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ASYMPTOTIC REPRESENTATION FOR REMAINDER OF QUASI-HERMITE-FEJER INTERPOLATION POLYNOMIAL

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Abstract

Let $Q_{2n+1}(f, x)$ be the quasi-Hermite-Fejer interpolation polynomial of function $f(x)\in C_{[-1,1]}$ based on the zeros of the Chebyshev polynomial of the second kind $U_n(x)=\frac{\sin((n+1)\arc\cos x)}{\sin(\arc\cos x)}$. In this paper, the uniform asymptotic representation for the quantity $|Q_{2n+1}(f,x)-f(x)|$ is given. A similar result for the Hermite-Fejer interpolation polynomial based on the zeros of the Chebyshev polynomial of tne first kind is also established.

§ 1. Introduction

Let

$$x_k=\cos\frac{k\pi}{n+1}\quad (k=1, 2, \cdots, n)$$

be the zeros of the Chebyshev polynomial of the second kind

$$U_n(x)=\frac{\sin(n+1)\arc\cos x)}{\sin(\arc\cos x)}\quad (-1\leqslant x\leqslant 1).$$

For a function $f\in C_{[-1,1]}$ we denote by $H_n(f, x)$ the Hermite-Fejer interpolation polynomial based on $\{x_k\}_{k=1}^n$. It is well known that sequence $\{H_n(f,x)\}_{n=1}^\infty$ converges to $f(x)$ pointwise in $(-1, 1)$ and uniformly on each closed sub-interval of $(-1,1)$. At the end points -1 and 1, however, the sequence may be divergent. For this, Szasz P.[1] introduced the quasi-Hermite-Fejer interpolation polynomial

$$Q_{2n+1}(f, x)=\frac{U_n^2(x)}{2(n+1)^2}\{(1+x)f(1)+(1-x)f(-1)\}$$
$$+\sum_{k=1}^{n} f(x_k)\frac{(1-x^2)(1-xx_k)U_n^2(x)}{(n+1)^2(x-x_k)}$$

and proved that the sequence $\{Q_{2n+1}(f, x)\}_{n=1}^\infty$ converges uniformly to $f(x)$ on $[-1, 1]$. A quantitative version of Szasz's result was given by Saxena R. B.[2], who proved that

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$$|Q_{2n+1}(f, x)-f(x)|\leqslant 5\omega\left(f, \frac{\ln n}{n}\right) \quad (n=2, 3, \cdots),$$

where $\omega(f, \delta)$ is the modulus of continuity of f on $[-1, 1]$. This inequality was improved by Saxena R. B. and Mathur K. K.[3], They showed that

$$|Q_{2n+1}(f, x)-f(x)|\leqslant \frac{c}{n}\sum_{k=1}^{n}\omega\left(f, \frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right) \quad (-1\leqslant x\leqslant 1),$$

where $c>0$ is an absolute constant. In a recent paper, Goodenough S. L. and Mills T. M.[4] considered the quantity

$$\Delta_n(x)=\sup_{f\in \mathrm{Lip}_1 1}|Q_{2n}(f, x)-f(x)| \quad (-1\leqslant x\leqslant 1)$$

and proved that if $x\in[-1, 1]$, then

$$\Delta_n(x)=\frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1}\left\{\frac{2}{\pi}(1-x^2)|U_n(x)|\ln n+O(1)\right\} \tag{1}$$

as $n\to\infty$. Observe that the formula (1) does not give uniform asymptotic representation of $\Delta_n(x)$ on the interval $[-1, 1]$. Thus it is natural to ask what is the uniform asymptotic representation for the quantity $\Delta_n(x)$ on the interval $[-1, 1]$. In this paper we shall give an answer to this question. The main result is the following

Theorem 1. *The asymptotic representation*

$$\Delta_n(x)=\frac{(1-x^2)U_n^2(x)}{n+1}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ln(1+(n+1)\arccos|x|)+|x| + \frac{1-x^2}{(n+1)|x-x_j|}+O(\sqrt{1-x^2})\right\}$$

holds uniformly on the interval $[-1, 1]$, *where* x_j *is the zero of* $U_n(x)$ *which is nearest to* x.

The proof of Theorem 1 will be given in § 2. In § 3 we shall establish similar result for the Hermite-Fejer interpolation polynomial based on the zeros of the Chebyshev polynomial of the first kind.

§ 2. Proof of the Theorem 1

Let $x\in[-1, 1]$. we may assume that x is not a node of interpolation for that case $\Delta(x)=0$ and there is nothing to prove. Since the function $f(t)=|t-x|(-1\leqslant t\leqslant 1)$ belongs to the class $\mathrm{Lip}_1 1$, we have

$$\Delta_n(x)=\frac{(1-x^2)U_n^2(x)}{(n+1)^2}+\sum_{k=1}^{n}|x_k-x|\frac{(1-x^2)(1-xx_k)U_n^2(x)}{(n+1)^2(x-x_k)^2}. \tag{2}$$

Set $x=\cos\theta$ $(0\leqslant\theta\leqslant\pi)$ and $\theta_k=\frac{k\pi}{n+1}(k=1, 2, \cdots, n)$, we have

$$\sum_{k=1}^{n}|x_k-x|\frac{(1-x^2)(1-xx_k)U_n^2(x)}{(n+1)^2(x-x_k)^2}=\frac{\sin^2(n+1)\theta}{(n+1)^2}A_n(\theta), \tag{3}$$

where

$$A_n(\theta)=\sum_{k=1}^{n}\left\{\left|\frac{\sin\frac{\theta_k+\theta}{2}}{2\sin\frac{\theta_k-\theta}{2}}\right|+\left|\frac{\sin\frac{\theta_k-\theta}{2}}{2\sin\frac{\theta_k+\theta}{2}}\right|\right\}.$$

Now let $0\leqslant x\leqslant 1$. If $x\in\left[\cos\frac{\theta_1}{2},1\right]$, then $0\leqslant\theta\leqslant\frac{\theta_1}{2}$. Hence

$$\frac{1}{\operatorname{tg}\frac{\theta_k-\theta}{2}}-\frac{1}{\operatorname{tg}\frac{\theta_k+\theta}{2}}=O\left(\frac{n+1}{k^2}\right), \tag{4}$$

where and in later $O(1)$ denote the bounded function of x and n. Since

$$\sin\frac{\theta_k\pm\theta}{2}=\pm\sin\theta\cos\frac{\theta_k\mp\theta}{2}+\cos\theta\sin\frac{\theta_k\mp\theta}{2},$$

we have

$$A_n(\theta)=n\cos\theta+\frac{\sin\theta}{2}\sum_{k=1}^{n}\left\{\frac{1}{\operatorname{tg}\frac{\theta_k-\theta}{2}}-\frac{1}{\operatorname{tg}\frac{\theta_k+\theta}{2}}\right\}.$$

From (4) we obtain

$$A_n(\theta)=n\cos\theta+O\{(n+1)\sin\theta\}. \tag{5}$$

Thus it follows that

$$\Delta_n(x)=\frac{(1-x^2)U_n^2(x)}{n+1}\{x+O(\sqrt{1-x^2})\}$$

by (2), (3) and (5).

If $x\in\left[0,\cos\frac{\theta_1}{2}\right]$, then $\frac{1}{2}\theta_1\leqslant\theta\leqslant\frac{\pi}{2}$. Choose the node $x_j=\cos\frac{j\pi}{n+1}$ such that

$$|x_j-x|\leqslant|x-x_k|\quad(k=1,2,\cdots,n),$$

then

$$|\theta-\theta_j|\leqslant\frac{\theta_1}{2}. \tag{7}$$

Set

$$I_k=\left|\frac{\sin\frac{\theta_k+\theta}{2}}{2\sin\frac{\theta_k-\theta}{2}}\right|+\left|\frac{\sin\frac{\theta_k-\theta}{2}}{2\sin\frac{\theta_k+\theta}{2}}\right|,$$

we have

$$A_n(\theta)=\sum_{k=1}^{j-1}I_k+\sum_{k=j+1}^{n}I_k+I_j, \tag{8}$$

and

$$I_k=x+\sqrt{1-x^2}\left\{\frac{n+1}{(k-j)\pi}-\frac{n+1}{(k+j)\pi}+O\left(1+\frac{n+1}{(k-j)^2}\right)\right\}\quad(j<k\leqslant n), \tag{9}$$

$$I_k=-x+\sqrt{1-x^2}\left\{\frac{n+1}{(j-k)\pi}+\frac{n+1}{(j+k)\pi}+O\left(1+\frac{n+1}{(j-k)^2}\right)\right\}\quad(1\leqslant k<j). \tag{10}$$

Furthermore

$$I_j=\frac{1-x^2}{|x-x_j|}+x\,\mathrm{sign}(x-x_j). \tag{11}$$

Substituting (9), (10) and (11) into (8), we obtain

$$A_n(\theta)=\frac{(n+1)\sqrt{1-x^2}}{\pi}\left\{2\ln j+\ln\frac{n-j}{n+j}+O(1)\right\}+(n+1-2j)x$$
$$+\frac{1-x^2}{|x-x_j|}+x\,\mathrm{sign}(x-x_j).$$

Since $j\leqslant\frac{n+1}{2}$, we have

$$A_n(\theta)=\frac{(n+1)\sqrt{1-x^2}}{\pi}(2\ln j+O(1))+(n+1-2j)x+\frac{1-x^2}{|x-x_j|}+x\,\mathrm{sign}(x-x_j). \tag{12}$$

Using the following inequalities

$$\ln j=\ln((n+1)\arccos x)+O(1),\ (x+1)=O\{(n+1)\sqrt{1-x^2}\},$$

and

$$\frac{j}{n+1}=O(\sqrt{1-x^2}),$$

we see that

$$A_n(\theta)=\frac{2(n+1)\sqrt{1-x^2}}{\pi}\ln((n+1)\arccos x)+(n+1)x$$
$$+\frac{1-x^2}{|x-x_j|}+O\{(n+1)\sqrt{1-x^2}\}.$$

Hence

$$\Delta_n(x)=\frac{(1-x^2)U_n^2(x)}{n+1}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ln((n+1)\arccos x)+x\right.$$
$$\left.+\frac{1-x^2}{(n+1)|x-x_j|}+O(\sqrt{1-x^2})\right\}. \tag{13}$$

Furthermore, it is easy to verify

$$\ln((n+1)\arccos x)=\ln((n+1)\arccos x+1)+O(1)\quad\text{for}\quad x\in\left[0,\cos\frac{\theta_1}{2}\right]$$

and

$$\ln((n+1)\arccos x+1)+\frac{\sqrt{1-x^2}}{(n+1)|x-x_1|}=O(1)\quad\text{for}\quad x\in\left[\cos\frac{\theta_1}{2},1\right].$$

Hence from (6) and (13) we obtain

$$\Delta_n(x)=\frac{(1-x^2)U_n^2(x)}{n+1}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ln((n+1)\arccos x+1)+x\right.$$
$$\left.+\frac{1-x^2}{(n+1)|x-x_j|}+O(\sqrt{1-x^2})\right\} \tag{14}$$

holds for $x\in[0, 1]$.

Similarly we can show

$$\Delta_n(x)=\frac{(1-x^2)U_n^2(x)}{n+1}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ln((n+1)\arccos|x|+1)-x\right.$$
$$\left.+\frac{1-x^2}{(n+1)|x-x_j|}+O(\sqrt{1-x^2})\right\} \tag{15}$$

holds for $x\in[-1, 0]$.

Finally (15), (14) imply

$$\Delta_n(x)=\frac{(1-x^2)U_n^2(x)}{n+1}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ln((n+1)\arccos|x|+1)+|x|\right.$$
$$\left.+\frac{1-x^2}{(n+1)|x-x_j|}+O(\sqrt{1-x^2})\right\}\quad(-1\leqslant x\leqslant 1)$$

and Theorem 1 follows.

Remarks. (I) It is easy to see that

$$\lim_{n\to\infty}\min_{|x|\leqslant 1}\frac{\frac{2}{\pi}\sqrt{1-x^2}\ln((n+1)\arccos|x|+1)+|x|}{\sqrt{1-x^2}}=\infty.$$

In fact, for $\sqrt{1-x^2}<\frac{1}{\sqrt{n+1}}$, we have

$$\frac{|x|}{\sqrt{1-x^2}}>\frac{1}{2}\ln(n+1),$$

and from the inequality

$$\sqrt{1-x^2}\leqslant\arccos|x|\leqslant\frac{\pi}{2}\sqrt{1-x^2},$$

we have

$$\ln((n+1)\arccos|x|+1)\geqslant\frac{1}{2}\ln(n+1)\quad\text{for}\quad\sqrt{1-x^2}\geqslant\frac{1}{\sqrt{n+1}}.$$

Hence for $x\in[-1,1]$,

$$\frac{\frac{2}{\pi}\sqrt{1-x^2}\ln((n+1)\arccos|x|+1)}{\sqrt{1-x^2}}\geqslant\frac{1}{\pi}\ln(n+1).$$

(II) Because of

$$\frac{(1-x^2)U_n(x)}{(n+1)|x-x_j|}=O(1),$$

theorem 1 implies

$$\Delta_n(x)=\frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1}\left\{\frac{2}{\pi}(1-x^2)|U_n(x)|\ln((n+1)\arccos|x|+1)\right.$$
$$\left.+|x|\sqrt{1-x^2}\,|U_n(x)|+O(\sqrt{1-x^2})\right\}.$$

From this and the estimation

$$\sqrt{1-x^2}\ln\left(\arccos|x|+\frac{1}{n+1}\right)=O(1)$$

we obtain immediately the result of Goodnaugh and Mills;

$$\Delta_n(x)=\frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1}\left\{\frac{2}{\pi}(1-x^2)|U_n(x)|\ln(n+1)+O(1)\right\}$$

holds uniformly on interval $[-1, 1]$.

§ 3. Remainder of Hermite–Fejér Interpolation

Let $f\in C_{[-1,1]}$. In this section, we shall consider the Hermite–Fejér interpolation polynomial $F_n(f, x)$ based on the zeros

$$\eta_k=\cos\frac{2k-1}{2n}\pi \quad (k=1, 2, \cdots, n)$$

of the Chebyshev polynomial of the first kind $T_n(x)=\cos(n\arccos x)$. For this case we have (see[5])

$$F_n(f, x)=\frac{1}{n^2}\sum_{k=1}^{n}f(\eta_k)\left\{\left(\frac{\sin n(\theta-\varphi_k)}{2\sin\frac{1}{2}(\theta-\varphi_k)}\right)^2+\left(\frac{\sin n(\theta+\varphi_k)}{2\sin\frac{1}{2}(\theta+\varphi_k)}\right)^2\right\}, \tag{16}$$

where $x=\cos\theta$ and $\varphi_k=\frac{2k-1}{2n}\pi$. On the asymptotic representation of quantity

$$\Delta_n^*(x)=\sup_{f\in\mathrm{Lip}_1 1}|F_n(f, x)-f(x)|,$$

we shall prove the following

Theorem 2. *The asymptotic representation*

$$\Delta_n^*(x)=\frac{T_n^2(x)}{n}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ln(n\arccos|x|+1)+|x|+\frac{1-x^2}{n|x-\eta_j|}+O(\sqrt{1-x^2})\right. \tag{17}$$

holds uniformly on the interval $[-1, 1]$, *where* η_j *is the zero of* $T_n(x)$ *which is nearest to* x.

Proof Let $0\leqslant x\leqslant 1$ and $x\neq\eta_k$ $(k=1, 2, \cdots, n)$. It is easy to see that there exists $\varphi_j\leqslant\frac{\pi}{2}$ such that

$$|\varphi_j-\theta|\leqslant\frac{\pi}{n}.$$

Hence $\eta_j=\cos\varphi_j$ is the zero of $T_n(x)$ which is nearest to $x=\cos\theta$. Set

$$I_k^*=\left|\frac{\sin\frac{1}{2}(\varphi_k+\theta)}{2\sin\frac{1}{2}(\varphi_k-\theta)}\right|+\left|\frac{\sin\frac{1}{2}(\varphi_k-\theta)}{2\sin\frac{1}{2}(\varphi_k+\theta)}\right|.$$

Similar to Theorem 1 we obtain that

$$\Delta_n^*(x)=\frac{T_n^2(x)}{n^2}\sum_{k=1}^{u}|x-\eta_k|\left(\frac{1}{2\sin^2\frac{1}{2}(\theta-\varphi_k)}+\frac{1}{2\sin^2\frac{1}{2}(\theta+\varphi_k)}\right)=\frac{T_n^2(x)}{n^2}\sum_{k=1}^{n}I_k^*, \tag{18}$$

and we can show

$$I_k^*=x+n\sqrt{1-x^2}\left\{\frac{1}{\pi(k-j)}-\frac{1}{\pi(k+j)}+O\left(\frac{1}{(k-j)^2}+\frac{1}{n}\right)\right\} \quad \text{for} \quad k>j, \tag{19}$$

and

$$I_k^*=-x+n\sqrt{1-x^2}\left\{\frac{1}{\pi(j-k)}+\frac{1}{\pi(k+j)}+O\left(\frac{1}{(j-k)^2}+\frac{1}{n}\right)\right\} \quad \text{for} \quad k<j. \tag{20}$$

Futhermore

$$I_j^*=\frac{1-x^2}{|x-\eta_j|}+x\,\mathrm{sign}(x-\eta_j). \tag{21}$$

Substiuting (19), (20) and (21) into (18) we have

$$\Delta_n^*(x)=\frac{T_n^2(x)}{n^2}\left\{\frac{2n}{\pi}\sqrt{1-x^2}\ln j+(n-2j+1)x+\frac{1-x^2}{|x-\eta_j|}\right.$$
$$\left.+x\,\mathrm{sign}(x-\eta_j)+O(n\sqrt{1-x^2})\right\}. \tag{22}$$

It is clear that for any $x\in\left[0, \cos\frac{\pi}{4n}\right]$,

$$\frac{2j-1}{n}=\frac{2\theta}{\pi}+O\left(\frac{1}{n}\right)=O(\sqrt{1-x^2}),$$

hence (22) implies

$$\Delta_n^*(x)=\frac{T_n^2(x)}{n}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ \ln(n\arccos x+1)+x+\frac{1-x^2}{n(x-\eta_j)}+O(\sqrt{1-x^2})\right\}. \quad (23)$$

For $x\in\left[\cos\frac{\pi}{4n}, 1\right]$, we have

$$I_k^*=x+O\left(\frac{n}{k^2}\sqrt{1-x^2}\right) \quad (k=1, 2, \cdots, n),$$

hence (18) implies

$$\Delta_n^*(x)=\frac{T_n^2(x)}{n}(x+O(\sqrt{1-x^2}).$$

But in this case

$$\frac{\sqrt{1-x^2}}{n|x-\eta_1|}+\ln(n\arccos x+1)=O(1),$$

whence

$$\Delta_n^*(x)=\frac{T_n^2(x)}{n}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ \ln(n\arccos x+1)+x+\frac{1-x^2}{n|x-\eta_1|}+O(\sqrt{1-x^2})\right\}. \quad (24)$$

Collecting the results (24) and (23) we obtain (17) holds uniformly on the interval [0, 1].

Similarly we can show that (17) holds uniformly on the interval [−1, 0]. Thus Theorem 2 is proved.

Remark. The remarks of Theorem 1 hold successively. In particular we have

$$\Delta_n^*(x)=\frac{T_n^2(x)}{n}\left\{\frac{2}{\pi}\sqrt{1-x^2}\ \ln(n\arccos|x|+1)+|x|\right\}+O\left\{\frac{|T_n(x)|\sqrt{1-x^2}}{n}\right\}$$

uniformly on the interval [−1, 1].

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