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ON LKUR SPACES

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Abstract

In this paper it is proved that if X is an LKUR space, then X has (H) property and if X^* is an LKUR space, then X^* has RNP. Also, if M is a Chebyshev subspace of LKUR space, then $P(M)$ is continuous.

§ 1. Introduction

In [1], F. Sullivan introduced KUR spaces and LKUR spaces. In this paper we study some properties of LKUR spaces.

Definition 1.[1] *A Banach space is said to be an LKUR space if for any* $\varepsilon>0$, $x\in S(X)\equiv\{x; x\in X, \|x\|=1\}$, $\exists\delta=\delta(x, \varepsilon)>0$, *such that for* $x_2, \cdots, x_{k+1}\in S(X)$, *if* $\|x+x_2+\cdots+x_{k+1}\|>(K+1)-\delta$, *then*

$$A(x, x_2, \cdots, x_{k+1})=\sup\left\{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1^*(x) & x_1^*(x_2) & & x_1^*(x_{k+1}) \\ \vdots & \vdots & & \vdots \\ x_k^*(x) & x_k^*(x_2) & & x_k^*(x_{k+1}) \end{vmatrix} \begin{matrix} x_i^*\in X^*; \\ \|x_i^*\|=1, \\ i=1, \cdots, k. \end{matrix}\right\}<\varepsilon.$$

Definition 2.[3] *A Banach space is said to have (H) property if* $(x_n)_{n=1}^{\infty}\subset S(X)$ and $x_n\xrightarrow{w}x\in S(X)$ imply $x_n\to x$.

Definition 3.[4] *If X is a Banach space, the dual space* X^* *is said to have* (**) *property, if* $(x_\alpha^*, \alpha\in D)\subset S(X^*)$ and $x_\alpha^*\xrightarrow{w^*}x^*\in S(X^*)$ *imply* $x_\alpha^*\to x^*$.

Definition 4.[5] *A Banach space X has the Radon-Nikodym property (RNP) if for every finite measurespace* (Ω, Σ, μ), *each* μ-*continuous vector measure* $G:\Sigma\to X$ *of bounded variation there exists* $g\in L_1(\mu, X)$ *such that* $G(E)=\int_E g\,d\mu$ *for all* $E\in\Sigma$.

In this paper we prove that if X is an LKUR space, then X has (H) property, and if X^* is an LKUR space, then X^* has (**) property and X^* has RNP.

In [1], F. Sullivan proved that if M is a Chebyshev subspace of L2UR space, then $P(M)$ is continuous, where $P(M)(x)=\{y\in M; d(x, y)=d(x, M)\}$. Now we generalize this result and prove that if M is a Chebyshev subspace of LKUR space, then $P(M)$ is continuous.

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§ 2. Theorems

Lemma 1.[6] *If $(x_\alpha, \alpha\in D)$ is a net of X, for every $\varepsilon>0$ there exists $\alpha_\varepsilon\in D$, such that the tail $\{x_\alpha: \alpha\geqslant\alpha_\varepsilon\}$ has a finite ε-net, then (x_α) is a relatively compact subset of X.*

Lemma 2.[2] *For all vectors $x_1, \cdots, x_k$ in X,*

$$A(x_1, \cdots, x_k)\geqslant \operatorname{dist}(x_1, [x_2, \cdots, x_k])A(x_2, \cdots, x_k),$$

where $[x_2, \cdots, x_k]$ denotes the affine span of the x_i's.

Theorem 1. *If X^* is an LKUR space, then X^* has $(**)$ property, and X^* has RNP.*

Proof Let (x^*_α) be a net of $S(X^*)$ such that $x^*_\alpha\xrightarrow{w^*} x^*\in S(X^*)$.

Let $y^*_\alpha=x^*_\alpha-x^*_\alpha$. If $\|y^*_\alpha\|\not\to 0$, then, since $0\leqslant\|y^*_\alpha\|\leqslant 2$, there exists a subnet (denoted by $\|y^*_\alpha\|$ again) such that $\|y^*_\alpha\|\to a>0$ $(a\leqslant 2)$, and obviously, $y^*_\alpha\xrightarrow{w^*} 0$.

We shall prove that for this subnet (y^*_α) and for each $\varepsilon>0$, there exists α_ε such that the tail $\{y^*_\alpha;\ \alpha\geqslant\alpha_\varepsilon\{$has a finite ε-net. It follows from Lemma 1 that the subnet (y^*_α) is a relatively compact subset of X^*. Therefore there exists a subnet $(y^*_{\alpha_\delta})$ of subnet (y^*_α) such that $y^*_{\alpha_\delta}\to z^*\in X^*$. On the other hand, $y^*_{\alpha_\delta}\xrightarrow{w^*} 0$, so $z^*=0$, which contradicts $\|y^*_{\alpha_\delta}\|\to a>0$. Therefore it is impossible that $\|y^*_\alpha\|\not\to 0$, hence $x^*_\alpha\to x^*$.

We divide the proof into three parts.

(I) In the case of the L2UR spaces:

Suppose $\varepsilon<a/2$. We choose $x\in S(X)$ such that $x^*(x)>1-\frac{1}{3}\delta\left(x^*, \frac{a\varepsilon}{8}\right)$, where $\delta\left(x^*, \frac{a\varepsilon}{8}\right)$ is the δ corresponding to x^* and $\frac{a\varepsilon}{8}$ in the definition of L2UR space.

Choose α_0 such that

$$x^*_\alpha(x)>1-\frac{1}{3}\delta\left(x^*, \frac{a\varepsilon}{8}\right), \tag{1}$$

$$0<a-\frac{a\varepsilon}{32}<\|y^*_\alpha\|<a+\frac{a\varepsilon}{32} \tag{2}$$

and

$$\frac{a}{\|y^*_\alpha\|}<\frac{3}{2} \tag{3}$$

for $\alpha\geqslant\alpha_0$.

Let $y^{**}_\alpha\in S(X^{**})$, such that $y^{**}_\alpha(y^*_\alpha)=\|y^*_\alpha\|$, $\forall\alpha$.

We have $\|x^*-y^*_\alpha\|=\|x^*-(x^*-x^*_\alpha)\|=\|x^*_\alpha\|=1$, and by (1),

$$3\geqslant\|x^*+(x^*-y^*_\alpha)+(x^*-y^*_\beta)\|=\|x^*+x^*_\alpha+x^*_\beta\|>3-\delta\left(x^*, \frac{a\varepsilon}{8}\right)$$

for α, $\beta\geqslant\alpha_0$.

Since X^* is an L2UR space, we have

$$a\frac{\varepsilon}{8}>A(x^*,\ x^*-y_\alpha^*,\ x^*-y_\beta^*)=A(0,\ y_\alpha^*,\ y_\beta^*)\geqslant\| \|y_\alpha^*\|y_\beta^*-y_\alpha^{**}(y_\beta^*)y_\alpha^*\|, \tag{4}$$

for $\alpha,\ \beta\geqslant\alpha_0$, and hence

$$a\frac{\varepsilon}{8}>|\,\|y_\alpha^*\|\,\|y_\beta^*\|-|y_\alpha^{**}(y_\beta^*)|\,\|y_\alpha^*\|\,|. \tag{5}$$

By Using (5) and (2), it is easy to see that

$$|\,\|y_\alpha^*\|-|y_\alpha^{**}(y_\beta^*)\,\|<\varepsilon/4 \tag{6}$$

for $\alpha,\ \beta\geqslant\alpha_0$.

If $\alpha\geqslant\alpha_0$ and $y_{\alpha_0}^{**}(y_\alpha^*)\geqslant 0$, then by (4), we have

$$a\frac{\varepsilon}{8}\geqslant\|y_{\alpha_0}^{**}(y_\alpha^*)y_{\alpha_0}^{**}-\|y_{\alpha_0}^*\|y_\alpha^*\|\geqslant\|y_{\alpha_0}^{**}\|\,\|y_{\alpha_0}^*-y_\alpha^*\|-\|y_{\alpha_0}^*\|\,|y_{\alpha_0}^{**}(y_\alpha^*)-\|y_{\alpha_0}^*\|\,|,$$

and, by (3), (6)

$$\|y_{\alpha_0}^*-y_\alpha^*\|\leqslant\frac{a\varepsilon}{8\|y_{\alpha_0}^*\|}+|y_{\alpha_0}^{**}(y_\alpha^*)-\|y_{\alpha_0}^*\|\,|<\varepsilon/2. \tag{7}$$

If $\alpha\geqslant\alpha_0$ and $y_{\alpha_0}^{**}(y_\alpha^*)<0$, then by (3), (4), and (6), we have

$$\|y_\alpha^*+y_{\alpha_0}^*\|<\varepsilon/2. \tag{8}$$

Now we consider three cases:

(i) If for all $\alpha>\alpha_0$, $y_{\alpha_0}^{**}(y_\alpha^*)\geqslant 0$, then, by (7), $(y_{\alpha_0}^*)$ is a $\frac{\varepsilon}{2}$-net of the tail $\{y_\alpha^*;\ \alpha\geqslant\alpha_0\}$.

(ii) If for all $\alpha>\alpha_0$, $y_{\alpha_0}^{**}(y_\alpha^*)<0$, then, by choosing $\alpha_1>\alpha_0$, $(y_{\alpha_0}^*,\ y_{\alpha_1}^*)$ is an ε-net of the tail $\{y_\alpha^*;\ \alpha\geqslant\alpha_0\}$. In fact, if $\alpha_2(\neq\alpha_1)>\alpha_0$, then, by (8)

$$\|y_{\alpha_2}^*-y_{\alpha_1}^*\|\leqslant\|y_{\alpha_2}^*+y_{\alpha_0}^*\|+\|y_{\alpha_1}^*+y_{\alpha_0}^*\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

(iii) If there exist α_1, $\alpha_2>\alpha_0$, such that $y_{\alpha_0}^{**}(y_{\alpha_1}^*)\geqslant 0$, $y_{\alpha_0}^{**}(y_{\alpha_2}^*)<0$, then, $(y_{\alpha_0}^*,\ y_{\alpha_2}^*)$ is an ε-net of the tail $\{y_\alpha^*;\ \alpha\geqslant\alpha_0\}$. In fact, if $\alpha>\alpha_0$, and $y_{\alpha_0}^{**}(y_\alpha^*)\geqslant 0$, then, by (7), we have $\|y_\alpha^*-y_{\alpha_0}^*\|<\frac{\varepsilon}{2}$; if $\alpha>\alpha_0$, and $y_{\alpha_0}^{**}(y_\alpha^*)<0$, then, by (8), we have $\|y_\alpha^*-y_{\alpha_2}^*\|<\varepsilon$.

In a word, the tail $\{y_\alpha^*;\ \alpha\geqslant\alpha_0\}$ has a finite ε-net.

(II) In the case of the L3UR spaces.

Choose δ corresponding to x^* and a $\left(\frac{\varepsilon}{16}\right)^2$ in the definition of L3UR space.

The notations and method are as above. There exists α_0 such that

$$a\left(\frac{\varepsilon}{16}\right)^2>A(0,\ y_\alpha^*,\ y_\beta^*,\ y_\gamma^*),$$

for $\alpha,\ \beta,\ \gamma\geqslant\alpha_0$.

By Using Lemma 2

$$a\left(\frac{\varepsilon}{16}\right)^2>\text{dist}\ (y_\gamma^*,\ \text{span}\ (y_\alpha^*,\ y_\beta^*))\ A(0,\ y_\alpha^*,\ y_\beta^*).$$

If for each pair $\alpha,\ \beta\geqslant\alpha_0$,

$$a\left(\frac{\varepsilon}{16}\right)>A(0,\ y_\alpha^*,\ y_\beta^*),$$

then, by (1), there exists $\alpha_\varepsilon>\alpha_0$ such that the tail $\{y^*_\alpha;\ \alpha\geqslant\alpha_\varepsilon\}$ has a finite ε-net.

Otherwise there exist $\alpha_1,\ \alpha_2\geqslant\alpha_0$ such that

$$A(0,\ y^*_{\alpha_1},\ y^*_{\alpha_2})\geqslant a\frac{\varepsilon}{16}.$$

Let $\alpha_\varepsilon>\alpha_1,\ \alpha_2$. Then if for $\alpha\geqslant\alpha_\varepsilon$,

$$\frac{\varepsilon}{16}\geqslant\operatorname{dist}(y^*_\alpha,\ \operatorname{span}(y^*_{\alpha_1},\ y^*_{\alpha_2})),$$

then there exists $z^*_\alpha\in\operatorname{span}(y^*_{\alpha_1},\ y^*_{\alpha_2})$such that $\|z^*_\alpha-y^*_\alpha\|=\operatorname{dist}(y^*_\alpha,\ \operatorname{span}(y^*_{\alpha_1},\ y^*_{\alpha_2}))$. Since (z^*_α) is bounded, then there exists $a\frac{\varepsilon}{2}$-net $(z^*_1,\ \cdots,\ z^*_n)$ of (z^*_α).

Therefore $(z^*_1,\ \cdots,\ z^*_n)$ is an ε-net of the tail $\{y^*_\alpha;\ \alpha\geqslant\alpha_\varepsilon\}$. In fact, if $\alpha\geqslant\alpha_\varepsilon$, then

$$\|y^*_\alpha-z^*_i\|\leqslant\|y^*_\alpha-z^*_\alpha\|+\|z^*_\alpha-z^*_i\|\leqslant\frac{\varepsilon}{16}+\frac{\varepsilon}{2}<\varepsilon\quad\text{for some } i,\ 1\leqslant i\leqslant n.$$

(III) In the case of the LKUR spaces ($k>3$):

Choose δ corresponding to x^* and $a\left(\frac{\varepsilon}{16}\right)^{k-1}$ in the definition of LKUR space.

Using the same argument, we get that if $\alpha_1,\ \cdots,\ \alpha_k\geqslant\alpha_0$,

$$a\left(\frac{\varepsilon}{16}\right)^{k-1}>A(0,\ y^*_{\alpha_1},\ \cdots,\ y^*_{\alpha_k})\geqslant\operatorname{dist}(y^*_{\alpha_k},\ \operatorname{span}(y^*_{\alpha_1},\ \cdots,\ y^*_{\alpha_{k-1}}))A(0,\ y^*_{\alpha_1},\ \cdots,\ y^*_{\alpha_{k-1}});$$

If whenever $\alpha_1,\ \cdots,\ \alpha_{k-1}\geqslant\alpha_0$,

$$a\left(\frac{\varepsilon}{16}\right)^{k-2}>A(0,\ y^*_{\alpha_1},\ \cdots,\ y^*_{\alpha_{k-1}}).$$

According to an inductive argument, there exists $\alpha_\varepsilon\geqslant\alpha_0$ such that the tail $\{y^*_\alpha;\ \alpha\geqslant\alpha_\varepsilon\}$ has a finite ε-net. Otherwise, there exist $\alpha_1,\ \cdots,\ \alpha_{k-1}\geqslant\alpha_0$ such that

$$A(0,\ y^*_{\alpha_1},\ \cdots,\ y^*_{\alpha_{k-1}})\geqslant a\left(\frac{\varepsilon}{16}\right)^{k-2}.$$

Then choose $\alpha_\varepsilon\geqslant\max(\alpha_1,\ \cdots,\ \alpha_{k-1})$. When $\alpha\geqslant\alpha_\varepsilon$ we have

$$\frac{\varepsilon}{16}\geqslant\operatorname{dist}(y^*_\alpha,\ \operatorname{span}(y^*_{\alpha_1},\ \cdots,\ y^*_{\alpha_{k-1}})).$$

Since dim span $(y^*_{\alpha_1},\ \cdots,\ y^*_{\alpha_{k-1}})\leqslant k-1$ and the bounded set in finite dimensional space is a relatively compact set, as the proof of L3UR spaces, we get that there exists a tail $\{y^*_\alpha;\ \alpha\geqslant\alpha_\varepsilon\}$ which has a finite ε-net.

In the end, we have proved that if X^* is an LKUR space, then X^* has (**) property, and, by [4], X^* has RNP. Q. E. D.

Theorem 2. *If X is an LKUR space, then X has (H) property.*

Proof Suppose $(x_n)\subset S(X)$ and $x_n\xrightarrow{w}x\in S(X)$. If $x_n\not\to x$, then there exists a subsequence (denoted by (x_n) again) such that

$$\operatorname{sep}(x_n)=\inf(\|x_n-x_m\|;\ n\neq m)>\varepsilon>0\quad\text{for some }\varepsilon>0.$$

As in the proof of Theorem 1, we get a subsequence (x_{n_i}) and an integer N_0 such that the set (x_{n_i}) has a finite $\frac{\varepsilon}{2}$-net, which contradicts sep $(x_n)>\varepsilon$. Q. E. D.

Theorem 3. *If M is a Chebyshev subspace of LKUR space, then $P(M)$ is continuous.*

Proof (I) If $x\in M$ and $x_n\to x$, then

$$\|P(M)x-P(M)x_n\|=\|x-P(M)x_n\|\leqslant\|x-x_n\|+\|P(M)x_n-x_n\|$$
$$\leqslant\|x-x_n\|+\text{dist}\ (x_n,\ M)\leqslant 2\|x_n-x\|\to 0.$$

(II) If $x\notin M$, then $x_n'\equiv\dfrac{x_n-P(M)_x}{\|x-P(M)x\|}\to x'=\dfrac{x-P(M)_x}{\|x-P(M)x\|}$ and

$$P(M)x_n\to P(M)x \quad \text{iff} \quad P(M)x_n'\to P(M)x'.$$

So we may assume that $\|x\|=1$ and $P(M)x=0$, and we need prove that if $x_n\to x$, then $P(M)x_n\to 0$.

Since M is a Chebyshev subspace, if $\{P(M)x_{n_i}\}$ is a convergence subsequence of $(P(M)x_n)$, then we have $P(M)x_{n_i}\to 0$. In fact, if $P(M)x_{n_i}\to m$, then

$$\|x\|=\|x-P(M)x\|\leqslant\|x-m\|=\lim\|x_{n_i}-P(M)x_{n_i}\|\leqslant\lim\|x_{n_i}\|=|x|,$$

therefore $m=0$. So if $P(M)x_n\not\to 0$, then there exists a subsequence of $\{P(M)x_n\}$ (denoted by $\{P(M)x_n\}$ again) such that sep $(P(M)x_n)>\varepsilon$, for some $\varepsilon>0$.

Since $\|x\|<\|x-P(M)x_n\|\leqslant\|x_n-P(M)x_n\|+\|x-x_n\|\leqslant\|x_n\|+\|x-x_n\|\to\|x\|$, we have $\|P(M)x_n-x\|\to\|x\|$ and $\|x+(x-P(M)x_n)+(x-P(M)x_m)\|\to 3$ when n, $m\to\infty$.

Let $y_n=P(M)x_n$, as in the proof of Theorem 1, we get a subsequence $\{P(M)x_{n_i}\}$ and an integer N_0 such that the set $(P(M)x_{n_i};\ i\geqslant N_0)$ has a finite $\frac{\varepsilon}{2}$-net, which contradicts sep $(P(M)x_n)>\varepsilon$. Q. E. D.

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