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ON A SURJECTIVITY FOR THE SUM OF TWO MAPPINGS OF MONOTONE TYPE

ZHAO YICHUN (赵义纯)*

Abstract

In this paper the sum $(T+S)$ of two nonlinear mappings is considered, where T is maximal monotone or generalized pseudomonotone and S is generalized pseudomonotone or of type (M). By using the concepts of T-boundedness, T-generalized pseudomonotone mappings and mappings of type $T-(M)$ introduced by the author, it is proved that $(T+S)$ is of type (M). A new surjectivity result for multivalued pseudo A-proper mappings is given. As a consequence, it is obtained that the coercive mappings of type (M) whose effective domain contains a dense linear subspace are surjectivity. In particular, the author answers affirmatively a part of Browder's question (see [1], p. 70).

It makes an important sense to study the surjectivity for the sum of two mappings of monotone type in the theory of monotone operators and its applications. Let X be a real Banach space, X^* its dual space, and let $T: X\to 2^{X^*}$ be a maximal monotone mapping. Browder posed the following open question[1]: Suppose that S is a bounded finitely continuous T-pseudomonotone mapping from X to X^* and $(T+S)$ is coercive; is it then true that $(T+S)$ is surjective? Hess and the author have researched into this question using different methods[2,3]. In addition, if T is weakly closed and S is of type (M), until now the best results on the surjectivity for $(T+S)$ belong to [4, 5]. When studying a surjectivity for the sum $(T+S)$ of two mappings of monotone type, all authors restricted T and S respectively, but did not make a connection between properties of T and S themselves. By the above reasons, we have introduced the notions on T-boundedness, T-generalized pseudomonotone mappings and mappings of type $T-(M)$ in [6]. We have proved that the quasi-bounded mapping S must be T-bounded and that generalized pseudomonotone mappings and T-pseudomonotone mappings in Browder's sense S must be T-generalized pseudomonotone, if T is maximal monotone. This paper is a continuation of [6]. In the first section of this paper, we shall simplify the sum of some mappings of monotone type by means of the notion on T-boundedness: The sum of two generalized pseudomonotone mappings is reduced to one; and the sum of a weakly

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* Department of Mathematics, Northeast University of Technology, Shenyang, China.

closed mapping and a mapping of type (M) is reduced to a mapping of type (M). In the second section of this paper, we shall first prove that quasi-bounded multivalued mappings of type (M) are weakly A-proper, and then give a surjectivity result for this kind of mappings. This not only extends a result in [7] but also answers affirmatively a part of Browder's question. It should be noted that the mappings studied here are not defined everywhere.

§ 1. On the Sum of Two Mappings of Monotone Type

Let the spaces X and X^* be as before and let $T:X\to 2^{X^*}$ be a mapping. We denote by $D(T)$ and $G(T)$ the effective domain and the graph of T and denote by "$\to$" and "$\rightharpoonup$" strong and weak convergences, respectively. $\mathcal{N}$ denotes the collection of all natural numbers. We consider the following hypotheses on the mappings:

(m_1) For each $x\in D(T)$, Tx is a nonempty bounded closed convex set of X^*;

(m_2) For any $[x_n, f_n]\in G(T)$ $(n\in\mathcal{N})$, if $x_n\rightharpoonup x_0$ in X, $f_n\rightharpoonup f_0$ in X^* with $x_0\in X$ and $f_0\in X^*$ and

$$\overline{\lim_n}(f_n,\ x_n-x_0)\leqslant 0,$$

then $[x_0,\ f_0]\in G(T)$. If, in addition, the assertion $(f_n,\ x_n)\to(f_0,\ x_0)$ holds, too, then we say that T satisfies the hypothesis (m_2');

(m_3) For each finite dimensional subspace F of X, T is upper semicontinuous as a mapping from F into 2^{X^*} relative to the weak topology of X^*;

(m_4) For any $[x_n, f_n]\in G(T)$ $(n\in\mathcal{N})$, if $x_n\rightharpoonup x_0$ in X, $f_n\rightharpoonup f_0$ in X^* with $x_0\in X$ and $f_0\in X^*$, then $[x_0, f_0]\in G(T)$.

The mapping T is said to be of type (M), generalized pseudomonotone or weakly closed, respectively, if it satisfies (m_1), and, (m_3) in addition, a corresponding hypothesis among (m_2), (m_2') or (m_4). It was known that maximal monotone $\Rightarrow$ generalized pseudomonotone $\Rightarrow$ of type (M) $\Leftarrow$ weakly closed. Obviously, if T and S satisfy (m_2) and (m_3), then their sum must be so. Thus, in order to show $(T+S)$ is some one of these types, it suffices to prove that $(T+S)$ satisfies the respective condition among (m_2), (m_2') and (m_4). As to the concepts for the quasi-boundedness of a mapping and the normalized dual map, see [4].

Definition 1[6]. *Let X be a Banach space, X^* its dual space, and let mappings T, S: $X\to 2^{X^*}$ and $\Omega\subset D(T)\cap D(S)\neq\phi$. A mapping S is said to be T-bounded on Ω if for any bounded sequence $\{x_n\}\subset\Omega$ when $f_n+g_n\rightharpoonup h(n\to\infty)$, where $f_n\in Tx_n$, $g_n\in Sx_n$ $(n\in\mathcal{N})$ and $h\in X^*$, $\{g_n\}$ is bounded.*

Clearly, if T is a bounded mapping (zero mapping), then arbitrary mappings S are T-bounded (0-bounded) on the effective domain $D(S)$. We have proved that if a mapping S is quasi-bounded, then S must be T-bounded on $D(T)\cap D(S)$ with respect to any monotone mapping T[3]. Thus, T-boundedness is a very weak concept.

Definition 2[6]. *Let spaces X and X^* be as in Definition 1, and mappings T, S: $X\to 2^{X^*}$ with $D(T)\cap D(S)\neq\phi$. A mapping S is said to be T-generalized pseudomonotone if for any sequence $\{x_n\}\subset D(T)\cap D(S)$ with $x_n\rightharpoonup x_0$, $g_n\rightharpoonup g_0$ and $\{f_n\}$ is bounded such that*

$$\overline{\lim_n}(f_n+g_n,\ x_n-x_0)\leqslant 0,$$

where $f_n\in Tx_n$, $g_n\in Sx_n (n\in\mathcal{N})$, we have $[x_0, g_0]\in G(S)$ and $(g_n, x_n)\to(g_0, x_0)(n\to\infty)$. S is said to be of type T-(M) if we do not require $(g_n, x_n)\to(g_0, x_0)$.

According to Definition 2, a generalized pseudomonotone mapping (a mapping of type (M)) in [4] must be 0-generalized pseudomonotone (0-of type (M)), where 0 is the zero mapping.

Lemma 1. *Let X be a real Banach space, T: $X\to 2^{X^*}$ generalized pseudomonotone. Suppose that $\{x_n\}\subset D(T)$, $x_n\rightharpoonup x_0$ and $f_n\rightharpoonup f_0$ $(n\to\infty)$ with $f_n\in Tx_n (n\in\mathcal{N})$. Then*

$$\underline{\lim_n}(f_n,\ x_n-x_0)\geqslant 0. \tag{1}$$

Proof If the inequality (1) does not hold, then

$$\underline{\lim_n}(f_n,\ x_n-x_0)<0. \tag{2}$$

By hypotheses, $\{(f_n, x_n-x_0)\}$ is a bounded numerical sequence. It follows that there exists its subsequence $\{(f_{n_j}, x_{n_j}-x_0)\}$ such that

$$\lim_j(f_{n_j},\ x_{n_j}-x_0)=\underline{\lim_n}(f_n,\ x_n-x_0)<0. \tag{3}$$

Since T is generalized pseudomonotone, we obtain $(f_{n_j}, x_{n_j})\to(f_0, x_0)$. This fact contradicts (3). Q. E. D.

Lemma 1 extends Lemma 1 in [6].

Theorem 1. *Let X be a real reflexive Banach space, T: $X\to 2^{X^*}$ generalized pseudomonotone. Suppose that S: $X\to 2^{X^*}$ is generalized pseudomonotone or T-pseudomonotone (in the Browder's sense in [1]) (of type (M)) and $D(T)\cap D(S)\neq\phi$. Then S is T-generalized pseudomonotone (of type $T-(M)$).*

Proof We shall show only the case when S is generalized pseudomonotone and T-pseudomonotone. If S is of type (M), the argument is similar. Let $\{x_n\}\subset D(T)\cap D(S)$ such that $x_n\rightharpoonup x_0$, $g_n\rightharpoonup g_0$ and $\{f_n\}$ is bounded with $f_n\in Tx_n$ $g_n\in Sx_n (n\in\mathcal{N})$ and

$$\overline{\lim_n}(f_n+g_n,\ x_n-x_0)\leqslant 0. \tag{4}$$

Since X is reflexive and $\{f_n\}$ is bounded, there exist $f_0\in X$ and its aubsequence $f_{n_j}\rightharpoonup f_0 (j\to\infty)$. (4) implies

$$\underline{\lim_j}(f_{n_j},\ x_{n_j}-x_0)+\overline{\lim_j}(g_{n_j},\ x_{n_j}-x_0)\leqslant 0.$$

Since S is either generalized pseudomonotone or T-pseudomonotone, we have $[x_0, g_0]$

$\in G(S)$ and $(g_{n_j}, x_{n_j})\to(g_0, x_0)\ (j\to\infty)$. Indeed, in the course of the above proof we have shown that to every subsequence $\{(g_{n_j}, x_{n_j})\}$ of $\{(g_n, x_n)\}$ there exists its subsequence $\{(g_{n_{j(k)}}, x_{n_{j(k)}})\}$ which converges to (g_0, x_0). Therefore, $(g_n, x_n)\to(g_0, x_0)$ $(n\to\infty)$. Q. E. D.

Corollary 1. *Theorem 1 in* [6].

In general, the sum of two generalized pseudomonotone mappings need not be generalized pseudomonotone, but we have

Theorem 2. *Let X be a real reflexive Banach space, T: $X\to 2^{X^*}$ generalized pseudomonotone. Suppose that S: $X\to 2^{X^*}$ is T-bounded and T-generalized pseudomonotone. Then $(T+S')$ is generalized pseudomonotone on $D(T)\cap D(S)$.*

Proof Let $\{x_n\}\subset D(T)\cap D(S)$ such that $x_n\rightharpoonup x_0$, $f_n+g_n\rightharpoonup h$ with $f_n\in Tx_n$, $g_n\in Sx_n (n\in\mathcal{N})$ and $h\in X^*$ and

$$\overline{\lim_n}(f_n+g_n, x_n-x_0)\leqslant 0. \tag{5}$$

Since S is T-bounded, $\{f_n\}$ and $\{g_n\}$ are bounded. We may assume $g_{n_j}\rightharpoonup g_0$ in X^* and $f_{n_j}\rightharpoonup h-g_0$ in X^* $(j\to\infty)$. Because of (5) and the fact that S is T-generalized pseudomonotone, we have $[x_0, g_0]\in G(S)$ and $(g_{n_j}, x_{n_j})\to(g_0, x_0)(j\to\infty)$. Hence, the inequality (5) becomes

$$\overline{\lim_j}(f_{n_j}, x_{n_j}-x_0)\leqslant 0.$$

Now, we conclude $[x_0, h-g_0]\in G(T)$ and $(f_{n_j}, x_{n_j})\to(h-g_0, x_0)$ since T is generalized pseudomonotone. Therefore, we obtain $[x_0, h]\in G(T+S)$ and $(f_{n_j}+g_{n_j}, x_{n_j})\to(h, x_0)(j\to\infty)$. By the same reason as in the proof of Theorem 1, we find $(f_n+g_n, x_n)\to(h, x_0)(n\to\infty)$. Q. E. D.

In combination with Theorem 1, we have

Corollary 1. *If T is generalized pseudomonotone and S is T-bounded generalized pseudomonotone or T-pseudomonotone, then $(T+S)$ is generalized pseudomonotone.*

Corollary 1 eliminates the assumptions of the boundedness on T and $D(T)=X$ in Lemma in [5, p. 212].

Corollary 2. *If T is generalized pseudomonotone and S is T-bounded generalized pseudomonotone or T-pseudomonotone which are multivalued and satisfy condition (m_1), suppose that there exists a dense linear subspace X_0 of X which is contained in $D(T)$ and $(T+S)$ is quasi-bounded and coercive, then $R(T+S)=X^*$.*

Proof By Corollary 1, $(T+S)$ is generalized pseudomonotone. Therefore, $R(T+S)=X^*$ by Theorem 5 in [8].

Corollary 3. *Let T: $X\to 2^{X^*}$ be a maximal monotone mapping and S: $X\to 2^{X^*}$ a quasi-bounded finitely continuous T-pseudomonotone which satisfies condition (m_1). If $(T+S)$ is coercive, then $(T+S)$ is surjective.*

Proof Since $D(T)=D(S)=X$, T is quasi-bounded and $0\in\operatorname{Int} D(T)$.

Therefore, $(T+S)$ is quasi-bounded. By Corollary 2, $(T+S)$ is surjective.

Remark 1. If $0 \in \operatorname{Int} D(T)$, T is quasi-bounded. The assumption that $D(T) = D(S) = X$ in Corollary 3 can be changed to $D(T+S) = X$. (see Corollary 2).

Remark 2. In general case, $(T+S)$ is coercive but is not certainly surjective. For example, let T, S: $R^1 \to R^1$ satisfy the assumption that for any x in R^1, $Tx = 0$ and $Sx = x$ for any $x \in D(S) = R^1_+ \cup \{0\}$. It is known easily that T is bounded maximal monotone, S is bounded T-pseudomonotone, and $(T+S)$ is coercive. But $R(T+S) = R^1_+ \cup \{0\} \neq R^1$. Corollaries 2 and 3 are pointed out by my post graduate Min Lequan.

In a similar fashion to the proof of Theorem 2 we obtain the following

Theorem 3. *Let X be a real reflexive Banach space, T: $X \to 2^{X^*}$ weakly closed. Suppose that a mapping S: $X \to 2^{X^*}$ is T-bounded and of type T-(M). Then $(T+S)$ is of type (M).*

In combination with Theorem 1, we have

Corollary 1. *If T is a weakly closed and maximal monotone mapping and S is a T-bounded mapping of type (M), then $(T+S)$ is of type (M).*

When T is generalized pseudomonotone (in particular, maximal monotone), Theorem 1 unifies two notions that S is generalized pseudomonotone and T-pseudomonotone by T-generalized pseudomonotone mappings. The assumptions in Corollary 1 of Theorem 2 is simpler than ones in Theorem 1 in [8]. Since a generalized pseudomonotone mapping must be of type (M), in order to study a surjectivity for the sum $(T+S)$ of two mappings of monotone type, by Theorems 2 and 3, it suffices to consider a surjectivity for a mapping of type (M).

§ 2. Results of a Surjectivity

In what follows we always assume that X is a real separable reflexive Banach space. For this kind of space, there is an injective approximation scheme $\Gamma = (\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ for (X, X^*), where $\{X_n\}$ is an increasing sequence of finite dimensional subspaces of X and $\rho(x, X_n) \to 0$ $(n \to \infty)$ for each $x \in X$, P_n: $X_n \to X$ is the injection mapping and $Q_n = P_n^*$ is the dual mapping of P_n. This scheme is assumed in this paper. For the concepts on a weakly (pseudo) A-proper mapping with respect to an injective approximation scheme, see [9, 10]. Let $\Omega \subset X$ and T: $X \to 2^{X^*}$. We write $\Omega_n = \Omega \cap X_n$ and $T_n = Q_n T P_n$.

The following theorem gives a very general result that mappings of monotone type are weakly A-proper.

Theorem 4. *Let a mapping T: $X \to 2^{X^*}$ be of type (M) and quasi-bounded. Then T is weakly A-proper with respect to an injective approximation scheme $\Gamma = (\{X_n\},$*

$\{X_n^*\}; \{P_n\}, \{Q_n\})$ *on* $D(T)$.

Proof Let $x_{n_j} \in D(T) \cap X_{n_j}$ with $\{x_{n_j}\}$ bounded and $h_{n_j} \in T_{n_j} x_{n_j}$ $(j \in \mathcal{N})$ satisfy

$$\|h_{n_j} - Q_{n_j} P\| \to 0 \quad (j \to \infty) \tag{6}$$

for some $p \in X^*$. Since P_{n_j}: $X_{n_j} \to X$ is an injection mapping and $T_{n_j} = Q_{n_j} T P_{n_j}$, we may take $f_{n_j} \in T x_{n_j}$ such that $h_{n_j} = Q_{n_j} f_{n_j}$. Hence, (6) becomes

$$\|Q_{n_j} f_{n_j} - Q_{n_j} p\| \to 0 \quad (j \to \infty). \tag{7}$$

Since $\|Q_{n_j}\| \leqslant 1$ $(j \in \mathcal{N})$, from (7) we know that $\{Q_{n_j} f_{n_j}\}$ is bounded. Hence, by the quasi-boundedness of T and

$$(f_{n_j}, x_{n_j}) = (f_{n_j}, P_{n_j} x_{n_j}) = (Q_{n_j} f_{n_j}, x_{n_j}) \leqslant \|Q_{n_j} f_{n_j}\| \|x_{n_j}\| \leqslant M_1 \|x_{n_j}\|,$$

where $M_1 = \sup_j \|Q_{n_j} f_{n_j}\|$, we see that $\{f_{n_j}\}$ is bounded.

For fixed X_n and each x in X_n, we have $x_{n_j} - x \in X_{n_j}$ as $n_j > n$. Consequentely, (7) implies that

$$|(f_{n_j} - p, x_{n_j} - x)| = |(f_{n_j} - p, P_{n_j}(x_{n_j} - x))| \leqslant \|Q_{n_j} f_{n_j} - Q_{n_j} p\| \cdot \|x_{n_j} - x\|$$
$$\leqslant (M + \|x\|) \|Q_{n_j} f_{n_j} - Q_{n_j} p\| \to 0 \quad (j \to \infty), \tag{8}$$

where $M = \sup_j \|x_{n_j}\|$. Indeed, to each $x \in X$, since $\rho(x, X_n) \to 0$ $(n \to \infty)$, we have from (8) and the boundedness of $\{f_{n_j}\}$

$$(f_{n_j} - p, x_{n_j} - x) \to 0 \quad (j \to \infty). \tag{9}$$

Since X is reflexive and $\{x_{n_j}\}$ is bounded, we may assume some of its subsequence $x_{n_{j(k)}} \rightharpoonup x_0 \in X$ $(k \to \infty)$. Setting $x = x_0$in (9), we obtain

$$(f_{n_{j(k)}} - p, x_{n_{j(k)}} - x_0) \to 0 \quad (k \to \infty). \tag{9'}$$

(9) and (9′) imply $(f_{n_{j(k)}} - p, x_0 - x) \to 0$. This means $f_{n_{j(k)}} \rightharpoonup p (k \to \infty)$. We have also from (9)

$$(f_{n_{j(k)}}, x_{n_{(jk)}} - x_0) \to 0 \quad (k \to \infty). \tag{10}$$

Since T is of type (M), we obtain, by (10), $[x_0, p] \in G(T)$, i. e., $x_0 \in D(T)$ and $p \in T x_0$. Thus, T is weakly A-proper. Q. E. D.

By Theorems 2 and 3 in the first section, we obtain

Corollary 1. *Theorem* 2 *in* [6]

To show a surjectivity of weakly A-proper mappings, we shall need the following

Lemma 2. *Let* $T: X \to 2^{X^*}$ *be weakly A-proper, and let* $\Omega (\subset D(T))$ *be a bounded set of* X *and* $p \in X^*$ *and* $p \overline{\in} T(\Omega)$. *Then there exist* $n_0 \in \mathcal{N}$ *and* $\alpha > 0$ *such that*

$$\rho(Q_n p, T_n(\Omega_n)) \geqslant \alpha \quad as \quad n \geqslant n_0.$$

In particular, $Q_n P \overline{\in} T_n(\Omega_n)$ $(n \geqslant n_0)$.

Proof If the assertion is false, there exist $\{\varepsilon_j\}$, $\varepsilon_j \to 0$ and $n_j \to \infty$ such that

$$\inf_{h \in T_{n_j}(\Omega_{n_j})} \|h - Q_{n_j} p\| = \rho(Q_{n_j} p, T_{n_j}(\Omega_{n_j})) < \varepsilon_j \quad (j \in \mathcal{N}).$$

It follows that there exist $x_{n_j} \in \Omega_{n_j} (\subset \Omega)$ and $h_{n_j} \in T_{n_j} x_{n_j}$ such that

$$\|h_{n_j} - Q_{n_j} p\| < \varepsilon_{n_j} \to 0 \quad (j \to \infty).$$

Since T is pseudo A-proper, there exists $x_0 \in \Omega$ satisfying $p \in T x_0$ This fact contradits

$p\overline{\in}T(\Omega)$. Q. E. D.

Theorem 5 (Theorem 3 in [6]). *Let $\Omega\subset X$ be a bounded set, $0\in\Omega$ and let Ω_n be an open symmetric set about the origin of X_n for each $n\in\mathcal{N}$. Suppose that $T: X\to 2^{X^*}$ is pseudo A-proper with respect to an injective apprcximation scheme $\Gamma=(\{X_n\},\{X_n^*\};\{P_n\},\{Q_n\})$ on $\overline{\Omega}$ and that for each $n\in\mathcal{N}$ it satisfies the following*

(i) *T_nx is a compact convex set of X_n^* for each $x\in\overline{\Omega}_n$;*

(ii) *$T_n: \overline{\Omega}_n\subset X_n\to 2^{X_n^*}$ is upper semicontinuous;*

(iii) *to each $p\in X^*$,*

$$(f_n, x)\geqslant(Q_np, x) \quad as \quad x\in\partial\Omega_n \text{ and } f_n\in T_nx.$$

Then there is $x_0\in\overline{\Omega}$ such that $p\in Tx_0$.

Proof Let $J: X\to 2^{X^*}$ be the normalized dual map. It is known easily that $J_n=Q_nJP_n$ is also the normalized dual map from X_n to X_n^* for each $n\in\mathcal{N}$. Hence, when $x\in X_n$ and $g_n\in J_nx$, we have $(g_n, x)=\|x\|^2$. Thus, in virtue of the hypothesis (iii), when $x\in\partial\Omega_n$ (according to the assumption on Ω_n, $x\neq0$), $f_n\in T_nx$ and $0\leqslant t<1$ for each $n\in\mathcal{N}$, we have

$$\begin{aligned}\|t(f_n-Q_np)+(1-t)g_n\|&\geqslant\frac{1}{\|x\|}(t(f_n-Q_np)+(1-t)g_n,x)\\&=\frac{t}{\|x\|}(f_n-Q_np, x)+(1-t)\|x\|\\&\geqslant(1-t)\|x\|>0.\end{aligned}\tag{11}$$

We are going to show that the equation $Q_np\in T_nx$ has a solution on $\overline{\Omega}_n$ for all $n\in\mathcal{N}$. Assume the contrary, then the equation $Q_{n_0}p\in T_{n_0}x$ has no solution on $\overline{\Omega}_{n_0}$ for some $n_0\in\mathcal{N}$. Consequentely, we have

$$\|f_{n_0}-Q_{n_0}p\|>0 \quad \text{as } x\in\partial\Omega_{n_0} \text{ and } f_{n_0}\in T_{n_0}x.$$

This together with (11) shows that for all $x\in\partial\Omega_{n_0}$ and $0\leqslant t\leqslant1$,

$$0\overline{\in}t(T_{n_0}x-Q_{n_0}p)+(1-t)J_{n_0}x.$$

In accordance with the hypothese (i) and (ii) of this theorem and the homotopy invariance of the Cellina-Lasota topological degree[11], we obtain

$$\deg_{L.S.}(T_{n_0}x-Q_{n_0}f,\ \Omega_{n_0},\ 0)=\deg_{L.S.}(J_{n_0},\ \Omega_{n_0},\ 0)=\{1\}\neq\{0\}.$$

Hence, there exisst $x_{n.}\in\Omega_{n.}\subset\overline{\Omega}_{n_0}$ such that $Q_{n_0}p\in T_{n_0}x_0$. This contradicts the fact that the equation $Q_{n_0}p\in T_{n_0}x$ has no solution on $\overline{\Omega}_{n_0}$. Therefore, to each $n\in\mathcal{N}$, $Q_np\in T_n(\overline{\Omega}_n)$. Since T is pseudo A-proper on $\overline{\Omega}$, the equation $p\in Tx$ is solvable on $\overline{\Omega}$ by Lemma 2. Q. E. D.

Corollary 1. *Let a mapping $T:X\to 2^{X^*}$ be of type (M) and quasibounded. Suppose that there exists a dense linear subspace X_0 of X such that $D(T)\supset X_0$. Suppose further that T is coercive, i. e.,*

$$\lim_{\|x\|\to\infty}\frac{(f, x)}{\|x\|}=+\infty \quad as\ [x, f]\in G(T).$$

Then $\{x|p\in Tx\}$ is a nonvoid weakly sequential compact set of X for any $p\in X^*$, in *particular*, $R(T)=X^*$.

Proof Since X_0 is a dense linear subspace of a separable space X, there exists an increasing sequence of finite dimensional subspace of $X: X_1\subset X_2\subset\cdots\subset X_n\subset\cdots$ such that $X_0=\bigcup_{n=1}^{\infty} X_n$, $\dim X_n=n$ and $\overline{X}_0=X$. So, we obtain an injective approximation scheme $\Gamma=(\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ by $\{X_n\}$. Since T is of type (M) and quasi-bounded, T is weakly A-proper with respect to Γ on $D(T)$ and moreover it is pseudo A-proper. By the hypothesis (m_1) on mappings of type (M) and the reflexivity of X, $T_n x=Q_n T P_n x$ is a compact convex set of X_n^* for each $x\in D(T)$ $(n\in\mathcal{N})$. Since the strong topology and the weak topology are equivalent in a finite dimensional space, T_n: $X_n\to 2^{X_n^*}$, by the hypothesis (m_3), is upper semicontinuous. Let $p\in X^*$, By the coercivity of T there exists a closed ball $B(0, r_p)$ such that $(f-p, x)>0$ as $x\in\partial B(0, r_p)\cap D(T)$, $f\in Tx$. We are going to show that the condition (iii) of Theorem 5 is satisfied. In fact, write $B_n(0, r_p)=B(0,r_p)\cap X_n$. Suppose $x\in\partial B_n(0, r_p)$ and $f_n\in T_n x$. By $x\in X_n$, we obtain $(p, x)=(p, P_n x)=(Q_n p, x)$. By $f_n\in T_n x$, there is $f\in Tx$ such that $f_n=Q_n f$. Hence, by $Q_n^*=P_n$, we obtain

$$(f_n, x)=(Q_n f, x)=(f, Q_n^* x)=(f, x)>(p, x)=(Q_n p, x).$$

By Theorem 5, we have $R(T)=X^*$. As for the fact that $\{x|p\in Tx\}$ is a weakly sequential compact set of X, it is deduced easily from the coercivity.

Q. E. D.

Corollary 2. *Suppose that mappings T and S satisfy the hypotheses of Theorem 2 or Corollary 1 to Theorem 3, and suppose further that there exists a dense linear subspace X_0 of X such that $D(T)\cap D(S)\supset X_0$ and $(T+S)$ is quasibounded coercive. Then* $R(T+S)=X^*$. Milojevic′ (Theorem 2.1 in [10]) gave a result similar to Theorem 5, there a projectionally complete scheme is assumed by him. But a general separable reflexive Banach space does not always have that scheme. Besides, he required that Ω is a bounded open set, whereas we require only that $D(T)$ contain a dense linear subspace of X. Our methods of the proof are different from those in [10]. Corollary 1 extends Theorem 5.2.3. in [7] to multivalued case and the hypothesis on the boundedness of a mapping is weakend. Corollary 2 gives a partially affirmative answer to a Browder′s question.

The stronger results can be obtained by using Yosida approximations, for example

Theorem 6 (Theorem 5 in [6]). *Let T: $X\to 2^{X^*}$ be maximal monotone and strongly quasi-bounded, and let S: $X\to 2^{X^*}$ be quasi-bounded generalized pseudomonotone. Suppose that there exists a dense linear subspace X_0 of X such that $D(S)\supset X_0$. Suppose further that S is coercive in the following sense, i. e., there is a real function $C(r)$: $R_+\to R_+$, $C(0)=0$ and $C(r)\to+\infty$ $(r\to+\infty)$ such that*

$$(g, x) \geqslant C(\|x\|)\|x\| \quad as\ [x, g] \in G(S). \tag{12}$$

Then $R(T+S) = X^*$.

Proof Since X is reflexive, we may assume that X and X^* are strictly convex by renormed theorem due to Asplund[12]. Hence, the normalized dual maps J and J^{-1} are singlevalued. We take $\varepsilon_n \to 0$, $\varepsilon_n > 0$ $(n \in \mathcal{N})$. Making Yosida approximations $T_{\varepsilon_n} = (T^{-1} + \varepsilon_n J^{-1})^{-1}$ of T, we see that T_{ε_n} is a bounded maximal monotone and singlevalued operator and $D(T_{\varepsilon_n}) = X$. Hence, the mapping S is T_{ε_n}-bounded. By Corollary 1 to Theorem 2, $(T_{\varepsilon_n} + S)$ is generalized pseudomonotone. Obviously, it is quasi-bounded. Let $p \in X^*$. From Theorem 5 there exists $x_{\varepsilon_n} \in D(S)$ such that

$$p \in (T_{\varepsilon_n} + S)x_{\varepsilon_n}.$$

Take $g_{\varepsilon_n} \in Sx_{\varepsilon_n}$ such that

$$T_{\varepsilon_n}x_{\varepsilon_n} + g_{\varepsilon_n} = p. \tag{13}$$

We know easily that there exists $r_p > 0$ such that $\|x_{\varepsilon_n}\| \leqslant r_p$ for all $n \in \mathcal{N}$ by the coercivity of S. Without loss of generality we may assume $x_{\varepsilon_n} \rightharpoonup x_0 \in X$. We write $T_n = T_{\varepsilon_n}$, $g_n = g_{\varepsilon_n}$, $x_n = x_{\varepsilon_n}$ and $u_n = x_n - J^{-1}T_n x_n$. By the definition of Yosida approximations, $T_n x_n \in Tu_n$. Now, we are going to show that $\{T_n x_n\}$ is bounded. $\varepsilon_n T_n x_n = J(x_n - u_n)$ implies that $\varepsilon_n(T_n x_n, x_n - u_n) = \|x_n - u_n\|^2 \geqslant 0$ $(n \in \mathcal{N})$. Hence, by this inequality, (12) and (13), we obtain

$$(T_n x_n, u_n) \leqslant (T_n x_n, x_n) = (p - g_n, x_n). \\ \leqslant \|p\| r_p.$$

It follows from strongly quasi-boundedness of T that $\{T_n x_n\}$ is bounded (see [8]). We know from (13) that $\{g_n\}$ is bounded, too. We may assume $g_n \rightharpoonup g_0 \in X^*$. We have from (13) $T_n x_n \rightharpoonup p - g_0$. Since J is a bounded mapping, from $J(x_n - u_n) = \varepsilon_n T_n x_n$ we obtain $\|x_n - u_n\| = \varepsilon_n \|T_n x_n\| \to 0$ $(n \to \infty)$. Thus, $u_n \rightharpoonup x_0$ $(n \to \infty)$.

Finally, we want to show $[x_0, g_0] \in G(S)$ and $[x_0, p - g_0] \in G(T)$. If so, we will complete the proof of Theorem 6. In virtue of Lemma 1, we find

$$\varliminf_n (T_n x_n, x_n - x_0) = \varliminf_n (T_n x_n, u_n - x_0) - \lim_n (T_n x_n, u_n - x_n) \geqslant 0.$$

By $T_n x_n + g_n = p$, $x_n \rightharpoonup x_0$ and the above inequality, we get

$$\varlimsup_n (g_n, x_n - x_0) \leqslant \lim_n (T_n x_n + g_n, x_n - x_0) - \varliminf_n (T_n x_n, x_n - x_0) \leqslant 0.$$

Since S is generalized pseudomonotone, $[x_0, g_0] \in G(S)$ and $(g_n, x_n - x_0) \to 0$. It follows from $(T_n x_n + g_n, x_n - x_0) \to 0$ and $u_n - x_n \to 0$ that

$$(T_n x_n, u_n - x_0) = (T_n x_n + g_n, u_n - x_0) + (g_n, u_n - x_n) + (g_n, x_n - x_0) \to 0 \quad (n \to \infty). \tag{14}$$

We remember $T_n x_n \in Tu_n$ and $T_n x_n \rightharpoonup p - g_0$. Since a maximal monotone mapping T must be generalized pseudomonotone, we obtain from (14) $[x_0, p - g_0] \in G(T)$, i. e., $p \in (T+S)x_0$.

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