed Lecture Notes, Univ. of Kansas, 1968.

[6] Li An min (李安民), On the Uniqueness theorems for the closed convex hypersurfaces in a space of constant curvature (to appear).