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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SECOND-ORDER NONLINEAR DIFFRENTIAL EQUATIONS

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Abstract

In this paper the authors study the oscillation and the asymptotic behavior of solutiosn to the second-order nonlinear differential equations

$$\ddot{x}\pm X(t, x, \dot{x})=0 \qquad (X_\pm)$$

and give necessary and sufficient conditions for Equation (X_+) to have a bounded nonoscillatory solution or to be oscillatory. There are four classes of solutions to Equation (X_-) with different asymptotic behavior. For each class of solution, the necessary or sufficient condition of the existence is obtained.

§ 1. Introduction

In this paper we study the oscillatory and asymptotic behavior of solutions to the second-order nonlinear differential equations

$$\ddot{x}\pm X(t, x, \dot{x})=0, \qquad (X_\pm)$$

where the function $X(t, x, y)$: $[0, \infty)\times R^2\to R$, $R=(-\infty, \infty)$, is continuous and satisfies the following condition:

(A) There exist continuous functions $a_i(t)$, $f_i(x)$ and $g_i(y)$, $i=1, 2$, such that

$$a_1(t)f_1(x)g_1(y)\leqslant X(t, x, y)\leqslant a_2(t)f_2(x)g_2(y),$$

where $a_i(t)\geqslant 0$ and $a_i(t)\not\equiv 0$ on any infinite subinterval of $[0, \infty)$; $xf_i(x)>0$ for $x\neq 0$; $g_i(y)>0$, $i=1, 2$.

A solution of Equations $(X_\pm)$ is called a proper solution if it is defined on a half-line $[t_0, \infty)$ for some $t_0\geqslant 0$. A proper solution is said to be oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$; otherwise it is said to be nonoscillatory. Throughout this paper, we are concerned only with proper solutions. In what follows, for simplicity, by a solution we mean a proper solution.

In section 2 we give the necessary and sufficient conditions for the existence of a nonoscillatory solution and for the oscillation of all solutions of Equation (X_+).

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These results renew the Atkinson's Theorem [1] and its generalizations [4,6,8]. A sufficiency theorem for Equation (X_+) to be oscillatory is also proved.

For Equation (X_-), section 3 is devoted to establishing necessary and/or sufficient conditions for the existence of each of the only four types of solutions with different asymptotic behavior. The results improve some of the results given in [2, 3, 5].

§ 2. The Oscillation of Solutions of Equation (X_+)

Before preceeding we shall require some lemmas, the proofs of which are omitted.

Lemma 1. *Every nonoscillatory solution $x(t)$ of Equation. (X_+) and its first derivative $\dot{x}(t)$ are eventually monotone, and $\dot{x}(t)$ tends to a finite limit as $t\to\infty$.*

Lemma 2. *The integral $\int^\infty ta_1(t)dt$ converges if and only if the integral $\int^\infty ta_2(t)$ dt does.*

We begin by presenting necessary and sufficient conditions of existence of a nonoscillatory bounded solution for Equation. (X_+).

Theorem 1. *There exists a nonoscillatory bounded solution of Equation. (X_+) if and only if*

$$\int_0^\infty ta_1(t)dt<\infty. \tag{1}$$

Proof Necessity. Let $x(t)$ be an eventually positive bounded solution of Equation(X_+). Choose $t_1\geqslant 0$ so large that $x(t)>0$ and $\dot{x}(t)>0$ for $t\geqslant t_1$. By Lemma 1 $\dot{x}(t)$ is nonincreasing for $t\geqslant t_1$, $\dot{x}_\infty=\lim\limits_{t\to\infty}\dot{x}(t)=0$, and $x(t)$ is increasing for $t\geqslant t_1$, $x_\infty=\lim\limits_{t\to\infty}x(t)>0$. So

$$0<x(t_1)\leqslant x(t)<x_\infty,\quad 0<\dot{x}(t)\leqslant\dot{x}(t_1) \tag{2}$$

for $t\geqslant t_1$. Integrating Equation (X_+) from s to $t\geqslant s\geqslant t_1$ and letting $t\to\infty$, we obtain

$$\dot{x}(s)=\int_s^\infty X(\tau,\ x(\tau),\ \dot{x}(\tau))d\tau. \tag{3}$$

Integrating (3) from t_1 to $t\geqslant t_1$ and using (A), we see that

$$\begin{aligned}x(t)-x(t_1)&\geqslant\int_{t_1}^t(\tau-t_1)X(\tau,\ x(\tau),\ \dot{x}(\tau))d\tau\\&\geqslant\int_{t_1}^t(\tau-t_1)a_1(\tau)f_1(x(\tau))g_1(\dot{x}(\tau))d\tau\\&\geqslant m_1n_1\int_{t_1}^t(\tau-t_1)a_1(\tau)d\tau,\end{aligned} \tag{4}$$

where $m_1=\min\limits_{x(t_1)\leqslant x\leqslant x_\infty}f_1(x)>0$, $n_1=\min\limits_{0\leqslant y\leqslant\dot{x}(t_1)}g_1(y)>0$.

By (2) and (4)

$$\int_{t_1}^{t}(\tau-t_1)a_1(\tau)d\tau \geqslant \frac{x_\infty}{m_1 n_1} \tag{5}$$

for $t \geqslant t_1$. It is easy to show that (5) implies (1).

When $x(t)$ is an eventually negative bounded solution, it can be shown by a similar argument that $\int_0^\infty t a_2(t)dt < \infty$. Then (1) holds by Lemma 2. So the necessity part of the proof is complete.

Sufficiency. Since (1) implies $\int_0^\infty t a_2(t)dt < \infty$, we can choose $t_1 \geqslant 0$ so large that

$$M_2 N_2 \int_{t_1}^{\infty} t a_2(t)dt \leqslant \frac{1}{2}, \tag{6}$$

where $M_2 = \max\limits_{\frac{1}{2}\leqslant x\leqslant 1} f_2(x) > 0$, $N_2 = \max\limits_{0\leqslant y\leqslant \frac{1}{2}} g_2(y) > 0$.

Let $\mathscr{F}$ be a set of all continuous functions $z(\cdot) = (x(\cdot), y(\cdot))$: $[t_1, \infty) \to R^2$. We regard $\mathscr{F}$ as a Fréchet space with the topology generated by uniform convergence on any compact subinterval of $[t_1, \infty)$. Let

$$B = \left\{ z \in \mathscr{F}: \frac{1}{2} \leqslant x(t) \leqslant 1,\ 0 \leqslant y(t) \leqslant \frac{1}{2},\ t \in [t_1, \infty) \right\}.$$

B is a closed, convex subset of $\mathscr{F}$. For any $z \in R$, define the operator $T: B \to \mathscr{F}$ by the relation $Tz = (Tx, Ty)$ and

$$\begin{aligned} (Tx)(t) &= 1 - \int_t^\infty (s-t) X(s, x(s), y(s))ds, \quad t \geqslant t_1, \\ (Ty)(t) &= \int_t^\infty X(s, x(s), y(s))ds, \quad t \geqslant t_1. \end{aligned} \tag{7}$$

It can be shown that $TB \subseteq B$, T is continuous and TB is precompact by the topology mentioned above. by Schauder–Tychonov's theorem, there is a $z \in B$ such that $z = Tz$. So

$$\begin{aligned} x(t) &= 1 - \int_t^\infty (s-t) X(s, x(s), y(s))ds, \quad t \geqslant t_1, \\ y(t) &= \int_t^\infty X(s, x(s), y(s))ds, \quad t \geqslant t_1. \end{aligned} \tag{8}$$

Obviously, the function $x(t)$ is an eventually positive bounded solution of Equation (X_+). The theorem is proved.

To study the oscillation of all solutions to Equation (X_+), we need the following definition. A function $X(t, x, y)$ is said to be locally superlinear in x at $|x| = \infty$, if there is a large $\alpha > 0$ such that $f_1(x)$ is nondecreasing for $x \geqslant \alpha$, $f_2(x)$ is nondecreasing for $x \leqslant -\alpha$, and

$$\int_\alpha^\infty \frac{dx}{f_1(x)} < \infty, \quad \int_{-\alpha}^{-\infty} \frac{dx}{f_2(x)} < \infty. \tag{9}$$

Theorem 2. *Suppose $X(t, x, y)$ is locally superlinear in x at $|x| = \infty$, then all solutions to Equation (X_+) are oscillatory if and only if*

$$\int_0^\infty ta_1(t)dt=\infty. \tag{10}$$

Proof By Lemma 2, the necessity follows from the sufficiency of Theorem 1. To prove the sufficiency, let $x(t)$ be an eventually positive solution to Equation (X_+). By Theorem 1, $x(t)$ must be unbounded. Consequently, $x(t)$ is eventually increasing and $x_\infty=\infty$. There is a $t_1\geqslant 0$ such that $x(t_1)=\alpha$, $x(t)$ is increasing and $\dot{x}(t)$ is nonincreasing for $t\geqslant t_1$. So

$$0<\alpha\leqslant x(t), \quad 0\leqslant \dot{x}_\infty\leqslant \dot{x}(t)\leqslant \dot{x}(t_1) \tag{11}$$

for $t_1\geqslant t_1$. Integrating Equation (X_+) from s to t_1, $t_1\leqslant s\leqslant t$, yields

$$\dot{x}(s)-\dot{x}(t)=\int_t^t X(\tau, x(\tau), \dot{x}(\tau))d\tau. \tag{12}$$

Letting $t\to\infty$ in (12) and using (A), we obtain

$$\begin{aligned}\dot{x}(s)\geqslant\dot{x}(s)-\dot{x}_\infty&=\int_s^\infty X(\tau, x(\tau), \dot{x}(\tau), \dot{x}(\tau))d\tau\\ &\geqslant\int_s^\infty a_1(\tau)f_1(x(\tau))g_1(\dot{x}(\tau))d\tau\\ &\geqslant n_1f_1(x(s))\int_s^\infty a_1(\tau)d\tau,\end{aligned} \tag{13}$$

where $n_1=\min\limits_{0\leqslant y\leqslant \dot{x}(t_1)} g_1(y)>0$. Dividing (13) by $f_1(x(s))$ and integrating from t_1 to t, we have

$$\int_\alpha^{x(t)}\frac{dr}{f_1(r)}\geqslant n_1\int_{t_1}^t ds\int_s^\infty a_1(\tau)d\tau\geqslant n_1\int_{t_1}^t (s-t_1)a_1(s)ds. \tag{14}$$

Let $t\to\infty$ in (14); we get $\int_{t_1}^\infty (s-t_1)a_1(s)ds<\infty$, which is a contradiction. The theorem is proved.

Remark 1. Theorems 1 and 2 extend and improve the well-known Atkinson's Theorem [1] and the results of [4, 6, 8]. Maoki and Wong[4, Theorem1] proved Theorem 2 in the case that $g_i(y)\equiv 1$, $f_i(x)$ are nondecreasing for $x\in R$. Wong [6, Theorems 1 and 2] discussed Theorem 1 for $X=a(t)f(x)g(y)$ with $f'(x)\geqslant 0$, $0<k\leqslant g(y)\leqslant K<\infty$, and proved Theorem 2 under additional assumption $\liminf\limits_{|x|\to\infty}\frac{|f(x)|}{|x|^p}>0 (p>1)$. If $g(y)\equiv 1$, we get Theorems 1 and 2 of [8]. Moreover, the proof of sufficiency part of Theorem 2 is much simpler than the one in [4].

Remark 2. The condition imposed in Theorem 2 on the monotonicity of functions $f_i(x)$ is weaker than the corresponding one in [4, 6, 7, 8]. If there exist a large positive number α and nondecreasing continuous functions $\varphi_1(x)$ on $[\alpha, \infty)$ and $\varphi_2(x)$ on $(-\infty, -\alpha]$ such that $f_1(x)\geqslant\varphi_1(x)$ for $x\geqslant\alpha$, $f_2(x)\leqslant\varphi_2(x)$ for $x\leqslant-\alpha$ and $\int_\alpha^\infty\frac{dx}{\varphi_1(x)}<\infty$, $\int_{-\infty}^{-\infty}\frac{dx}{\varphi_2(x)}<\infty$, then the conclusion of Theorem 2 holds. This seems to be better. But, instead of $f_i(x)$ we can choose functions $\bar{f}_i(x)$ such that

$\bar{f}_1(x)=\varphi_1(x)$ for $x\geqslant\alpha$, $\bar{f}_2(x)=\varphi_2(x)$ for $x\leqslant-\alpha$ and (A) holds for $\bar{f}_i(x)$. Therefore, X is still locally superlinear in x at $|x|=\infty$. In particular, taking $\varphi_1(x)=\mu_1x^p$, $\varphi_2(x)=\mu_2x^p$, μ_1, $\mu_2>0$, $p>1$, we obtain a condition corresponding to condition $\liminf_{|x|\to\infty}\frac{|f(x)|}{|x|^p}>0$ used in [6, 8].

Theorem 3. *If, in addition, the following assumption*

(B) $\liminf_{x\to\infty} f_1(x)>0$ and $\liminf_{x\to-\infty} f_2(x)<0$

is satisfied, then

$$\int_0^\infty a_1(t)dt=\infty \tag{15}$$

implies that all solutions to Equation (X_+) *are oscillatory.*

Proof Let solution $x(t)>0$ eventually. A parallel argument is valid for the case that $x(t)<0$ eventually. By Lemma 1, $x(t)$ satisfies (11) and is increasing for $t\geqslant t_1$. Since (B) holds, there is a positive number m_1 such that $f_1(x(t))\geqslant m_1$ for $t\geqslant t_1$. Thus

$$\dot{x}(t_1)-\dot{x}(t)=\int_{t_1}^t X(\tau, x(\tau), \dot{x}(\tau))d\tau\geqslant m_1n_1\int_{t_1}^t a_1(\tau)d\tau, \tag{16}$$

where $n_1=\min_{0<y<\dot{x}(t_1)} g_1(y)>0$. Let $t\to\infty$ in (16); we have $\int_{t_1}^\infty a_1(t)dt\leqslant\frac{1}{m_1n_1}[\dot{x}(t_1)-\dot{x}_\infty]$ $<\infty$. This proves the theorem.

Remark 3. Wong [7] got a suffiicient condition $\int^\infty a(t)dt=\infty$ for the equation $x''(t)+a(t)f(x(t))g(x'(t))=0$ to be oscillatory, where $g(x')>0$, $f'(x)\geqslant0$, $xf(x)>0$ for $x\neq0$, and $a(t)$ may become negative for some t.

§ 3. The Asymptotic Behavior of Solutions of Equation (X_-)

It is well known that every solution to Equation (X_-) is eventually monotone. So either

I. $\dot{x}_\infty=0$, $x_\infty=0$;

II. $\dot{x}_\infty=0$, $x_\infty=c_1\neq0$;

III. $\dot{x}_\infty=\pm c_2^2\neq0$, $x_\infty=\pm\infty$;

or

IV. $\dot{x}_\infty=\pm\infty$, $x_\infty=\pm\infty$.

A solution is said to be of I-type, II-type, III-type and IV-type, if it behaves as above respectively.

We first present some results concerned with bounded solutions of Equation (X_-).

Theorem 4. *There exists a II-type solution to Equation (X_-) if and only if*

$$\int_0^\infty ta_1(t)dt<\infty. \tag{17}$$

Proof Necessity. Suppose $x(t)$ is a II-type solution with $x_\infty>0$. One argues similarly for the case $x_\infty<0$. It is easy to see that $x(t)>0$, $\dot{x}(t)<0$ for $t\geqslant0$, and

$$0<x_\infty<x(t)\leqslant x_0=x(0),\ \dot{x}(0)=\dot{x}_0\leqslant(t)<0 \tag{18}$$

for $t\geqslant0$. Since $\dot{x}_\infty=0$, we have

$$\dot{x}(\mathrm{t})=-\int_t^\infty X(\tau,\ x(\tau),\ \dot{x}\ (\tau))d\tau,\ t\geqslant0 \tag{19}$$

and for any $t\geqslant0$,

$$\begin{aligned} x(t)&=x_0-\int_0^t ds\int_s^\infty X(\tau,\ x(\tau),\ \dot{x}(\tau))d\tau \\ &\leqslant x_0-\int_1^t \tau a_1(\tau)f_1(x(\tau))g_1(\dot{x}(\tau))d\tau \\ &\leqslant x_0-m_1n_1\int_0^t \tau a_1(\tau)d\tau, \end{aligned} \tag{20}$$

where $m_1=\min\limits_{x_\infty\leqslant x\leqslant x_0} f_1(x)>0$, $n_1=\min\limits_{\dot{x}_0\leqslant y\leqslant 0} g_1(y)>0$. So

$$\int_0^t \tau a_1(\tau)d\tau\leqslant\frac{x_0-x_\infty}{m_1n_1}$$

for any $t\geqslant0$. This implies (17).

Sufficiency The proof is similar to theone of sufficiency part of Theorem 1. So we give only a sketch of the proof. By Lemma 2 we can take $t_1\geqslant0$ large enough such that

$$\int_{t_1}^\infty ta_2(t)dt\leqslant\frac{1}{2M_2N_2}, \tag{21}$$

where $M_2=\max\limits_{1\leqslant x\leqslant2} f_2(x)>0$, $N_2=\max\limits_{-\frac{1}{2}\leqslant y\leqslant 0} g_2(y)>0$. Let $\mathscr{F}$ be a Fréchet space defined as in the proof of Theorem 1. Let

$$E=\left\{z\in\mathscr{F}:\ 1\leqslant x(t)\leqslant2,\ \ -\frac{1}{2}\leqslant y(t)\leqslant1,\ t\geqslant t_1\right\}.$$

E is a closed, convex subset of $\mathscr{F}$. For any $z\in E$, define the operator T: $E\to\mathscr{F}$ by the relation $Tz=(Tx,\ Ty)$ and

$$\begin{aligned} (Tx)(t)&=1+\int_t^\infty(s-t)X(s,\ x(s),\ y(s))ds,\ t\geqslant t_1, \\ (Ty)(t)&=-\int_t^\infty X(s,\ x(s),\ y(s))ds,\ t\geqslant t_1. \end{aligned} \tag{22}$$

Then we can show that $TE\subseteq E$, T is continuous and TE is precompact by the topology of compact convergence on $[t_1,\ \infty)$. So T has a fixed point $z\in E$ satisfying

$$\begin{aligned} x(t)&=1+\int_t^\infty(s-t)X(s,\ x(s),\ y(s))ds,\ t\geqslant t_1, \\ y(t)&=-\int_t^\infty X(s,\ x(s),\ y(s))ds,\ t\geqslant t_1. \end{aligned} \tag{23}$$

Thus, we get a II-type solution $x(t)$. The proof of the theorem is complete.

Corollary. *Every bounded solution to Equation* (X_-) *is of I-type if and only if*

$$\int_0^\infty ta_1(t)dt=\infty. \tag{24}$$

To make every nontrivial bounded solution to Equation (X_-) be of II-type, we should confine the function X within necessary limits. A function $X(t, x, y)$ is said to be locally superlinear in x at $x=0$, if there is a small $\alpha>0$ such that $f_2(x)$ is nondecreasing on $(0, \alpha)$, $f_1(x)$ is nondecreasing on $(-\alpha, 0)$, and

$$\int_0^\alpha \frac{dx}{f_2(x)}=\infty, \quad \int_0^{-\alpha} \frac{dx}{f_1(x)}=\infty. \tag{25}$$

Theorem 5. *If* $X(t, x, y)$ *is locally superlinear in* x *at* $x=0$, *then every nontrivial bounded solution to Equation* (X_-) *is of II-type if and only if*

$$\int_0^\infty ta_1(t)dt<\infty. \tag{26}$$

Proof. Obviously, only the sufficiency must be proved. Let $x(t)$ be a nontrivial I-type solution and let $x(t)>0$ for $t\geqslant 0$. In the case $x(t)<0$ the proof is similar. Since $\ddot{x}(t)\geqslant 0$ and $\dot{x}(t)$ is nondecreasing for $t\geqslant 0$, $\dot{x}(t)$ must be negative for $t\geqslant 0$ and

$$0<x(t)\leqslant x_0=x(0), \quad \dot{x}(0)=\dot{x}_0\leqslant \dot{x}(t)<0 \tag{27}$$

for $t\geqslant 0$. By Lemma 2, $\int_0^\infty ta_2(t)dt<\infty$. Since $\dot{x}_\infty=0$, Equation (19) holds. Since $x_\infty=0$, there is a $t_1\geqslant 0$ such that $x(t_1)=\alpha$ and $0<x(t)\leqslant\alpha$ for $t\geqslant t_1$. Using the monotonicity of $x(t)$ and the local superlinearity of X and noting that $f_2(x(t))$ is nonincreasing for $t\geqslant t_1$, from (19) we obtain

$$-\dot{x}(t)\leqslant \int_t^\infty a_2(\tau)f_2(x(\tau))g_2(\dot{x}(\tau))d\tau \leqslant N_2 f_2(x(t))\int_t^\infty a_2(\tau)d\tau \tag{28}$$

for $t\geqslant t_1$, where $N_2=\max\limits_{\dot{x}_0\leqslant y\leqslant 0} g_2(y)>0$. Dividing (28) by $f_2(x(t))$ and integrating from t_1 to t produce

$$\int_{x(t)}^\alpha \frac{dr}{f_2(r)}\leqslant N_2\int_{t_1}^t ds\int_s^\infty a_2(\tau)d\tau \leqslant N_2\int_{t_1}^\infty (\tau-t_1)a_2(\tau)d\tau \tag{29}$$

for $t>t_1$. Letting $t\to\infty$ in (29) gives $\int_0^\alpha \frac{dr}{f_2(r)}<\infty$. This contradiction proves the theorem.

Remark 4. If there exist nondecreasing continuous functions $\varphi_2(x)$ on $(0, \alpha)$ and $\varphi_1(x)$ on $(-\alpha, 0)$ such that $f_2(x)\leqslant\varphi_2(x)$ for $t\in(0, \alpha)$ and $f_1(x)\geqslant\varphi_1(x)$ for $t\in(-\alpha, 0)$, and $\int_0^\alpha \frac{dx}{\varphi_2(x)}=\int_0^{-\alpha}\frac{dx}{\varphi_1(x)}=\infty$, where α is a small positive number, then the conclusion of Theorem 5 is true (see Remark 2). Especially for $\varphi_1(x)=\mu_1 x$, $\varphi_2(x)=\mu_2 x$, $\mu_1, \mu_2>0$, we get conditions $\limsup\limits_{x\to 0^+}\frac{f_2(x)}{x}<\infty$ and $\limsup\limits_{x\to 0^-}\frac{f_1(x)}{x}<\infty$ which are also available to Theorem 5, where $f_i(x)$ may not be monotone near $x=0$.

Remark 5. For $a_1(t)=a_2(t)$, $g_i(\dot{x})=1$, $i=1, 2$,

$$f_1(x)=\begin{cases} a|x|^n x, & x>0, \\ b|x|^n x, & x\leqslant 0, \end{cases} \quad f_2(x)=\begin{cases} b|x|^n x, & x>0, \\ a|x|^n x, & x\leqslant 0, \end{cases}$$

where $n>0$, $b\geqslant a>0$, Theorem 5 and the corollary of Theorem 4 coincide with Taliaferro's Theorems [3, Theorems 4 and 5]. If, in addition, $a=b$, we get a result established by Wong [Theorem 1.1].

Notice that all nontrivial bounded solutions to Equation (X_-) are either of I-type or of II-type if function $X(t, x, y)$ is locally superlinear in x at $x=0$. We give an example to illustrate.

Example Consider the differential equation

$$\ddot{x}=2(t+1)^{-8/3}x^{1/3},\ t\geqslant 0. \tag{30}$$

Here, $a(t)=2(t+1)^{-8/3}$ satisfies $\int^\infty ta(t)dt<\infty$ and $f(x)=x^{1/3}$ is sublinear. It follows from Theorem 4 that Equation (30) has at least one II-type solution. But Theorem 5 is false for Equation (30) which possesses $x=(t+1)^{-1}$ as a I-type solution.

We are now in a position to investigate the existence of unbounded solutions to Equation (X_-).

Theorem 6. *Suppose functions $f_1(x)$ and $f_2(x)$ are nondecreasing. Then there exists a III-type solution to Equation (X_-) if for some $\alpha>0$ either*

$$\int_0^\infty a_2(t)f_2(\alpha t)dt<\infty \tag{31}$$

or

$$\int_0^\infty a_1(t)\,|f_1(-\alpha t)|\,dt<\infty; \tag{32}$$

and only if for some $\alpha>0$ either

$$\int_0^\infty a_1(t)f_1(\alpha t)dt<\infty \tag{33}$$

or

$$\int_0^\infty a_2(t)\,|f_2(-\alpha t)|\,dt<\infty. \tag{34}$$

Proof Let (31) hold. Take $t_1>0$ large enough such that

$$\int_{t_1}^\infty a_2(t)f_2(\alpha t)dt<\frac{\alpha}{2M_2}, \tag{35}$$

where $M_2=\max\limits_{0<y<\alpha} g_2(y)>0$. Let $x(t)$ be any solution to the equations

$$\begin{aligned} &\dot{x}(t)=\frac{\alpha}{2}+\int_{t_1}^t X(s, x(s), \dot{x}(s))ds,\ t\geqslant t_1, \\ &x(t_1)=0. \end{aligned} \tag{36}$$

We claim that $\dot{x}(t)<\alpha$ for $t\geqslant t_1$. If not, then there exists $t_2>t_1$ such that $\dot{x}(t_2)=\alpha$ and $\frac{1}{2}\alpha\leqslant\dot{x}(t)<\alpha$ for $t\in[t_1, t_2)$. Therefore

$$x(t)=\int_{t_1}^{t}\dot{x}(s)ds\leqslant\alpha(t-t_1)<\alpha t. \tag{37}$$

Since $f_2(x)$ is nondecreasing, from (36) we have

$$\alpha=\dot{x}(t_2)=\frac{1}{2}\alpha+\int_{t_1}^{t_2}X(s,\ x(s),\ \dot{x}(s))ds$$
$$\leqslant\frac{1}{2}\alpha+M_2\int_{t_1}^{t_2}a_2(s)f_2(\alpha s)ds<\frac{1}{2}\alpha+M_2\frac{\alpha}{2M_2}=\alpha, \tag{38}$$

which is a contradiction. So $\dot{x}(t)<\alpha$ for $t\geqslant t_1$.

Because $\dot{x}(t)$ is nondecreasing for $t>t_1$, the limit $\dot{x}_\infty$ exists and is a finite positive number. Obviously, as a solution to Equation (X_-), $x(t)$ is aslo of III-type.

Similarly, (32) leads to the existence of an eventually negative III-type solution. So the first part of the proof is complete.

Let $x(t)$ be a III-tyhe solution with $\dot{x}_\infty>0$, $x_\infty=\infty$. Then there is a $t_1>0$ such that $x(t)$ is positive and increasing, $\dot{x}(t)$ is positive and nondecreasing for $t\geqslant t_1$, and

$$0<\beta=\dot{x}(t_1)\leqslant\dot{x}(t)<\dot{x}_\infty \tag{39}$$

for $t\geqslant t_1$. If $t\geqslant 2t_1$, then

$$x(t)=x(t_1)+\int_{t_1}^{t}\dot{x}(s)ds\geqslant x(t_1)+\beta(t-t_1)$$
$$\geqslant x(t_1)+\frac{1}{2}\beta t>\frac{1}{2}\beta t. \tag{40}$$

On the other hand, owing to condition (A) and (40) we have

$$\dot{x}(t)-\dot{x}(2t_1)=\int_{2t_1}^{t}X(s,\ x(s),\ \dot{x}(s))ds$$
$$\geqslant n_1\int_{2t_1}^{t}a_1(s)f_1\Big(\frac{1}{2}\beta s\Big)ds \tag{41}$$

for $t>2t_1$, where $n_1=\min\limits_{\beta\leqslant y\leqslant\dot{x}^\infty}g_1(y)>0$. Letting $t\to\infty$ in (41) gives

$$\int_{2t_1}^{\infty}a_6(s)f_1\Big(\frac{1}{2}\beta s\Big)ds\leqslant\frac{1}{n_1}(\dot{x}_\infty-\beta). \tag{42}$$

The inequality (42) proves that (33) holds for $\alpha=\frac{1}{2}\beta$.

Similarly, a III-type solution $x(t)$ with $\dot{x}_\infty<0$ and $x_\infty=-\infty$ leads to (34). The theorem is proved.

Corollary. *Suppose $f_1(x)$ and $f_2(x)$ are nondecreasing. Then every unbounded solution to Equation (X_-) is of IV-type if for any $\alpha>0$*

$$\int_0^\infty a_i(t)\,|f_i((-1)^{i+1}\alpha t)|\,dt=\infty,\quad i=1,\ 2;$$

and only if for any $\alpha>0$,

$$\int_0^\infty a_i(t)\,|f_i((-1)^{i}\alpha t)|\,dt=\infty,\quad i=1,\ 2.$$

Remark 6. In the case $X=a(t)f(x)g(\dot{x})$ Theorem 6 says that there exists a III-type solution if and only if either $\int_0^\infty a(t)f(\alpha t)dt<\infty$ or $\int_0^\infty a(t)\,|f(-\alpha t)|\,dt<\infty$

for some $\alpha>0$, and the corollary says that every unbounded solution is of IV–type if and only if $\int_0^\infty a(t)\,|f(\pm\alpha t)|\,dt=\infty$ for any $\alpha>0$. Combining the latter result with the corollary of Theorem 4, we can obtain necessary and sufficient condition to ensure that every solution is either of I–type or of IV–type. This result improves Theorem 3.1 in [2] established by one of the authers of the present paper and for $f(x)=x^\lambda(\lambda>1)$ and $g(\dot{x})\equiv 1$ coincides with Taliaferro's result [5, Theoerm 2.4].

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