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POSITIVE MARTINGALES AND RANDOM MEASURES

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Abstract

Given $Q_n(t)$ $(n=0, 1, \cdots)$, a positive martingale indexed by t $(t\in T$, compact metric space) and a measure $\sigma\in M^+(T)$, the random measure $Q\sigma$ is defined as a limit of $Q_n\sigma$. In general $EQ\sigma\leqslant\sigma$. Conditions are given to insure either $EQ\sigma=0$ (degeneracy) or $EQ\sigma=\sigma$ (full action). In the particular case when $Q_n(t)$ a product of independent weight functions, σ is decomposed into a sum of two mutually singular measures, $\sigma=\sigma'+\sigma''$, such that Q acts fully on σ' and is degeneoate on σ'', and the operator EQ is a projection. Examples and applications aoe given (random coveoings, B. Mandelbrot's martingales, multiplicative chaos).

The following situation appears in several circumstances, such as random covering[5,11,7,2,12,16,17,24,25,27,28,6,10], random models of turbulence[18,19,22,13,20,15,21], random geometrical constructions[21,23,26], multiplicative chaos[14].

(T, d) is a compact metric space and $(\Omega, \mathscr{A}, P)$ a probability space. We are given an increasing sequence of σ-fields in $\mathscr{A}$, $(\mathscr{C}_n)_{n\in\mathbb{N}}$, and a sequence of random functions $Q_n(t, \omega)(n\in N,\ t\in T,\ \omega\in\Omega$, the probability space) such that for each t the sequence $(Q_n(t, \omega))_{n\in\mathbb{N}}$ is a positive martingale adapted to $(\mathscr{C}_n)_{n\in\mathbb{N}}$ (positive means $\geqslant 0$) and for almost all ω the functions $Q_n(\cdot, \omega)$ are positive Borel functions on T. To be short, we write $(Q_n)_{n\in\mathbb{N}}$ and we call such a sequence a positive T-martingale. Given a positive Radon measure σ on T (we write $\sigma\in M^+(T)$), we consider the sequence $Q_n\sigma$, and we are looking for a random limit, S. Theorem 1 shows how this is possible. Let us remark that the interesting case is when the martingales $Q_n(t, \cdot)$ are degenerate, that is, converge to 0 almost surely, whenever $t\in T$.

It may happen that $(Q_n\sigma)_{n\in\mathbb{N}}$ is also degenerate, that is $S=0$. In the opposite direction, it may happen that the expectation of S is the expectation of $Q_n\sigma$ (both are measures on T). Theorem 2 says that the general case can be decomposed into these extreme cases.

We indicate the main methods for studying the random measure S, and we give

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a rather general theorem for complete degeneracy, that is, $S=0$ whatever $\sigma\in M^+(T)$ (Theorem 3). This ends the general part (§ 1).

In § 2 we study the most interesting situation, when $Q_n=P_1P_2\cdots P_n$, a product of independent weights of expectation 1. In this case the decomposition theorem has a much stronger form, and the operator $\sigma\to ES$ from $M^+(T)$ into itself is a projection (Theorem 4).

Then we introduce the Peyriére probability on $(T\times\Omega)$ (it is not a tensorial product of measures), a useful tool for investigating the local properties of S (a.s.), and the basic Theorem 5.

The § 3 is devoted to a few examples and comments.

§ 1. The General Theory

Let us define $Q_n(t)=Q_n(t,\ \cdot)$ and consider its expectation

$$q(t)=EQ_n(t). \tag{1}$$

We suppose $q\in L^1(\sigma)$, that is

$$\int_T q(t)d\sigma(t)<\infty. \tag{2}$$

We write $C(T)$ for the space of continuous functions on T; weak convergence of measures means weak*-convergence in the dual of $C(T)$.

Theorem 1. *Assuming* (2), *the random measures $Q_n\sigma$ converge weakly a.s. to a random measure S. Moreover, given a finite or countable family of Borel sets B_j on T, we have*

$$\forall j\quad S(B_j)=\lim_{n\to\infty}(Q_n\sigma)(B_j)\qquad \text{a.s.}. \tag{3}$$

Proof Let Φ be a countable family of bounded Borel functions on T. When $\varphi\in\Phi$ the sequence $\int\varphi Q_n d\sigma$ converges a. s. (we use (2)). Let us write

$$S(\varphi)=\lim_{n\to\infty}\int\varphi Q_n d\sigma\quad \text{a.s.}. \tag{4}$$

Let Φ_0 be a countable denseo subset of $C(T)$, containing 1. The measures $Q_n\sigma$ are norm bounded and converge on Φ_0 a.s., therefore they converge weakly and the weak limit S satisfies a. s.

$$\int\varphi dS=\lim_{n\to\infty}\int\varphi\, Q_n\, d\sigma \tag{5}$$

for all $\varphi\in C(T)$. Given the B_j, let Φ be the union of Φ_0 and $\{1_{B_j}\}$. Writing (4) for $\varphi\in B$ we get (3).

Let us remark that the probability of the event

$$(S(B)=\lim_{n\to\infty}(Q_n\sigma)(B)\ \text{for all Borel sets}\ B) \tag{6}$$

may be zero (and actually is zero in most interesting cases, when S is a.s. singular with respect to σ).

From now on let us write

$$Q\sigma = S = \lim_{n\to\infty} Q_n\sigma. \tag{7}$$

Q is an operator with maps $M^+(T)$ (and $M(T)$ as well) into random measures. EQ is the operator which maps σ into ES (therefore, $M^+(T)$ into $M^+(T)$).

There are two extreme cases. The first is $Q\sigma=0$ (a. s.); we say that Q is degenerate on σ. The second is when the martingale $(Q_n\,\sigma)(B)$ converges in $L^1(\Omega)$ for each given Borel set B (or, the same, when $B=T$); this means

$$E(Q\sigma)(B) = (q\sigma)(B) \tag{8}$$

(q being defined in (1)); we write simply

$$EQ\sigma = q\sigma \tag{9}$$

and say that Q is fully acting on σ. Here is a simple observation, which we shall imporve later in the second part.

Theorem 2. *Given $(Q_n)_{n\in\mathbb{N}}$ and σ, there is a unique decomposition of (Q_n) as a sum of two positive T–martingales*

$$Q_n = Q'_n + Q''_n \tag{10}$$

such that the corresponding operators Q' and Q'' are respectively fully acting and degenerate on σ. Assuming moreover $q(t)=1$ on T, the operator EQ is a contraction of $M^+(T)$.

Proof Let $\mathscr{B}_0$ be a countable set of Borel sets in T, such that $\mathscr{B}_0$ is a Boole algebra and $\mathscr{B}_0$ generates the Borel σ–field of T; it is well known that a positive measure on $\mathscr{B}_0$ has a unique extension to the Borel σ–field. Let us write $Q_n\sigma=S_n$ and

$$E(S\,|\,\mathscr{C}_n) = S'_n \tag{11}$$

meaning that

$$E(S(B)\,|\,\mathscr{C}_n) = S'_n(B) \tag{12}$$

for each $B\in\mathscr{B}_0$. Obviously S'_n is a.s. a positive measure on $\mathscr{B}_0$ and

$$S'_n(B) \leqslant S_n(B), \tag{13}$$

therefore

$$\begin{cases} S'_n = Q'_n\sigma, \\ 0\leqslant Q'_n \leqslant Q_n \end{cases} \tag{14}$$

and $(Q'_n)_{n\in\mathbb{N}}$ is a positive T–martingale. Moreover (12) implies that $S'_n(B)$ tends to $S(B)$ in $L^1(\Omega)$, which implies (and is equivalent to)

$$EQ'\sigma = q'\sigma \tag{15}$$

with the obvious notation $q'(t) = EQ'_n(t)$. Moreover

$$E(Q-Q')\sigma = 0. \tag{16}$$

(15) means that Q' acts fully on σ and (16) means that Q'' is degenerate on σ. When

we suppose $q(t)=1$ on T we have

$$EQ\sigma=EQ'\sigma\leqslant\sigma, \tag{17}$$

therefore EQ is a contraction.

In general we cannot say more. Given $\sigma\in M^+(T)$, $q(t)$ and $0\leqslant q'(t)\leqslant q(t)$ $(t\in T)$ we can built a T-martingale $(Q_n)_{n\in\mathbb{N}}$ satisfying (9) and (15). Given a contraction of $M^+(T)$, we can write it as EQ for a convenient T-martingale $(Q_n)_{n\in\mathbb{N}}$ such that $q(t)=1$ on T.

Let us describe now the main methods to prove either full action or degeneracy. In order to prove full action we consider $h>1$ and the submartingale $E((Q_n\sigma)(T))^h$. If

$$E((Q_n\sigma))T))^h=O(1) \quad (\text{for some } h>1), \tag{18}$$

then Q acts fully on σ. This is particularly manageable when $h=2$, in the form

$$\iint_{T^2} E(Q_n(t)Q_n(s))d\sigma(t)d\sigma(s)=O(1).$$

If moreover $k(t, s)$ is a Borel function on T^2 which is either positive or bounded, the formula

$$E\iint k(t, s)dS(t)dS(s)=\lim_{n\to\infty}\iint_{T^2} E(Q_n(t)Q_n(s))k(t, s)d\sigma(t)d\sigma(s)$$

allows to study some a.s. properties of the random measure S.

In order to prove degeneracy we consider $0<h<1$ and the supermartingale $E((Q_n\sigma)(T))^h$. If

$$E((Q_n\sigma)(T))^h=o(1) \quad (\text{for some } h<1), \tag{19}$$

then Q is degenerate on σ. Let us use (19) in order to get a sufficient condition for complete degeneracy, that is, $Q\sigma=0$ (a. s.) for every $\sigma\in M^+(T)$.

Theorem 3. *Let α be a positive number such that* $\text{meas}_\alpha T<\infty$, $0<h<1$ *and* $C>0$. *Suppose*

$$E\sup_{t\in B}(Q_n(t))^h\leqslant C(\text{diam } B)^{(1-h)\alpha} \tag{20}$$

for all balls B and some $n=n(B)$ depending on B. Then Q is completely degenerate, that is, $Q\sigma=0$ a.s. for all $\sigma\in M^+(T)$.

Proof Changing C if necessary (20) holds for all Borel sets B. Let us decompose T into a finite union of disjoint Borel sets, B_j, and choose $n_j=n(B_j)$. Writing $S_n=Q_n\sigma$ as usual and assuming $n\geqslant n_j$ we have

$$\begin{aligned}\Sigma E(S_n(B_j))^h&\leqslant\Sigma E(S_{n_j}(B_j))^h\leqslant\Sigma E\sup_{t\in B_j}(Q_{n_j}(t))^h(\sigma(B_j))^h\\&\leqslant\Sigma(E\sup_{t\in B_j}(Q_{n_j}(t))^h)^{\frac{1}{1-h}})^{1-h}(\Sigma\sigma(B_j))^h\\&\leqslant C(\Sigma(\text{diam } B_j)^\alpha)^{1-h}(\sigma(T))^h\end{aligned} \tag{21}$$

by using the submartingale property, Hölder's inequality and (20). Now, using the numerical inequality

$$(\Sigma a_j)^h \leqslant \left(\frac{\sup a_j}{\Sigma a_j}\right)^{1-h} \Sigma a_j^h \qquad (a_j \geqslant 0) \tag{22}$$

we have

$$E(S_n(T))^{h/2} = E(\Sigma S_n(B_j))^{h/2} \leqslant \left(E\left(\frac{\sup S_n(B_j)}{S_n(T)}\right)^{\frac{1-h}{2}} (\Sigma(S_n(B_j))^h)^{1/2}\right)$$

$$\leqslant \left(E\left(\frac{\sup S_n(B_j)}{S_n(T)}\right)^{1-h} E\,\Sigma(S_n(B_j))^h\right)^{1/2}. \tag{23}$$

According to (21) and the assumption $\mathrm{meas}_\alpha T < \infty$ a sufficient condition for (19) $\left(\text{with } \frac{h}{2} \text{ instead of } h\right)$ is

$$\lim_{\rho\to 0} \overline{\lim_{n\to\infty}}\, E\left(\frac{S_n(B_\rho)}{S_n(T)}\right)^{1-h} = 0, \tag{24}$$

where B_ρ denotes an arbitrary ball of radius ρ. Moreover we can assume $S(T)>0$ a. s. (if it is not the case, we can replace $S_n(T)$ by $1_{S(T)>0}\, S_n(T)$ in (19) and (23) and restricting the probability space to the event $S(T)$ in (24) we are led to the same computations). (24) will follow from

$$\lim_{\rho\to 0} \overline{\lim_{n\to\infty}}\, \frac{S_n(B_\rho)}{S_n(T)} = 0 \quad \text{a.s.}, \tag{25}$$

Suppose now that (25) does not hold. Then there exist $\varepsilon>0$, a sequence of balls B_{ρ_j} $(\rho_j \to 0)$ and a doubly indexed sequence $n_{jk} (\lim_{k\to\infty} n_{jk} = \infty)$ such that

$$P(S_{n_{jk}}(B_{\rho_j}) > \varepsilon S_{n_{jk}}(T)) > \varepsilon.$$

Moreover we can suppose that $\overline{\lim}\, B_{\rho_j}$ contains one point at most. Taking limits we obtain

$$P(S(B_{\rho_j}) \geqslant \varepsilon S(T)) \geqslant \varepsilon,$$

hence

$$P(S(\overline{\lim}\, B_{\rho_j}) \geqslant \varepsilon S(T)) \geqslant \varepsilon.$$

Therefore $\overline{\lim}\, B_{\rho_j}$ consists of one point, t, and, due to $S(T)>0$ a.s.

$$P(S(t)>0)>0.$$

This implies $\inf Q_n(t) > 0$, which contradicts assumption (20). The contradiction proves (25).

Let us remark that the proof is simpler when we assume $\mathrm{meas}_\alpha T = 0$, using (21) directly together with

$$(S_n(T))^h \leqslant \Sigma(S_n(B_j))^h \tag{26}$$

in order to get (19).

We shall see an application of Theorem 3 in the examples.

Let us remark that there are stronger conditions than complete degeneracy, such as

$$\lim_{n\to\infty} \sup_{t\in T} Q_n(t) = 0 \quad \text{a.s.} \tag{27}$$

We may call (27) "strong complete degeneracy". We shall see an example of this (random covering).

§ 2. Independent Multiplications

From now on we suppose

$$Q_n = P_1 P_2 \cdots P_n \quad (n \in \mathbb{N}), \tag{28}$$

where the $P_n = P_n(t, \omega)$ are independent positive random functions such that $P_n(\cdot, \omega)$ is borelian for almost all ω and

$$EP_n(t, \cdot) = 1 \tag{29}$$

for all $t \in T$. Then $(Q_n)_{n\in\mathbb{N}}$ is a T-martingale with $q(t) = 1$ $(t \in T)$. Here is an improvement of Theorem 2.

Theorem 4. *Given $(P_n)_{n\in\mathbb{N}}$ as above and $\sigma \in M^+(T)$ there exists a Borel set B such that*

$$E(S \mid \mathscr{C}_n) = 1_B Q_n \sigma. \tag{30}$$

σ can be decomposed as a sum of two mutually singular measures, $\sigma = \sigma' + \sigma''$ (where $\sigma' = 1_B \sigma$), such that Q acts fully on σ' and is degenerate on σ''. The operator EQ maps σ into σ', and it is a projection.

Before proving Theorem 4, let us observe that, given σ, we have as a consequence of the theorem

$$\begin{cases} Q'_n(t, \omega) = 1_B(t) Q_n(t, \omega) \\ Q''_n(t, \omega) = (1 - 1_B(t)) Q_n(t, \omega) \end{cases} \tag{31}$$

with the notations of Theorem 2, and also

$$q'(t) = 1_B(t). \tag{32}$$

Now B depends on σ Usually the operator EQ will kill a "singular" part of σ (that is, σ''), and keep a "regular" part of σ (that is, σ'); for example, σ'' may be the part of σ which is carried by Borel sets of dimension $\leqslant \alpha$, a given number. In all examples below, EQ has this character of a regulasing operator.

Proof of Theorem 4 Given n, let us consider the T-martingale

$$Q_m^{(n)} = P_{n+1} P_{n+2} \cdots P_{n+m} \quad (m \in \mathbb{Z}) \tag{33}$$

and the corresponding operator $Q^{(n)}$. Clearly with obvious notations

$$E(Q\sigma \mid \mathscr{C}_n) = E(P_1 P_2 \cdots P_n Q^{(n)} \sigma \mid \mathscr{C}_n) = P_1 P_2 \cdots P_n E(Q^{(n)} \sigma). \tag{34}$$

Writing (as in (15) and (16))

$$\sigma' = EQ\sigma = q'\sigma \tag{35}$$

and considering this as the commun expectation of both members of (34) we obtain

$$E(S \mid \mathscr{C}_n) = q' Q_n \sigma \tag{36}$$

(that is

$$\begin{cases} Q'_n = q'Q_n \\ Q''_n = (1-q')Q_n \end{cases} \tag{37}$$

with the notations of Theorem 2). For every Borel set A in T

$$\begin{cases} \lim_{n\to\infty} \int_A q'Q_n d\sigma = \int_A dS \quad \text{(a.s. and in } L^1(\Omega)) \\ \lim_{n\to\infty} \int_A (1-q')Q_n d\sigma = 0 \quad \text{(a.s.)}. \end{cases} \tag{38}$$

It follows that the intersection of the sets $\{t\in T \mid q'(t)>0\}$ and $\{t\in T \mid 1-q'(t)>0\}$ has zero σ-measure, therefore $q'=1_B\sigma$ for some Borel set B. Then (36) reads (30), σ' $(=1_B\sigma)$ and σ'' $(=(1-1_B)\sigma)$ are mutually singular, $(Q_n\sigma')(T)$ $(=(Q'_n\sigma)(T))$ converges in $L^1(\Omega)$ and $(Q_n\sigma'')(T)$ $(=(Q''_n\sigma)(T))$ converges to 0 a.s. Finally $Q\sigma'=Q\sigma$, therefore EQ is a projection.

Let us suppose now $\sigma=EQ\sigma$, that is, Q acts fully on σ, and moreover σ is a probability measure (we write $\sigma\in M_1^+(T)$). There is a unique probability measure $\mathscr{Q}$ on the σ-field generated by the $B\times A$ (B: Borel set in T, A: event in Ω) which satisfies

$$\int_{T\times\Omega} f(t,\omega)d\mathscr{Q}(t,\omega) = E\int_T f(t,\omega)dS(t) \tag{39}$$

for all positive measurable functions $f(t,\omega)$. By definition it is the Peyrière probability. We also write $E_Q f$ for the first member of (39).

Theorem 5. *Assuming $\sigma\in M_1^+(T)$, $\sigma=EQ\sigma$, and moreover that the distribution of $P_n(t)$ does not depend on t $(t\in T)$, the P_n $(=P_n(t,\omega))$ are $\mathscr{Q}$-independent.*

Proof We have to show

$$E_Q \prod_1^N f_n(P_n) = \prod_1^N E_Q f_n(P_n)$$

for all $N\in\mathbb{N}$ and positive Borel functions f_n defined on $\mathbb{R}^+$ $(n=1, 2, \cdots, N)$. Using (39) and the previous notation (33) we have

$$E_Q \prod_1^N f_n(P_n) = E\int_T \prod_1^N (P_n f_n(P_n))d(Q^{(n)}\sigma) = \int_T \prod_1^N E(P_n f_n(P_n))d\sigma \tag{40}$$

and the assumption on the distribution of $P_n(t)$ implies that $E(P_n f_n(P_n))$ does not depend on t, therefore

$$E_Q \prod_1^N f_n(P_n) = \prod_1^N E(P_n f_n(P_n)) = \prod_1^N E_Q f_n(P_n), \tag{41}$$

what we had to prove.

As an application let us suppose that the distribution of $P_n(t)$ does not depend on n (and does not depend on t either), that is, all $P_n(t)$ have same distribution as a given positive random variable P such that $EP=1$. Then

$$\lim_{n\to\infty} (P_1P_2\cdots P_n(t))^{1/n} = \exp E(P\log P) \quad S\text{-a.s.} \tag{42}$$

with probability 1. This is nothing but the law of large numbers applied to the $\log P_n$ in the probability space $(T\times\Omega, \mathscr{Q})$.

§ 3. Examples

1. Random coverings.

Suppose $T=\mathbb{T}=\mathbb{R}/\mathbb{Z}$, $1>l_1\geqslant l_2\geqslant l_3\geqslant\cdots\geqslant l_n\cdots>0$, and

$$P_n(t, \omega)=\frac{1-\chi_n(t-\omega_n)}{1-l_n}, \tag{43}$$

where $\chi_n=1_{[0, l_n]}$, and ω_n are independent random variables equally distributed on $\mathbb{T}$. The martingale

$$\int_{\mathbb{T}} P_1\cdots P_n(t)dt \tag{44}$$

converges in $L^2(\Omega)$ if and only if

$$\iint_{\mathbb{T}^2}\prod_1^\infty \frac{E((1-\chi_n(t-\omega_n))(1-\chi_n(s-\omega_n))}{(1-l_n)^2}\,dt<\infty. \tag{45}$$

Writing

$$\Delta_n(t)=\chi_n*\check{\chi}_n(t)=\int_{\mathbb{T}}\chi(t+u)\chi(u)du \tag{46}$$

(Δ_n is the triangle function supported by $[-l_n, l_n]$, $\Delta(0)=l_n$), (45) can be written as

$$\begin{cases}\sum_1^\infty l_n^2<\infty,\\ \int_{\mathbb{T}}\exp\sum_1^\infty \Delta_n(t)dt<\infty.\end{cases} \tag{47}$$

A remarkable theorem of L. Shepp says that there is strong complete degeneracy when (47) does not hold. This has a nice interpretation when we consider the random intervals

$$I_n=[0, l_n]+\omega_n. \tag{48}$$

Strong complete degeneracy means

$$P\left(\bigcup_1^\infty I_n=\mathbb{T}\right)=1 \tag{49}$$

and full action implies

$$P\left(\bigcup_1^\infty I_n=\mathbb{T}\right)<1. \tag{50}$$

(47) can also be written as

$$\sum_1^\infty n^{-2}\exp(l_1+\cdots+l_n)<\infty \quad \text{(see [25])}. \tag{51}$$

There are many variations around this theme (see [12], second edition). The analogue of (45) is always a sufficient condition for non-covering (that is, the probability of non-covering is strictly positive). Is it also a necessary condition as in the case we just considered? This seems to be open, even in the case of covering a ball by random balls in an euclidean space of dimension $\geqslant 2$.

In the case of non-covering, a closer investigation of the random measure S leads to a precise estimate of the Hausdorff dimension of the subset of $\mathbb{T}$ which is not covered infinitely many times:

$$\dim(\mathbb{T}\setminus\overline{\lim}\, I_n) = 1 - \overline{\lim}\, \frac{l_1+\cdots+l_n}{\log n} \quad \text{a.s. [12] first edition.} \tag{52}$$

2. Some martingales of Benoit Mandelbrot [19,22,13,20,15].

Suppose $T=\{1, 2, \cdots, c\}^N$. Let W be a positive random variable such that $EW=1$, and let $W_{i_1 i_2\cdots i_n}$ be independent copies of W ($n\in\mathbb{N}$, $i_j\in\{1, 2, \cdots c\}$). We define $i_j(t)$ as the j-th coordinate of $t\in T$ and

$$P_n(t) = W_{i_1(t)i_2(t)\cdots i_n(t)}. \tag{53}$$

We consider σ = Haar measure on T (considered as the group $(\mathbb{Z}/c\mathbb{Z})^N$). Then we have the following results.

Q is degenerate on σ, that is $EQ\sigma=0$, if $E(W\log W)\geqslant \log c$.

Q acts fully on σ, that is $EQ\sigma=\sigma$, if $E(W\log W)<\log c$.

$(Q_n\sigma)(T)$ converges to $(Q\sigma)(T)$ in $L^h(\Omega)$ if and only if $E(W^h)<c^h$ $(h>1)$.

Moreover, if Q acts fully on σ and if the distribution of W is not too sparse, we have a.s.

$$\lim_{n\to\infty} \frac{\log(Q\sigma)(I_n(t))}{\log \sigma(I_n(t))} = D = 1 - E\left(\frac{W\log W}{\log c}\right)$$

except on t-set of $Q\sigma$-measure 0. As a consequence, $Q\sigma$ is concentrated on Borel sets of dimension D, and all Borel sets of dimension $<D$ have $Q\sigma$-measure zero, a.s. The probability of Peyriere was introduced in this context[15].

3. Log normal operators and multiplicate chaos.

This again relies on an intuition of B. Mandelbrot ([18, 21] p. 380). The theory is developed in [14]. Here

$$P_n(t) = \exp(X_n(t) - \frac{1}{2} EX_n^2(t)),$$

where the $X_n(t)$ are independent gaussian centered random functions on T. The distribution of the $P_n(\cdot)$ depends only on the correlation functions

$$p_n(t, s) = E(X_n(t)X_n(s)) \quad (t\in T,\ s\in T). \tag{54}$$

A basic fact is that the distribution of the operator Q (that is, all joint distributions of $(Q\sigma_1(B_1)$, $(Q\sigma_2)(B_2)$, $\cdots (Q\sigma_n)(B_n)$, for all choices of n, $\sigma_1, \cdots \sigma_n$, $B_1, \cdots B_n$) depends only on

$$q(t, s) = \sum_1^\infty p_n(t, s) \leqslant \infty \tag{55}$$

whenever the $p_n(t, s)$ are positive. A case of particular interest is when $\exp(-yd^2(t, s))$ is a correlation function (in other words, a kernel of positive type) for all $y>$

0; this is true when (T, d) is embedded in a euclidian space with the euclidean metric. An important example then is

$$q(t, s)=\frac{u}{2}\int_1^\infty \exp(-yd^2(t, s))\frac{dy}{y}=u\log^+\frac{1}{d(t, s)}+O(1) \tag{56}$$

with $u>0$ given.

Let us add the following assumption (satisfied in the particular case mentioned above). Writing $N(\varepsilon, B)$ for the smallest number of balls of diameter ε whose union contains the given set B, we assume

$$\sup_B N\left(\frac{1}{2}\operatorname{diam} B, B\right)<\infty \tag{57}$$

the supremum being taken on all balls in T. This condition appears in different contexts in [1] and [3]. It implies that Hausdorff dimension and capacitarian dimension are the same for Borel subsets of T, and it implies also that the Dudley integral $J(B, d)$ satisfies

$$J(B, d)=\int_0^\infty \sqrt{\log N(\varepsilon, B)}\,d\varepsilon<C\operatorname{diam} B \tag{58}$$

for all balls B, C depending only on (T, d).

Assuming (57), Q is completely degenerate when

$$\dim T<\frac{u}{2}. \tag{59}$$

If (assuming (57) again)

$$\dim T>\frac{u}{2}, \tag{60}$$

Q is not completely degenerate and the infimum of the dimensions of random Borel sets on which the non-vanishing random measures $Q\sigma$ are concentrated ($\sigma\in M^+(T)$) is

$$\dim T-\frac{u}{2}\quad\text{(see [14])}. \tag{61}$$

For the first port of this statement we need only (58) instead of (57). Let us sketch the proof (and correct in this way a mistake in the proof given in [14], where formula (182) is not correct). It relies on Theorem 3. Choosing α and $0<h_0<h<1$ such that

$$\frac{uh}{2}>\frac{uh_0}{2}=\alpha>\dim T \tag{62}$$

we shall prove (20), defining

$$\begin{cases} Q_n(t)=\exp(Y_n(t)-\frac{1}{2}EY_n^2(t)) \\ E(Y_n(t)Y_n(s))=q_n(t,s)=\frac{u}{2}\int_1^n \exp(-yd^2(t, s))\,\frac{dy}{y}. \end{cases} \tag{63}$$

We choose

$$n=n(B)=(\operatorname{diam} B)^{-2}+O(1). \tag{64}$$

In order to prove (20) we choose $p>1$, $q>1$, such that

$$\frac{1}{p}+\frac{1}{q}=1 \tag{65}$$

(later on, we take q near 1). Choosing $s\in B$ we have

$$\begin{aligned} E\sup_{t\in B}(Q_n(t))^h &= E\left(\sup_{t\in B}\left(\frac{Q_n(t)}{Q_n(s)}\right)^h (Q_n(s))^h\right) \\ &\leqslant \left(E\sup_{t\in B}\left(\frac{Q_n(t)}{Q_n(s)}\right)^{hp}\right)^{\frac{1}{p}} (E(Q_n(s))^{hq})^{\frac{1}{q}} \\ &= (E\sup_{t\in B}\exp(hp(Y_n(t)-Y_n(s)))^{\frac{1}{p}} n^{-\frac{u}{4}(h-qh^2)}. \end{aligned} \tag{66}$$

According to properties of gaussian processes[4] we have

$$E\sup_{t\in B}\exp(hp(Y_n(t)-Y_n(s))\leqslant e^{CJ(B,hp\delta)}, \tag{67}$$

where

$$\delta^2(t,s)=E(|Y_n(t)-Y_n(s)|^2)=u\int_1^n(1-e^{-yd^2(t,s)})\frac{dy}{y}\leqslant und^2(t,s). \tag{68}$$

Using (64), (66), (67), (68) and the assumption (58) we get

$$E\sup_{t\in B}(Q_n(t))^h\leqslant C(\mathrm{diam}\,B)^{\frac{u}{2}h(1-qh)}, \tag{69}$$

where C depends only on (T, d) and q. Choosing now q such that

$$h\frac{1-qh}{1-h}=h_0 \tag{70}$$

we obtain (20), which ends the proof.

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