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TORSION THEORIES AND PRIMARY DECOMPOSITION OF MODULES

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Abstract

In this paper, the author obtains the following result: If R is a left semi-artinian max ring, then the primary decomposition theorem holds for all R-modules if and only if Ext $(S_1, S_2)=0$ for any non-isomorphic simple modules S_1 and S_2. By mecns of the primary decomposition of modules, the author proves: All torsion theories of $_R\mu$ are simple type if and only if R isa left semi-artinian max ring and Ext $(S_1, S_2)=0$ for any non-isomorphic simple modules S_1, S_2.

It is well known that many torsion-theoretic results are obtained for hereditary torsion theories. Thus it is desirable to characterize ring R such that all torsion theories of $_R\mu$, the category of unitary left R–modules, are hereditary. K. Ohtake[1] has given an answer to this problem in the case of commutative rings. We shall answer the question in the case of left semi-artinian rings, and characterize ring R such that every torsion theory of $_R\mu$ is simple type. In § 2 we establish a primary decomposition therem, which is a generalization of Bronowitz and Teply's result[2]. It will be of interest in other situations.

§ 1. Preliminaries

Throughout this paper, R denotes an associative ring with identity and $_R\mu$ the category of unitary left R–modules.

In [3], Dickson defined a torsion theory on $_R\mu$ to be pair $(\mathscr{T}, \mathscr{F})$ of classes of modules such that:

(1) $\mathscr{T}\cap\mathscr{F}=0$;

(2) $N\subseteq M$ and $M\in\mathscr{T}$ imply $M/N\in\mathscr{T}$;

(3) $N\subseteq M$ and $M\in\mathscr{F}$ imply $N\in\mathscr{F}$:

(4) For each $M\in {_R\mu}$, there exists a unique submodule M_t of M such that $M_t\in\mathscr{T}$ and $M/M_t\in\mathscr{F}$.

The class $\mathscr{T}$ is called the torsion class and $\mathscr{F}$ the torsionfree class. M_t is called

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the $\mathscr{T}$-torsion submodule of M. A torsion class $\mathscr{T}$, and the associated torsion theory $(\mathscr{T}, \mathscr{F})$, is called hereditary if $\mathscr{T}$ is closed under submodules.

Let φ be a complete set of representatives of the non-isomorphic simple R-modules. A torsion theory $(\mathscr{T}, \mathscr{F})$ is said to be simple type if $\mathscr{T}$ is generated by a subset of φ[2,4]. For each $S\in\varphi$, we say that the R-modulet M is S-primary, if each non-zero homomorphic image of M contains an isomorphic copy of S. A ring R is said to have primary decomposition[5,6] if, for each $M\in\mathscr{T}_\varphi$, the torsion class generated by φ, $M=\oplus_{s\in\varphi}M_s$, where M_s is the S-primary submodule of M.

Goldie's torsion class $\mathscr{Y}$ for ${}_R\mu$ [7,8] is the smallest torsion class containing all modules M/N, where N is an essential submodule of M. The torsionfree class $\mathscr{N}$ associated with $\mathscr{Y}$ is precisely the class of non-singular modules. $(\mathscr{Y}, \mathscr{N})$ is called Goldie's torsion theory.

A ring R is called left semi-artinian [9] if non-zero R-modules have non-zero soscles. In this case $\mathscr{T}_\varphi={}_R\mu$. A ring R is said to be a left max ring if every left module has a maximal submodule.

§ 2. The Primary Decomposition

Lemma 2.1.[5] *A ring R has u primary decomposition if and only if, for any R-module M and simple R-module $S_0\in\varphi$, the extension of the form*

$$0\to\sum_{s\neq s_0}M_s\to X\to S_0\to 0$$

is split exactly.

In this section, our main result is the following

Theorem 2.2. *Let Rcea left semiartinian max ring, then R has a primary decomposition if and only Extfor any non-isomorphic simple modules and S_2.*

To prove the theorem, the following two lemmas will be needed.

Lemma 2.3. *Let R be a ring, $X\in{}_R\mu$ and $I=\bigcap(L\subseteq X\colon L\neq 0,\ X/L\in\mathscr{Y})$. Then $I\in\mathscr{N}$. Here $(\mathscr{Y}, \mathscr{N})$ is the Goldie's torsion theory for ${}_R\mu$,*

Proof By Zorn's lemma, there exists a submodule J of X maximal with respect to $J\cap\mathscr{Y}(I)=0$, then $J+\mathscr{Y}(I)$ is essential in X. Consider the exact sequence

$$0\to\frac{J+\mathscr{Y}(I)}{J}\to\frac{X}{J}\to\frac{X}{J+\mathscr{Y}(I)}\to 0.$$

Since $\mathscr{Y}$ is closed under extension, $X/J\in\mathscr{Y}$. $J\supseteq I$ by the definition of I, and hence $\mathscr{Y}(I)=0$ since $J\supseteq I\supseteq\mathscr{Y}(I)$.

Lemma 2.4. *Let $S_0\in\varphi$. Then* Hom $(S_0, \prod_A M_s/\sum_A M_s)=0$ *for any module M, where M_s is the S-primary submodule of M, and A may be any subset of $\varphi-\{S_0\}$.*

Proof Suppose, to the contrary, that S_0 is isomorphic to the submodule $R((x_s)$

$+\sum_A M_s)$ of $\prod_A M_s/\sum_A M_s$. Let $N=(\sum_A M_s:(x_\alpha))$. Then $R/N\cong S_0$. For every $n\in N$, $n(x_\alpha)\in\sum_A M_s$, and so $nx_\alpha=0$ for all but finitely many $\alpha\in A$. Hence $N\subseteq\sum_{\alpha\in A}(0:x_\alpha)$. Thus $N=\sum_{\alpha\in A}(0:x_\alpha)$, since $R\neq\sum_{\alpha\in A}(0:x_\alpha)$.

Finaly, $S_0\cong R/N=R/\sum_{\alpha\in A}(0:x_\alpha)$. This is a contradiction since $R/(0:x_\alpha)\cong S\not\cong S_0$ for each $\alpha\in A$. Hence $\mathrm{Hom}\,(S_0,\prod_A M_s/\sum_A M_s)=0$.

Proof of the theorem 2.2

The necessity is obvious by Lemma 2.1.

We show that the condition in the theorem is sufficient.

Let M be an R-module and $S_0\in\varphi$. Let $X=\prod_{s\neq s_0} M_s$ and $Y=\sum_{s\neq s_0} M_s$. By Lemma 2.1 it is sufficient to show $\mathrm{Ext}(S_0, Y)=0$.

Consider the exact sequence

$$0\to Y\to X\to X/Y\to 0.$$

We get the exact sequence

$$\mathrm{Hom}(S_0, X/Y)\to\mathrm{Ext}(S_0, Y)\to\mathrm{Ext}(S_0, X).$$

By Lemma 2.4 $\mathrm{Hom}(S_0, X/Y)=0$. Thus it is sufficient to show $\mathrm{Ext}(S_0, X)=0$.

By noting that $\mathrm{Ext}(S_0, X)=\mathrm{Ext}(S_0,\prod_{s\neq s_0} M_s)=\prod_{s\neq s_0}\mathrm{Ext}(S_0, M_s)$, it is clear that $\mathrm{Ext}(S_0, X)=0$ if $\mathrm{Ext}(S_0, M_s)=0$ for all S-prinary modules, where $S\in\varphi$, $S\neq S_0$.

Consider an exact sequence

$$(*)\qquad 0\to M_s\to W\to S_0\to 0.$$

If M_s is not essential in W, then there exists a snbmodule N of W such that $N\cap M_s=0$. Thus $W=M_s\oplus N$ and hence W has a submodule N isomorphic to S_0. This means that the sequence $(*)$ splits.

If M_s is essential in W, then $I=\cap\{L\subseteq W\colon L\neq 0,\ W/L\in\mathscr{Y}\}$ is in $\mathscr{N}$ by Lemma 2.3, and $S\subseteq I$ since R is left semi-artinian and M_s is S-primary, thus $S\in\mathscr{N}$. By [10], S is a projective module. Since R is a max ring, there exists a maximal submodule K_s of M_s such that $M_s/K_s\cong S$, and hence $M_s=S\oplus K_s$ since S is projective. It is not difficult to show that $M_s=M_0\oplus K_s$ since S is projective. It is not difficult to show that $M_s=M_0\oplus\mathrm{Soc}\,M_s$ since $\mathrm{Soc}\,M_s=\Sigma S$. Since R is left semi-artinian, M_s is an essentilal extension of $\mathrm{Soc}\,M_s$, and hence $M_s=\mathrm{Soc}\,M_s$.

As before, we get an exact sequence

$$\mathrm{Hom}(S_0, \Pi S/\Sigma S)\to\mathrm{Ext}(S_0, \Sigma S)\to\mathrm{Ext}(S_0, \Pi S).$$

By Lemma 2.4, $\mathrm{Hom}(S_0, \Pi S/\Sigma S)=0$.

On the other hand, $\mathrm{Ext}(S_0, \Pi S)=\Pi\mathrm{Ext}(S_0, S)=0$, since $S\neq S_0$. Hence $\mathrm{Ext}(S_0, M_s)=0$. Thus the sequence $(*)$ splits and so the proof is complete.

§ 3. Torsion Theory

Theorem 3.1. *All torsion theories of* $_R\mu$ *are simple type if and only if*

(1) *R is a left semi-artinian ring;*

(2) *R is a max ring;*

(3) $\mathrm{Ext}(S_1, S_2)=0$ *for any non-isomorphic simple modules* S_1, S_2.

Proof Necessity. See [2].

Sufficiency. Let $(\mathscr{T}, \mathscr{T}_+)$ be a torsion theory. It is sufficient to show that $(\mathscr{T}, \mathscr{T}_+)$ is hereditary since R is left semi-artinian.

Let $M\in\mathscr{T}$, and let S be a simple shbmodule of M. If we choose $N\sqsubseteq M$, maximal with respect to $N\cap S=0$, then S is essential in M/N. By Theorem 2.2, R has a primary decomposition. So M/N is a S-primary module. By (2), there exists a maximal submodule $M'/N\sqsubseteq M/N$, then $M/M'\cong S$, and heace $S\in\mathscr{T}$ since $\mathscr{T}$ is closed under homomorphic image.

Let $\varDelta=\{S$: S is a simple R-module and $S\sqsubseteq M/N$ for some $N\sqsubseteq M\}$. Then $M\subset\mathscr{T}_\varDelta$ since R is left semi-artinian, where $\mathscr{T}_\varDelta$ is the torsion theory generated by $\varDelta$. By the precoding paragraphs, $\varDelta\sqsubseteq\mathscr{T}$; so $\mathscr{T}_\varDelta\sqsubseteq\mathscr{T}$. It follows that any submodule of M is in $\mathscr{T}_\varDelta$ since $\mathscr{T}_\varDelta$ is hereditary. Thus $\mathscr{T}$ is hereditary and the proof is complete.

In the process of the proof of Theorem 3.1, we also prove the following

Theorem 3.2. *Let R be a left semi-artinian ring. Then all torsion theories are hereditary if and only if*

(i) *R is a max ring;*

(ii) $\mathrm{Ext}(S_1, S_2)=0$ *for any non-isomorphic simple modules* S_1 *and* S_2.

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